Another method of calculation already indicated by BERTHELOT in 1856 viz. from the molecular-volumes and the molecular-refractions of the substances that have interacted, at equal temperatures, led to better results. It has already been remarked by others, that the ethereal salts of the fatty acids in general often seem to be formed without great change of volume and this seems equally to be the case with triacylins. For caprin the result of this calculation of the molecular-volume even perfectly agrees with the one really found; it does not result therefrom that this should necessarily be the case with the other terms as well, and in the subjoined table given by Mr. SCHEY their deviations are shown.

A third method of calculation starts from what is found for one of the terms and as all of them rise or fall with an equal difference of composition, it takes into account the average value of this difference. Starting from caprin, as the purest product, the values calculated on this base and those for molecular-volumes and molecular-refraction concurred pretty well, as will be seen from subjoined tables.

As to the meltingpoints it was found that they were quite or nearly quite equal for capric acid and for tricaprin, while for the lower terms of the triacylins the meltingpoint is below that of the acid, for the higher ones on the other hand above it, which does not agree with what BERTHELOT thought to have found.

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Physics. — *„On the Vibrations of Electrified Systems, placed in a Magnetic Field".* By Prof. H. A. LORENTZ.

(Read in the meeting of January 28th 1899).

§ 1. Many spectral lines show the ZEEMAN-effect according to the well known elementary theory, and are thus changed into triplets or, if viewed along the lines of force, into doublets, yet there are a rather large number of cases, in which the phenomena are more complicated. CORNU [1]) found that the line D_1 becomes a quartet, whose outer and inner components are polarized, the first parallel and the latter perpendicularly to the lines of force. Similar quartets have been observed in other cases. Sometimes [2]), in triplets and quar-

[1]) CORNU, Comptes rendus, T. 126.

[2]) BECQUEREL and DESLANDRES, Comptes rendus, T 127, p. 18.

tets, the inner and outer lines have interchanged their ordinary states of polarization. Finally MICHELSON, PRESTON [1]) and other physicists have seen a division of some lines into 5, 6 or even more components.

I shall examine in this paper, to what extent such multiple lines may be explained by appropriate assumptions concerning the way in which light is emitted. Of course I am perfectly aware of the possibility that my interpretation of the facts will have to be replaced by a more adequate one. The special form of my hypotheses has the less value, as in the only case in which I have endeavoured to account for all the peculiarities of the phenomena, I have succeeded but poorly, at the best.

§ 2. Since the components, into which a line is broken up by the magnetic force, are in many cases as sharp as the original line itself, it must be admitted that the periods of all the luminous particles of the source of light are modified in the same way. This is only possible in two ways. Either, in the magnetic field, all the particles must take the same direction, or the modification of the periods must remain unchanged, into whatever position the particles may be turned. The first assumption leads however to some difficulties [2]). I shall therefore suppose the luminous particles to be spherical bodies, having the same properties in all directions. This may be true, even though the chemical atoms be of a much less simple structure; indeed, the vibrating spherical ion may very well be only a part, perhaps a very small part, of the whole atom [3]).

It has been shown in a former paper [4]) that a triplet may be observed if, among the principal modes of vibration of the system, there be three, for which, outside the magnetic field, the time of vibration is the same, or, as we may say, if the system have three equivalent degrees of freedom. Afterwards Mr. PANNEKOEK [5]) remarked that a quartet may appear if there be, in the same sense, four equivalent degrees of liberty, and in general, a n-fold line, if n of the principal modes have equal periods.

Now, spherical systems, vibrating in one of their higher modes, have indeed more than three equivalent degrees of freedom.

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[1]) MICHELSON, Phil. Mag., Vol. 45, p. 348. PRESTON, ibidem, p. 325.

[2]) See LORENTZ, Verslag der Verg. Akad. van Wetensch. VI, p. 197, and Arch. néerl., Sér. 2, T. 2, p. 5.

[3]) See LORENTZ, Verslag der Verg. Akademie van Wetenschappen VI, p. 514.

[4]) Wied. Ann. Bd. 63, p. 278.

[5]) Verslag der Verg. Akademie van Wetenschappen VII, p. 120. Proceedings Royal Acad. Vol. I, p. 96

§ 3. I shall consider in the first place an infinitely thin spherical shell of radius a, charged, in the state of equilibrium, to a uniform surface-density σ. The surface-density of the ponderable matter itself will be denoted by ϱ. We shall suppose that the points of the shell can only be displaced along its surface, that the elements carry their charge along with them, and finally that, after a displacement, each element is acted on by an elastic force, which is brought into play merely by the displacement of the element itself, and not by the relative displacement of adjacent elements.

When the motions are infinitely small, the elastic force may be taken proportional to the displacement $\mathfrak{a}$. Let it be

$$-k^2 \mathfrak{a}$$

per unit area, the constant k^2 having the same value all over the sphere.

The only connexion between the different parts of the shell will consist in their mutual electric forces. If the wave-length of the emitted radiations be very large in comparison with the radius of the sphere, we have merely to consider the ordinary *electrostatic* actions, depending solely on the configuration of the system. Hence there will be no resistances proportional to the velocities, and consequently no damping. In fact, it is well known that the damping which, in some degree, must always be caused by the loss of energy, accompanying the radiation, may be neglected when the wave-length is very much larger than the dimensions of the vibrating system.

