Citation:

J.C. Kluyver, On the reducible hyperelliptic Integrals, in:
KNAW, Proceedings, 1, 1898-1899, Amsterdam, 1899, pp. 449-465

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN TE AMSTERDAM.

PROCEEDINGS OF THE MEETING

of Saturday April 22th 1899.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige Afdeeling van Zaterdag 25 Maart 1899 Dl VII)

The following papers were read:

Mathematics. — "*On reducible hyperelliptic Integrals*". By Prof. J. C. KLUYVER.

(Read in the Meeting of March 25th 1899)

In some cases an Abelian integral of the first kind and of deficiency p can be reduced to an elliptic integral. According to a theorem of WEIERSTRASS [1]) when such a reducible integral presents itself, it is possible to transform by a substitution of order r the ϑ-function of the first order into an other one of order r, so that the $p-1$ constituents $\tau'_{12}, \tau'_{13}, \ldots \tau'_{1p}$ in the first row of the period matrix all assume the value zero. If, conversely, it is possible by a substitution of order r to find a ϑ-function whose period matrix shows this peculiarity, at least *one* of the integral

[1]) KOWALEVSKI, *Acta Math.* IV, p 395.

of the first kind is reducible, and it is possible to construct rational functions on the RIEMANN surface T, which are doubly periodic functions of this integral.

Let us suppose that in the ϑ-function of p variables $\vartheta(u;\tau)$, where u denotes the p normal integrals and τ denotes the given period matrix, we make a substitution of order r associated with the Abelian matrix [1])

$$\begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix}$$

of $2p$ rows and $2p$ columns of integers, in such a way that the separate matrices $\alpha, \beta, \alpha', \beta'$ satisfy the equations

$$\alpha' = 0 \ , \ \overline{\alpha}\beta' = \overline{\beta}'\alpha = \alpha\overline{\beta}' = \beta'\overline{\alpha} = r \ , \ \overline{\beta}\beta' - \overline{\beta}'\beta = 0.$$

According to this substitution the integrals u are replaced by other integrals w determined by the equation

$$u = \alpha w,$$

and $\vartheta(u;\tau)$ of the first order becomes a function $\vartheta_r(w;\tau')$ of order r with a period matrix τ' which can be derived from the equation

$$\alpha\tau' = \beta + \tau\beta'.$$

From the above relations we can immediately calculate the increments Ω, taken by the integrals w, when by describing some closed curve on T the normal integrals u are increased by

$$\omega = k + \tau k',$$

where k and k' denote two columnletters.

In the first place we find

$$\alpha\Omega = \omega = k + \tau k',$$

and also by multiplying by the matrix $\overline{\beta}'$

$$\overline{\beta}'\alpha\Omega = r\Omega = \overline{\beta}'k + \overline{\beta}'\tau k'.$$

---

[1]) For the notation compare: BAKER, *Abel's theorem and the allied theory*. Cambridge 1897.

From

$$\alpha \tau' = \beta + \tau \beta'$$

we have

$$\overline{\beta'} \tau = \tau' \overline{\alpha} - \overline{\beta},$$

so that we get for the system of the increments Ω of the integrals w

$$r \Omega = \overline{\beta'} k - \overline{\beta} k' + \tau' \alpha k'.$$

By supposing that in the first row of the matrix τ' the constituents $\tau'_{12}, \tau'_{13} \ldots . \tau'_{1p}$ are all equal to zero we shall find that every increment Ω_1 of the integral w_1 is expressed by

$$r \Omega_1 = \beta'_1 k - \beta_1 k' + \tau'_{11} \alpha_1 k',$$

where the first columns of the matrices β', β, α are denoted by $\beta'_1, \beta_1, \alpha_1$.

Hence the moduli of periodicity of the integral $r w_1$ corresponding to any closed circuit are always multiples of 1 and τ'_{11} and therefore this integral must be an elliptic integral.

It may be noticed that in the case $p = 2$, the same conclusion holds for the integral $r w_2$, so that for $p = 2$ there exist two reducible integrals or there is none.

Assuming $r w_1$ to be an elliptic integral we can easily find how many zeros the function $\Theta(r w_1 ; \tau'_{11})$, of the single variable $r w_1$ and the period τ'_{11}, possesses on the surface T. We have only to calculate the value of the integral

$$\frac{1}{2 \pi i} \int_{\widehat{T'}} d \log \Theta (r w_1 ; \tau'_{11}),$$

taken round the boundary of the simply connected surface T', into which T is resolved by the customary p pairs of cross-cuts A_h and B_h. On opposite edges of a cross-cut A_h the variable $r w_1$ has values the difference of which amounts to β'_{h1}, so that on both edges $d \log \Theta (r w_1 ; \tau_{11})$ has the same value and the integrals taken in opposite directions round these edges, destroy one another.