§ 4. In the absence of magnetic forces the shell can vibrate in the following way.

Let Y_h be a surface harmonic of order h. Then the displacement of a point of the sphere is

$$p \frac{\partial Y_h}{\partial l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Here l is the direction in the surface in which Y_h increases most rapidly, and $\frac{\partial Y_h}{\partial l}$ is to be regarded as a vector in this direction l. The coefficient p is the same all over the sphere; it has the form

$$q \cos (n_h t + c) , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

so that n_h is the frequency of the vibrations.

In consequence of the displacements (1), the electric density will have changed from σ to

$$\sigma + h(h+1)\frac{\sigma}{a^2} p\, Y_h\,.$$

Hence, there will be an electric force

$$-4\pi V^2 \frac{h(h+1)}{2h+1} \cdot \frac{\sigma}{a} p \frac{\partial Y_h}{\partial l}$$

($V =$ velocity of light) along the surface and, as the density differs from σ by an infinitely small amount, we may write for the force per unit area

$$-4\pi V^2 \frac{h(h+1)}{2h+1} \cdot \frac{\sigma^2}{a} p \frac{\partial Y_h}{\partial l}\,.$$

The equation of motion becomes

$$\varrho \ddot{p} \frac{\partial Y_h}{\partial l} = -k^2 p \frac{\partial Y_h}{\partial l} - 4\pi V^2 \frac{h(h+1)}{2h+1} \frac{\sigma^2}{a} p \frac{\partial Y_h}{\partial l},$$

and the frequency n_h is determined by

$$\varrho n_h^2 = k^2 + 4\pi V^2 \frac{h(h+1)}{2h+1} \cdot \frac{\sigma^2}{a} \quad . \quad . \quad . \quad . \quad (3)$$

Thus, we see that the frequency is the same for all motions corresponding to a harmonic of order h, no matter what particular harmonic of this order may be chosen.

If we put $h = 1$, we obtain the frequency of the slowest vibrations; $h = 2$ corresponds to the first of the higher types of motion, and so on. However each of the different types includes a certain number of different modes of motion.

In the motion we have considered there is a kinetic energy

$$T = \tfrac{1}{2} \varrho \dot{p}^2 \int \left(\frac{\partial Y_h}{\partial l}\right)^2 d\omega\,,$$

$d\omega$ being an element of the sphere, and the integration extending all over the surface.

In virtue of the properties of spherical harmonics we may also write

$$T = \tfrac{1}{2}\, h\,(h+1)\frac{\varrho}{a^2}\, \dot{p}^2 \int Y_h^2\, d\omega \, .$$

The potential energy is given by

$$U = 2\,\pi\, V^2 \frac{h^2 (h+1)^2}{2\,h+1}\, \frac{\sigma^2}{a^3}\, p^2 \int Y_h^2\, d\omega + \tfrac{1}{2}\, k^2 p^2 \int \left(\frac{\partial\, Y_h}{\partial\, l}\right)^2 d\omega =$$

$$= \left[2\,\pi\, V^2 \frac{h^2 (h+1)^2}{2\,h+1}\, \frac{\sigma^2}{a^3} + \tfrac{1}{2}\, h\,(h+1)\, \frac{1}{a^2}\, k^2\right] p^2 \int Y_h^2\, d\omega \, .$$

If we put

$$A_h = 4\,\pi\, V^2 \frac{h^2\,(h+1)^2}{2\,h+1}\, \frac{\sigma^2}{a^3} + h\,(h+1)\, \frac{k^2}{a^2}$$

and

$$B_h = h\,(h+1)\, \frac{\varrho}{a^2}\, ,$$

these formulae may be replaced by

$$T = \tfrac{1}{2}\, B_h\, \dot{p}^2 \int Y_h^2\, d\,\omega$$

and

$$U = \tfrac{1}{2}\, A_h\, p^2 \int Y_h^2\, d\,\omega \, .$$

§ 5. We shall now take for h a determinate number and consider only vibrations corresponding to harmonics of this order. These motions of the common frequency n_h may differ from one another by the position of the *poles* of the harmonic Y_h. Moreover, vibrations depending on different functions Y_h may be superposed with any amplitudes and phases we like.

However, not all of these modes of motion are mutually independent. Since any surface harmonic of order h may be decomposed into $2\,h+1$ particular harmonics of the same order, there are only $2\,h+1$ equivalent degrees of freedom, for which the frequency is n_h. As for those $2\,h+1$ *fundamental* harmonics, as we shall call them, they need only satisfy the condition that none of them can be expressed in terms of the other ones. After having chosen these functions, which I shall denote by

$$Y_{h1}\ ,\quad Y_{h2}\ ,\quad Y_{h3}\ \text{etc}\ ,$$

we may write for the displacement in the most general case we shall have to consider

$$\alpha = p_1 \frac{\partial Y_{h1}}{\partial l} + p_2 \frac{\partial Y_{h2}}{\partial l} + p_3 \frac{\partial Y_{h3}}{\partial l} + \text{etc.}, \quad . \quad . \quad . \quad (4)$$

where each term represents a vector along the surface in the manner indicated in § 4, so that l has different meanings in the different terms.

The potential and the kinetic energy will now be found to be

$$U = \tfrac{1}{2} a_{11} p_1^2 + \tfrac{1}{2} a_{22} p_2^2 + \tfrac{1}{2} a_{33} p_3^2 + \text{ etc.} \ldots +$$

$$+ a_{12} p_1 p_2 + a_{13} p_1 p_3 + \text{ etc.},$$

$$T = \tfrac{1}{2} b_{11} \dot{p}_1^2 + \tfrac{1}{2} b_{22} \dot{p}_2^2 + \tfrac{1}{2} b_{33} \dot{p}_3^2 + \text{ etc.} \ldots +$$

$$+ b_{12} \dot{p}_1 \dot{p}_2 + b_{13} \dot{p}_1 \dot{p}_3 + \text{ etc.},$$

where

$$a_{\mu\mu} = A_h \int Y^2_{h\mu} \, d\omega, \qquad a_{\mu\nu} = A_h \int Y_{h\mu} \, Y_{h\nu} \, d\omega,$$

$$b_{\mu\mu} = B_h \int Y^2_{h\mu} \, d\omega, \qquad b_{\mu\nu} = B_h \int Y_{h\mu} \, Y_{h\nu} \, d\omega \, .$$

As long as we limit the investigation to the vibrations of order h, we may ignore the other degrees of freedom; we may then consider the $2\,h + 1$ coefficients $p_1, p_2, p_3 \ldots$ as the general coordinates. The equation of motion for the coordinate p_μ will be

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{p}_\mu}\right) = - \frac{\partial U}{\partial p_\mu} \, ;$$

it will take the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{p}_\mu}\right) = - \frac{\partial U}{\partial p_\mu} + Q_\mu, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

if, besides the forces which we have considered thus far, there are other forces whose general components are Q_μ.

§ 6. If an electrified system be vibrating in a magnetic field, its parts will be acted on by electromagnetic forces proportional to the charges. Per unit charge these forces are given by the vector product of the velocity and the magnetic force in the field.

Let there be a mode of motion A, with frequency n, before there is any magnetic force, and let F be the electromagnetic forces arising from this motion as soon as the field is produced. The direction of these forces will obviously change with the frequency n, and to determine their action on the system is a problem of „resonance" or of „forced vibrations". In general, the system will respond to the forces F by a motion in several of its other fundamental modes. In fact, any particular motion B will certainly be excited if only the forces F do a positive or negative work in an infinitely small displacement corresponding to that mode B.

Since the electromagnetic forces are perpendicular to the velocities, the forces F will do no work if the infinitely small displacement belong to the mode A itself. A direct influence of the forces F on the motion A which gave rise to them is thereby excluded.

As to the other modes, all depends on their frequency. If the frequency n' of a motion B be considerably different from n, the forced vibration B, if it exist at all, will be very insignificant, for experience shows the forces F to be very feeble as compared with the other forces acting in the system. As well as the forces F themselves, the amplitude of the forced vibrations B will be proportional to the strength H of the field. Hence, the electromagnetic forces F', which exist in consequence of the vibration B, will be of the order H^2, and it will be permitted to neglect their reaction on the original motion.

The case is quite different as soon as the frequency of B is equal to that of A. The amplitude of the new motion B will then rise to a much higher value; as may be deduced from the equations of the problem, it will reach the same order of magnitude as the amplitude of A itself. The influence of the forces F' on the original motion will likewise be much greater than in the former case.

One may see by a simple reasoning that this influence will consist in a modification of the period. Since the forces F have the same phase as the velocities in the motion A, there will be a difference of phase of $^1/_4$ period between them and the displacements A. On the other hand, the displacements in the motion B have the same or the opposite phase as the forces F, and the phase of the forces F' will differ by $^1/_4$ period from that of the displacements B. These latter forces will therefore have the same or the opposite phase as the displacements A, and this is precisely what is required, if they are to change the frequency of A.

We see at the same time that the simultaneous motions A and B will differ in phase by $^1/_4$ period. This is the reason why circularly polarized light can be emitted in the direction of the lines of force.

§ 7. The foregoing reasoning shows that, in the magnetic field, the vibrations of order h will never be perceptibly modified by the vibrations of different orders. We may therefore continue to consider them by themselves. Now, the meaning of the term Q_μ in equation (5) is this, that $Q_\mu\, \delta p_\mu$ is the work of the electromagnetic forces corresponding to the infinitely small displacement determined by δp_μ. But the electromagnetic forces are linear functions of the velocities; consequently, Q_μ will take the form

$$Q_\mu = \Sigma_\nu\, \varepsilon_{\mu\nu}\, \dot{p}_\nu . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The coefficients ε are easily calculated. Let the centre of the shell be the origin of coordinates, the axis of z having the direction of the magnetic force H. Then, if r be the distance to the centre, and

$$W_{h\mu} = r^h\, Y_{h\mu}$$

the solid harmonic of degree h, corresponding to the surface harmonic $Y_{h\mu}$, I find

$$\varepsilon_{\mu\nu} = \frac{H\sigma}{a^{2h+2}} \int z \begin{vmatrix} x, & y, & z, \\ \dfrac{\partial W_{h\mu}}{\partial x}, & \dfrac{\partial W_{h\mu}}{\partial y} & \dfrac{\partial W_{h\mu}}{\partial z} \\ \dfrac{\partial W_{h\nu}}{\partial x}, & \dfrac{\partial W_{h\nu}}{\partial y}, & \dfrac{\partial W_{h\nu}}{\partial z} \end{vmatrix} d\omega \quad . \quad . \quad . \quad . \quad (7)$$

I shall suppose that the axis of y points to the place the observer occupies when viewing the phenomena across the lines of force.