31*

On the contrary by crossing B_h the integral rw_1 increases by

$$- \beta_{h1} + \tau'_{11}\, \alpha_{h1},$$

and as

$$d \log \bar{\vartheta} (rw_1 - \beta_{h1} + \tau'_{11}\, \alpha_{h1} \,;\, \tau'_{11}) = d \log \vartheta (rw_1 \,;\, \tau'_{11}) - 2 \pi i\, \alpha_{h1}\, d\, rw_1,$$

we finally get

$$\frac{1}{2\pi i} \int_{\widehat{T}} d \log \vartheta (rw_1 \,;\, \tau'_{11}) = - \sum_h \alpha_{h1} \int_{\widehat{B_h}} d\, rw_1 = \sum_h \alpha_{h1}\, \beta'_{h1} = r.$$

Consequently the function $\vartheta(rw_1 ;\, \tau'_{11})$ admits r zeros on T', so that any quotient of the squares of two thetas must be a uniform function of position on the undissected surface T with r double zeros and r double poles. Hence as soon as one of the integrals of the first kind W is reducible, there exist four adjoint curves R_1, R_2, R_3 and R_4, belonging to a pencil, which each separately, letting alone any possible common points of intersection with the fundamental curve f, touch — or at least intersect in two coincident points — the latter r times. The three quotients $R_1 : R_4$, $R_2 : R_4$, $R_3 : R_4$ being quotients of the squares of two thetas save as to some constant factor, may be taken equal to $\bar{p}W - \varepsilon_1$, $\bar{p}W - \varepsilon_2$, $\bar{p}W - \varepsilon_3$, whence

$$\bar{p}'W = \frac{2}{R_4^2} \sqrt{R_1 R_2 R_3 R_4}.$$

The function $\bar{p}'W$ being however likewise uniform on T it must be possible to replace the product of the four functions R by the square of a rational function F, otherwise said: through the $4r$ points of contact of the curves R there can be made to pass an adjoint curve F, the order of which is the double of the order of the curves R, which touches the fundamental curve f in the common points of intersection with the curves R and which for the rest intersects f only in the double points. The elliptic integral itself is now given by the equation

$$dW = \frac{R_4 \dfrac{dR_1}{dx} - R_1 \dfrac{dR_4}{dv}}{\sqrt{4 R_1 R_2 R_3 R_4}}\, dx,$$

or, when homogeneous variables x, y, z are introduced, by

$$\varrho\, dw = \frac{J(x, y\ z)}{F} \cdot \frac{dx}{\frac{\partial f}{\partial y}},$$

where $J(x, y, z)$ denotes the Jacobian of f and of the pencil of the curves R.

Meanwhile it is clear that when a reducible integral W presents itself the curves R are not yet uniquely determined. The lower limit of the integral W is still arbitrary and what is said of the squares of thetas with the argument W is also applicable to the squares of thetas with the argument $W + \alpha$. Hence the functions R can be replaced by

$$\frac{S_1}{S_4} = \wp(W + \alpha) - \varepsilon_1 \ , \ \frac{S_2}{S_4} = \wp(W + \alpha) - \varepsilon_2 \ , \ \frac{S_3}{S_4} = \wp(W + \alpha) - \varepsilon_3 \ ,$$

which functions can be expressed rationally in R_1, R_2, R_3, R_4 and F. Evidently the constant α may be regulated in such a way that one of the curves S, e. g. S_1, touches f in a given point x', y', and by the prescription of this point the remaining r—1 points of contact of S_1 are completely determined. Thus we infer that the existence of an elliptic integral W implies an involutory grouping of the points of the curve f in such a way, that the r points of any group may be regarded as the points of contact of some curve S.

The fact that the system of the curves R depends on an arbitrary parameter is important when we consider hyperelliptic curves. For then in the equation of the curve f *one* of the coordinates, say y, occurs only in the second power and the rational functions contain no power of y higher than the first. So it is always possible to choose the constant α in such a manner that in the ratio $S_1 : S_4$, and then also in the two others $S_2 : S_4$ and $S_3 : S_4$, the term containing y is wanting. In other words: in the case of an hyperelliptic curve admitting a reducible integral we can suppose beforehand that each of the curves R, by means of which the reduction has been effected, breaks up into a group of r right lines, drawn through the multiple point of the curve.