It will sometimes be found convenient to distinguish the fundamental harmonics by suffixes, indicating the position of their poles. Thus Y_x will be the surface harmonic of the first order whose pole is determined by the intersection of the axis of x with the sphere; Y_{xy} the harmonic of the second order, having its poles on the axes of x and y, Y_{xx} the zonal harmonic, whose poles coincide on the axis of x. If these notations be used, the suffix which indicates the order of the harmonic may be omitted, By (7) we see that

$$\varepsilon_{\mu\mu} = 0 \ , \qquad \varepsilon_{\nu\mu} = -\varepsilon_{\mu\nu} .$$

These relations would hold for all electrified systems, vibrating in the magnetic field.

§ 8. We shall begin by examining the vibrations, depending on harmonics of the first order.

Let the fundamental harmonics be

$$Y_{11} = Y_x \ , \qquad Y_{12} = Y_y \ , \qquad Y_{13} = Y_z \ ,$$
$$W_{11} = x \ , \qquad W_{12} = y \ , \qquad W_{13} = z \ .$$

Then:

$$a_{11} = a_{22} = a_{33} = {}^4/_3\, \pi\, a^2 A_1,$$

$$a_{12} = a_{23} = a_{31} = 0,$$

$$b_{11} = b_{22} = b_{33} = {}^4/_3\, \pi\, a^2 B_1 = {}^8/_3\, \pi\, \varrho,$$

$$b_{12} = b_{23} = b_{31} = 0,$$

$$\varepsilon_{12} = {}^4/_3\, \pi\, H\, \sigma \ , \qquad \varepsilon_{13} = \varepsilon_{23} = 0 \ ,$$

and, if we replace a_{11}, b_{11} and ε_{12} by α_1, β_1, ε_1,

$$\beta_1 \ddot{p}_1 = -\alpha_1 p_1 + \varepsilon_1 \dot{p}_2 \ , \quad \ldots \ldots \quad (8)$$

$$\beta_1 \ddot{p}_2 = -\alpha_1 p_2 - \varepsilon_1 \dot{p}_1 \ , \quad \ldots \ldots \quad (9)$$

$$\beta_1 \ddot{p}_3 = -\alpha_1 p_3 .$$

From these equations we conclude that, for $H = 0$ and $\varepsilon_1 = 0$, all vibrations have the frequency n_1, given by

$$n_1^2 = \frac{\alpha_1}{\beta_1} = \frac{A_1}{B_1} \ ,$$

which follows also from (3).

When there is a magnetic field, the vibrations corresponding to Y_z will still have this frequency n_1, but besides these there will be two motions with a modified time of vibration. In order to find them, we may suppose that p_1 and p_2 contain the factor e^{int} multiplied by quantities independent of the time. Then, neglecting quantities of the order of H^2, we may satisfy (8) and (9), by assuming either

$$p_2 = +\, i p_1, \qquad n = n_1 + n'_1$$

or

$$p_2 = -\, i p_1, \qquad n = n_1 - n'_1 .$$

In these formulae

$$n'_1 = \frac{\varepsilon_1}{2\,\beta_1} = \frac{H\,\sigma}{4\,\varrho},$$

or, writing e for the total charge $4\,\pi a^2 \sigma$, and m for the total mass $4\,\pi a^2 \varrho$,

$$n'_1 = \frac{H\,\mathrm{e}}{4\,m} .$$

In the two modes of motion, which correspond to $p_2 = +ip_1$, and $p_2 = -\,i p_1$, and the expressions for which are got in the ordinary way by taking only the real parts of the complex quantities, the coexisting Y_x- and Y_y-vibrations will show a difference of phase of $1/4$ period. This difference will have opposite signs for the two modes.

The vibrations corresponding to surface harmonics of the first order may be roughly described as oscillations of the entire charge in the direction of one of the axes of coordinates, or, to speak more correctly, in these vibrations there exists at every instant an „electric moment" parallel to one of the axes. Thus it appears that the mode of motion we have now examined closely resembles the one that is assumed in the elementary theory of the ZEEMAN-effect and it is but natural that we should again be led to the triplets and doublets of this theory. Only, for equal values of e and m the change n'_1 of the frequency is half of what it would be in the elementary explanation.

§ 9. In investigating the vibrations of the second order we shall introduce two new axes OX' and OY', which are got by rotating OX and OY in their plane and the first of which bisects the angle XOY. We shall take for the fundamental harmonics:

$$Y_{21} = Y_{xy}, \quad Y_{22} = Y_{x'y'}, \quad Y_{23} = Y_{xz}, \quad Y_{24} = Y_{yz}, \quad Y_{25} = Y_{zz}.$$

We may really do so, because any harmonic of the second order may be decomposed into these five functions. Now we have the o lowing solid harmonics

$$W_{21} = {}^3/_2\, x\, y, \quad W_{22} = {}^3/_2\, x'\, y' = {}^3/_4\, (y^2 - x^2),$$

$$W_{23} = {}^3/_2\, x\, z, \quad W_{24} = {}^3/_2\, y\, z$$

$$W_{25} = {}^1/_2\, (-x^2 - y^2 + 2\, z^2)\ ,$$

and, putting

$${}^1/_5\, \pi\, a^2\, A_2 = \alpha_2, \quad {}^1/_5\, \pi\, a^2\, B_2 = \beta_2, \quad {}^3/_5\, \pi\, H\, \sigma = \varepsilon_2\ ,$$

the following values of the coefficients:

$$a_{11} = a_{22} = a_{33} = a_{44} = 3\, \alpha_2, \quad a_{55} = 4\, \alpha_2$$

$$b_{11} = b_{22} = b_{33} = b_{44} = 3\, \beta_2, \quad b_{55} = 4\, \beta_2$$

$$\varepsilon_{12} = +\, 2\, \varepsilon_2, \qquad \varepsilon_{21} = -\, 2\, \varepsilon_2$$

$$\varepsilon_{34} = +\, \varepsilon_2, \qquad \varepsilon_{43} = -\, \varepsilon_2.$$

All coefficients that have not been mentioned here have the value 0.

The equations of motion become

$$3\, \beta_2\, \ddot{p}_1 = -\, 3\, \alpha_2\, p_1 + 2\, \varepsilon_2\, \dot{p}_2 \quad \ldots\ldots \quad (10)$$

$$3\, \beta_2\, \ddot{p}_2 = -\, 3\, \alpha_2\, p_2 - 2\, \varepsilon_2\, \dot{p}_1 \quad \ldots\ldots \quad (11)$$

$$3\, \beta_2\, \ddot{p}_3 = -\, 3\, \alpha_2\, p_3 + \varepsilon_2\, \dot{p}_4 \quad \ldots\ldots \quad (12)$$

$$3\, \beta_2\, \ddot{p}_4 = -\, 3\, \alpha_2\, p_4 - \varepsilon_2\, \dot{p}_3 \quad \ldots\ldots \quad (13)$$

$$\beta_2\, \ddot{p}_5 = -\, \alpha_2\, p_5.$$

It appears that, in the absence of a magnetic force, all vibrations of the second order will have a common frequency n_2, given by

$$n_2{}^2 = \frac{\alpha_2}{\beta_2} = \frac{A_2}{B_2}.$$

When the shell is placed in the magnetic field, it will be only for the Y_{zz}-vibrations that this frequency remains unchanged, and there will be *four* motions with a slightly increased or diminished frequency.

Operating again with expressions that contain the factor e^{int}, we can satisfy (10) and (11) by the values

$$p_2 = + i p_1 \quad , \quad n = n_2 + n_2' ,$$

and likewise by

$$p_2 = - i p_1 \quad , \quad n = n_2 - n_2' ,$$

the change in the frequency being given by

$$n_2' = \frac{\varepsilon_2}{3\beta_2} = \frac{H\sigma}{6\varrho} = \frac{He}{6m}.$$

In both cases we have to do with a combination of a Y_{xy}- and a $Y_{x'y'}$-vibration, the two vibrations having equal amplitudes, and differing in phase by $^1/_4$ period.

From (12) and (13) we deduce the possibility of two similar combinations of a Y_{xz}- and a Y_{yz}-vibration; the frequency is

$$n_2 + \tfrac{1}{2} n_2' ,$$

for one combination, and

$$n_2 - \tfrac{1}{2} n_2' ,$$

for the other.

§ 10. Similar results are obtained by supposing that a charge is distributed with uniform volume-density σ over a spherical space and that each element of volume, after having undergone a displacement $\mathfrak{a}$ from its position of equilibrium, is acted on by an elastic force, proportional to the displacement. Let $k^2 \mathfrak{a}$ be this force per unit volume, ϱ the uniform volume-density of the ponderable matter, and let us suppose that this density is invariable and that, besides the charge σ, the sphere contains an equal charge of opposite sign that is *immovable*. Then, outside the magnetic field, a motion represented by

$$\mathfrak{a} = p \frac{\partial W_h}{\partial l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

may take place.

By l I have now indicated the direction in space in which the solid harmonic W_h increases most rapidly, and the differential coefficient is to be understood as a vector in that direction. The factor p is still of the form (2), and for the frequency I find

$$\varrho\, n_h^2 = k^2 + 4\,\pi\, V^2 \frac{h\, \sigma^2}{2\,h+1}\,.$$

This formula is of some interest in connection with an important phenomenon that presents itself in the series of spectral lines. If, namely, the number h is made to increase indefinitely, the frequency n_h approaches to a determinate limit.

It appears from (14) that in the present case, as well as in the former one, each type of motion corresponds to a certain spherical harmonic. Hence, all the reasonings of the foregoing articles may be repeated with only a slight modification.

I shall not dwell at length on this subject; suffice it to say, that in the magnetic field the vibrations of the first order have the three frequencies

$$n_1 \quad \text{and} \quad n_1 \pm \frac{H\,\mathrm{e}}{2\,m}\,,$$

whereas the frequencies of the motions of the second order are

$$n_2\,, \quad n_2 \pm \frac{H\,\mathrm{e}}{2\,m} \quad \text{and} \quad n_2 \pm \frac{H\,\mathrm{e}}{4\,m}\,.$$

In these expressions e again denotes the total charge, and m the total mass.