In the preceding the existence of the funtions R proved to be a necessary consequence of the reducibility of one of the integrals of the first kind; conversely, if the existence of the functions R is established at least one of the integrals is reducible.

For in this supposition the elliptic integral

$$\int \frac{d\,\dfrac{R_1}{R_4}}{\sqrt{\dfrac{R_1}{R_4}\cdot\dfrac{R_2}{R_4}\cdot\dfrac{R_3}{R_4}}}$$

is an integral of the first kind belonging to the curve, because it can assume the form

$$\int \frac{J(x, y, z)}{F} \cdot \frac{dx}{\dfrac{\partial f}{\partial y}}.$$

As for reducible non-hyperelliptic integrals the case $p = 3$, $r = 2$ has been treated by SOPHIE KOWALEVSKI. The curve f is here the general quartic upon which, r being equal to 2, an involutory correspondence one to one exists. It can be shown that in this case the curve can be transformed into itself by a reciprocal projective transformation of the plane. Consequently four double tangents of the curve pass through the centre of the transformation, so that its equation can always be thrown into the form

$$f = xy\,(ax + by)\,(cx + dy) - K^2 = 0.$$

Evidently the four double tangents passing through the origin can be identified with the curves R. For each of these tangents touches f in two points, together they belong to a pencil and the eight points of contact lie on the conic K.

Accordingly we get for the elliptic integral

$$\frac{\bar{p}W - \varepsilon_1}{\dfrac{x}{d}} = \frac{\bar{p}W - \varepsilon_2}{\dfrac{y}{c}} = \frac{\bar{p}W - \varepsilon_3}{\dfrac{ax + by}{cx + dy}} = \frac{1}{cx + dy},$$

and the integral itself is

$$W = \int \frac{\dfrac{\partial K}{\partial z}}{\dfrac{\partial f}{\partial y}}\, dx\ .$$

The simplest case of the reducible hyperelliptic integral, $p = 2$, $r = 2$, was already known to LEGENDRE. Again the curve f is of the fourth order, but it has now a node from which six tangents can be drawn to the curve. If there is a reducible integral each of the first three functions R is made up by a pair of these tangents, as fourth function R_4 we must take one of the two double elements, counted twice, of the involution which is now necessarily formed by the three pairs of tangents.

The equation of the curve being

$$f = xy^2 - (x-1)(x-kl)(x-k)(x-l) = 0$$

we get in this way

$$R_1 = x,\ R_2 = (x-1)(x-kl),\ R_3 = (x-k)(x-l),\ R_4 = (x \pm \sqrt{kl})^2$$

and the reducible integral is

$$W = \int \frac{(x \mp \sqrt{kl})\, dx}{xy} .$$

In accordance with what resulted from the theorem of WEIERSTRASS for the case $p = 2$ two independent reducible integrals are obtained.

Also the case $p = 2$, $r = 3$ has been considered from various sides. As before the integral is relative to a nodal quartic f the equation of which we take in the form

$$(x - a_1)(x - c_1)\, y^2 = (x - a_2)(x - a_3)(x - c_2)(x - c_3) .$$

BURKHARDT[1]) has pointed out the invariant relation existing between the binary cubics $(x-a_1)(x-a_2)(x-a_3)$ and $(x-c_1)(x-c_2)(x-c_3)$ when one of the integrals is reducible. Previously GOURSAT[2]) had treated a more or less particular case of the reducibility and finally BURNSIDE[3]) indicated in connection with his more general researches a remarkable form which the reducible integral can always assume.

After the deduction of some of their results a few remarks will be added.

The curves R, each of which breaks up into three right lines, must

---

[1]) *Math. Annal.*, Vol. 36, 1890, p. 410

[2]) *Compt. Rendus*, 100, 1885, p. 622.