§ 11. The fundamental electromagnetic equations for the surrounding ether enable us to determine the vibrations emitted by the systems whose motion we have examined. The expressions for the components of the dielectric displacement will contain terms inversely proportional to the distance r, but also other terms varying as the second and higher powers of r^{-1}. Now, it is clear that only the terms of the first kind are to be taken into account when we treat of the emission of light. If these terms are calculated for the vibrations of the first and the second order, they are found in the latter case to contain the factor $\frac{a}{\lambda}$, a being again the radius of the sphere, and λ the wave-length of the emitted radiations. If, therefore, the displacements on the sphere itself in the Y_2-vibrations were of the same order of magnitude as those in the Y_1-vibrations, the light produced by the first would be very much feebler than that which is due to the latter. All determinations of molecular dimensions tend to show that $\frac{a}{\lambda}$ is an excessively small fraction; if it were otherwise, there would be so much damping that the spectral lines could not be as sharp as they are.

Now one might believe that on the sphere itself the amplitude of the Y_2-vibrations were so much greater than that of the Y_1-vibrations, that the motions of the second order could produce a perceptible amount of light, notwithstanding the smallness of the factor $\frac{a}{\lambda}$. Assuming this for an instant, improbable though it seemed, and determining by my formulae, for the shell as well as for the solid sphere, the properties of the emitted rays of light, I was led precisely to CORNU's quartet if I supposed the observations to take place across the lines of force, the middle line of the quintet vanishing altogether. This seemed very promising at first sight, but, considering the radiation along the lines of force, I found that in this case it ought to be the two inner lines of the quartet that remained, and not the outer ones, as observation has shown to be the case. This suffices to banish all idea that the influence of the factor $\frac{a}{\lambda}$ might be compensated by a large amplitude in the sphere. We cannot but take for granted that the vibrations corresponding to harmonics of the second order are incapable of radiating. This is due to the circumstance that in adjacents parts of the sphere there are equal and opposite displacements of equal charges.

Of course, the vibrations of still higher orders will be equally incapable of producing rays, and similar remarks will apply to systems of a totally different nature. Thus, the higher tones of a sounding body whose dimensions are very much smaller than the wave-length, will be very feebly heard, and it is for a similar reason that the tone of a tuning fork has to be reinforced by a resonance case. After all it seems very probable that the light of a flame is in every case caused by vibrations in which there is a variable electric moment in a definite direction, and which may in so far be called of the first order, though they need not depend precisely on a spherical harmonic. If this principle be admitted, it may be shown that, along the lines of force, only those components remain visible which are polarized in the direction of these lines, when viewed across the field.

§ 12. Seeking for some means by which the vibrations of the second order might be made to reveal themselves in the spectrum, and by which therefore the multiple lines in the ZEEMAN-effect migh perhaps be explained, I have been led to the assumption that in a source of light there exist not only the primary vibrations we have so far considered, but also secondary vibrations which are produced in the way of VON HELMHOLTZ's combination tones. This assumption is by no means a new one. Many years ago, Mr. V. A. JULIUS [1]) has remarked that the many equal differences existing between the frequencies of different lines of a spectrum, seem to indicate the presence of such secondary vibrations. Indeed, it seems difficult to conceive another cause for the constancy of the difference of frequencies which is found e.g. in the doublets of the alkali metals. It ought to be remarked that secondary vibrations, the word being taken in its widest sense, may arise in very different ways. The displacements may be so large that the elastic forces — and in our spheres also the electric forces — are no longer proportional to the elongations. Or, perhaps, the vibrations will cause the superficial density of the charged shell to vary to such a degree, that the convection current cannot be reckoned proportional to the velocity and the *original* density. Moreover, two vibrating particles may act upon each other and each or one of them may thus be made to vibrate as a whole. This case would present itself e.g., if there were *two* concentric spherical shells, each of them capable of vibrating in the way we have examined. They might have different frequencies, or even one of them might have the frequency 0; i.e., one sphere might be charged to an invariable density proportional to some surface harmonic.

It is not necessary to make any special assumption concerning the mechanism by which the secondary vibrations are produced. It will suffice to assume that the system is perfectly symmetrical all around the centre of a particle and that, if in one primary vibration we have to do with expressions of the form:

$$q \cos (nt + c), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and in a second one with similar expressions:

$$q' \cos (n't + c'), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

the derived vibrations will depend on the product

[1]) V. A. JULIUS, De lineaire spectra der elementen. Verh. der Kon. Akad. v. Wetensch., Deel 26.

$$g\, q' \cos(nt + c) \cos(n't + c') = \tfrac{1}{2}\, q\, q' \cos[(n - n')\, t + (c - c')] +$$
$$+ \tfrac{1}{2}\, q\, q' \cos[(n + n')\, t + (c + c')].$$

Of the two vibrations, I shall only consider the one whose frequency is $n - n'$.

§ 13. It is easily seen, and may be verified by working out some example, that we can obtain a secondary vibration of the first order, i. e. one which really emits light, by combining a vibration of the second order with one of the first order, these primary motions being executed either by the same sphere, or by two concentric shells.

Let us now imagine the three vibrations corresponding to the functions Y_x, Y_y and Y_z, and the five vibrations determined by Y_{xy}, $Y_{x'y'} = \frac{1}{2}(Y_{yy} - Y_{xx})$, Y_{xz}, Y_{yz}, Y_{zz}. Let the factor p that has been introduced in § 4 be of the form (15) for one of the former vibrations, and of the form (16) for one of the latter. By considering the symmetry of the system, it may be shown that a secondary vibration in the direction of one of the axes of coordinates can only be produced by the combination of these two, if, among the three indices of the two harmonics, the one that corresponds to the axis considered, appear an uneven number of times. Thus the mutual action of a Y_{xy}- and a Y_x-vibration will call forth only a vibration in the direction of OY.

Another question is to determine the amplitudes of the derived vibrations taking place along OX, OY and OZ. In every special case the amplitude must be proportional to qq'; we may therefore denote it by multiplying qq' by a certain amplitude factor.

These factors are not independent of one another; they may all be expressed in terms of one of them. To understand this, it must be kept in mind that, if the first of the two combined primaries a and b be decomposed into some components, say into a_1, a_2, etc., the secondary vibration $\{a, b\}$ may be considered as the resultant of $\{a_1, b\}$, $\{a_2, b\}$, etc. If we denote the amplitude factors by $[Y_{xx}, Y_x]_x$, etc., the last index indicating the direction of the secondary vibration, we shall have

$$[Y_{yy}, Y_x]_x = [Y_{zz}, Y_x]_x\,,$$

and

$$[Y_{xx}, Y_x]_x + [Y_{yy}, Y_x]_x + [Y_{zz}, Y_x]_x = 0\,.$$

The last formula is a consequence of the relation

$$Y_{xx} + Y_{yy} + Y_{zz} = 0\,.$$

Let us put

$$[Y_{xx}, Y_x]_\mathrm{x} = \varkappa .$$

Then

$$[Y_{yy}, Y_x]_\mathrm{x} = -\tfrac{1}{2}\varkappa, \quad [Y_{zz}, Y_x]_\mathrm{x} = -\tfrac{1}{2}\varkappa,$$

so that the amplitude factor is now known in all cases in which the harmonic of the second order is a zonal one whose pole coincides with that of the other harmonic or is 90° distant from it. All other cases may be reduced to these two by suitable decomposition of the harmonics. In this way I find the values of the amplitude factors inscribed in the following table; the letters x, y, z again serve to indicate the direction of the secondary vibration.

	Y_{xy}	$Y_{x'y'}$	Y_{xz}	Y_{yz}	Y_{zz}
Y_x	$+{}^3/_4\,\varkappa$ (y)	$-{}^3/_4\,\varkappa$ (x)	$+{}^3/_4\,\varkappa$ (z)	0	$-\frac{1}{2}\varkappa$ (x)
Y_y	$+{}^3/_4\,\varkappa$ (x)	$+{}^3/_4\,\varkappa$ (y)	0	$+{}^3/_4\,\varkappa$ (z)	$-\frac{1}{2}\varkappa$ (y)
Y_z	0	0	$+{}^3/_4\,\varkappa$ (x)	$+{}^3/_4\,\varkappa$ (y)	$+\varkappa$ (z)

§ 14. In the magnetic field there are three modes of motion of the first order, whose frequencies are

$$n_1 + n'_1\ , \quad n_1 - n'_1\ , \quad n_1 \,. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

We shall call the amplitudes of the variable p_1 (§ 8) in the first two motions, and that of the variable p_3 in the last one

$$q_1\ ,\quad q_2\ ,\quad q_3 .$$

Then there are five motions of the second order, having the frequencies

$$n_2 + n'_2\ ,\quad n_2 - n_2\ ,\quad n_2 + \tfrac{1}{2} n'_2\ ,\quad n_2 - \tfrac{1}{2} n'_2\ ,\quad n_2 \,.\,. \quad (18)$$

Let

$$q'_1\ ,\quad q'_2\ ,\quad q'_3\ ,\quad q'_4\ ,\quad q'_5$$

respectively be the amplitudes of p_1 (§ 9) for the two first motions, of p_3 for the third and fourth, and of p_5 for the last vibration.

We shall now take as an example the combination of the first of the vibrations (17) with the first of (18).

The motion of the second order consists in a Y_{xy}- and a $Y_{x'y'}$-vibration for which we may respectively write

$$q'_1 \cos\left[(n_2 + n'_2)\, t + c'\right]$$

and

$$q'_1 \cos\left[(n_2 + n'_2)\, t + c' + \tfrac{1}{2}\pi\right].$$

On the other hand, there are at the same time a Y_x-vibration

$$q_1 \cos\left[(n_1 + n_1')\, t + c\right],$$

and a Y_y-vibration

$$q_1 \cos\left[(n_1 + n_1')\, t + c + \tfrac{1}{2}\pi\right].$$

Consulting the small table of the last Art., I find a vibration

$$\begin{aligned} &{}^3/_4\, \varkappa\, q_1 q_1' \cos\left[(n_2 - n_1 + n_2' - n_1')\, t + c' - c - \tfrac{1}{2}\pi\right] - \\ &- {}^3/_4\, \varkappa\, q_1 q_1' \cos\left[(n_2 - n_1 + n_2' - n_1')\, t + c' - c + \tfrac{1}{2}\pi\right] = \\ &= {}^3/_2\, \varkappa\, q_1 q_1' \cos\left[(n_2 - n_1 + n_2' - n_1')\, t + c' - c - \tfrac{1}{2}\pi\right] \end{aligned}$$

parallel to OX, and a vibration

$$\begin{aligned} &{}^3/_4\, \varkappa\, q_1 q_1' \cos\left[(n_2 - n_1 + n_2' - n_1')\, t + c' - c\right] + \\ &+ {}^3/_4\, \varkappa\, q_1 q_1' \cos\left[(n_2 - n_1 + n_2' - n_1')\, t + c' - c\right] = \\ &= {}^3/_2\, \varkappa\, q_1 q_1' \cos\left[(n_2 - n_1 + n_2' - n_1')\, t + c' - c\right) \end{aligned}$$

in the direction of OY Hence, across the lines of force we shall see light whose vibrations are perpendicular to the lines of force and whose intensity may be put proportional to $\frac{9}{4}\varkappa^2 q_1^2 q_1'^2$. Since there is a difference of phase of $^1/_4$ period between the two secondary vibrations, both together will produce circularly polarized light along the lines of force.

By similar reasoning it is found that the second of (17) and the second of (18) do not produce any secondary vibrations.

Examining all the 15 combinations, I find the following results, as regards the radiation across the lines of force.

A. There will be seen in the spectrum the following lines, whose vibrations are parallel to the lines of force.

1. A central line whose frequency is $n_2 - n_1$, and whose intensity is proportional to

$$q_3^2\, q'^2_5\ [12].$$

2. Two side lines, each at a distance of $\frac{1}{2} n_2'-n_1'$ from the preceding one. Their intensities are

$$\frac{9}{4} q_1^2 q_3'^2 \ [9] \quad \text{and} \quad \frac{9}{4} q_2^2 q_4'^2 \ [9] \ .$$

B. The following lines will be produced by vibrations perpendicular to the lines of force.

1. Two lines at distances $n_2'-n_1'$ from A, 1. Intensities:

$$\frac{9}{4} q_1^2 q_1'^2 \ [9] \quad \text{and} \quad \frac{9}{4} q_2^2 q_2'^2 \ [9] \ .$$

2. Two lines at distances $\frac{1}{2} n'_2$ from A, 1. Intensities:

$$\frac{9}{16} q_3^2 q'^2_3 \left[\frac{9}{2}\right] \text{ and } \frac{9}{16} q_3^2 q'^2_4 \left[\frac{9}{2}\right].$$

3. Two lines at distances n'_1 from A, 1. Intensities:

$$\frac{1}{4} q_1^2 q'^2_5 \left[\frac{3}{2}\right] \text{ and } \frac{1}{4} q_2^2 q'^2_5 \left[\frac{3}{2}\right].$$

In the observations along the lines of force, the lines B only will be seen, with the same relative intensities. They will then be circularly polarized.

Of couise, the source of light will contain innumerable molecules for which the quantities q and q' will have widely different values. Assuming that both the vibrations of the first and those of the second order take place indifferently in all directions, and that even a particular vibration of one kind may be equally accompanied by vibrations of the other kind in all possible directions, I find for the relative intensities the numbers inclosed in brackets.

Perhaps the way in which the ions are made to vibrate will be unfavourable to the existence at the same time of certain particular vibrations of the first and the second order; some of the derived vibrations would then have a smaller intensity than the one indicated. As to the middle line A, 1, it must always be weakened by absorption in the exterior parts of the source. Yet, in the case of luminous particles of a symmetrical structure, it seems impossible that this central line should ever vanish altogether.

§ 15. If there were no ZEEMAN-effect for the vibrations of the first order, we should have $n'_1 = 0$, and the lines B, 3 would form a single line in the middle, whose intensity would be 3. If in this case, for one reason or another, this line B, 3, and the lines A, 1 and B, 2 were to disappear or to become imperceptible, we should only see A, 2 and B, 1 and this would be a quartet as has been observed by CORNU.

The case $n'_2 = {}^2/_3\, n'_1$ (§ 9) is likewise of some interest. B, 1 and B, 2 would then form a single pair, each of whose components would have the intensity ${}^{27}/_2$. The distance of these strong lines would be half that of the lines A, 2, and, if it were not for A, 1 and B, 3 we should have a quartet, the outer components of which would be polarized perpendicularly to, and the inner components in the direction of the lines of force. Quartets of this kind have been really observed.

§ 16. The following remarks remain to be made.

1. Since the frequency of the secondary vibrations is wholly determined by that of the primary ones, we need not trouble ourselves about a direct ZEEMAN-effect in these secondary vibrations.

2. Any explanation of the spectral lines must account for their *reversibility*. Consequently, the foregoing theory, which attributes some lines to derived vibrations, will hold only, if a system can be made to vibrate by the action of forces, whose period corresponds not to a primary, but to a secondary vibration of the system. I believe this to be really possible, but for want of space, I shall not now insist on this point.

3. If one wishes to apply the above considerations to vibrations of an order, higher than the second, one soon perceives that it is impossible to obtain a motion of the first order by combining these higher modes with vibrations of the first order.

Vibrations that are capable of radiating may however be derived from two vibrations whose order differs by unity. If now the primary motions showed the peculiarity that has been mentioned in § 10 and has been observed in the series of spectral lines, this peculiarity would also present itself in those derived vibrations whose frequency is the sum of the frequencies of the primaries; it would not exist in the secondary vibrations corresponding to the difference of these frequencies. I must acknowledge however that this conception of the series of spectral lines seems hardly reconcilable to the fact of so large a number of lines becoming simple triplets in the magnetic field.

(March 22th, 1899.)