[3]) *Proc. Lond. Math. Soc*, 23, 1892, p. 173.

be required to touch — or at least to intersect in two coincident points — the curve f three times. From the node six tangents, the inflexional tangents included, can be drawn to the curve, three of which $(x-a_1)$, $(x-a_2)$, $(x-a_3)$ make up together the curve R_4; by joining to each of the remaining tangents $(x-c_1)$, $(x-c_2)$, $(x-c_3)$ a line through the node $(x-b_1)$, $(x-b_2)$, $(x-b_3)$, counted twice, the four functions thus obtained

$$R_1=(x-c_1)(x-b_1)^2,\quad R_2=(x-c_2)(x-b_2)^2,$$

$$R_3=(x-c_3)(x-b_3)^2,\quad R_4=(x-a_1)(x-a_2)(x-a_3)$$

indeed satisfy all the demands, if only we take care to choose the quantities b in such a way that the four functions are in involution. With the aid of this condition we can eliminate the b's, after which still one relation remains between the a's and the c's. Hence of a reducible integral five branch-points out of six may be chosen arbitrarily.

Reducible integrals of the kind considered here are easy to construct if we observe that the four binary cubics belong to the system of first polars of a binary biquadratic α_x^4. Among these polars there are four of the form $(x-c)(x-b)^2$, having a double point, so besides R_1, R_2, R_3 also $(x-c_4)(x-b_4)^3$ belongs to the system and the four quantities b are at once recognised as the roots of the Hessian Δ_x^4. Also c_1, c_2, c_3, c_4 are the roots of a covariant, found by the following consideration. The four points y, whose first polars $\alpha_y\,\alpha_x^3$ contain a double point b, are the roots of the covariant[1] $3\,i\,\Delta_x^4-2\,j\,\alpha_x^4$, where i and j denote the two invariants of α_x^4. The result of the elimination of y between this covariant and $\alpha_x\,\alpha_y^3$, which result is of the 12th order in x and of the 8th order in the coefficients α, must be a covariant having the quantities b for double roots and the quantities c for single roots. After division by the square of Δ_x^4 a covariant of the 4th order in the coefficients α will remain, necessarily of the form $\lambda\,i\,\Delta_x^4+\mu\,j\,\alpha_x^4$, the roots of which are c_1, c_2, c_3, c_4. To determine the coefficients λ and μ, we consider the special case

$$\alpha_x^4=\frac{2\,i}{3\,j}\,x^2\,(x^2-1),\quad \Delta_x^4=-\frac{2}{3}\,(2\,x^4+1).$$

In this case α_x^4 has a double point, the four values of c are

---

[1]) CLEBSCH-LINDEMANN. *Vorlesungen*, I, p 231.

evidently 0 and ∞, each taken twice, and, as

$$i \Delta_x^4 + 2 j \alpha_x^4 = -2 i x^2 ,$$

we must take $\lambda = 1$ and $\mu = 2$. So in general the quantities c are determined by the equation

$$i \Delta_x^4 + 2 j \alpha_x^4 = 0 .$$

As soon as we regard as known *one* root c_4 of this equation we can construct a reducible integral. For then we can put

$$\alpha_z \alpha_x^3 = (x - a_1)(x - a_2)(x - a_3) ,$$

$$\frac{1}{x - c_4} (i \Delta_x^4 + 2 j \alpha_x^4) = (x - c_1)(x - c_2)(x - c_3) ,$$

whence it follows that

$$R_4 \frac{dR_1}{dx} - R_1 \frac{dR_4}{dx} = \varrho \Delta_x^4 ,$$

$$\sqrt{R_1 R_2 R_3 R_4} = \frac{\Delta_x^4}{x - b_4} \sqrt{\alpha_z \alpha_x^3 \cdot \frac{1}{x - c_4} (i \Delta_x^4 + 2 j \alpha_x^4)} ,$$

so that

$$W = \int \frac{(x - b_4)\, dx}{\sqrt{\alpha_z \alpha_x^3 \cdot \frac{1}{x - c_4} (i \Delta_x^4 + 2 j \alpha_x^4)}}$$

is an integral of the first kind relative to the quartic f, which is transformed by the substitution

$$\frac{\bar{p} W - \varepsilon_1}{\varrho_1 (x - c_1)(x - b_1)^2} = \frac{\bar{p} W - \varepsilon_2}{\varrho_2 (x - c_2)(x - b_2)^2} =$$

$$= \frac{\bar{p} W - \varepsilon_3}{\varrho_3 (x - c_3)(x - b_3)^2} = \frac{1}{\varrho_4 (x - a_1)(x - a_2)(x - a_3)}$$

into an elliptic integral with the variable $p W$.

It is now clear that in this way it is possible to construct the integral out of five assigned branch-points. If the five branch-points a_1, a_2, a_3, c_1, c_2, are given, a binary biquadratic α_x^4 must be sought which has a given cubic A_x^3 with the roots a_1, a_2, a_3 among its first polars. So there is an identical equation of the form

$$\alpha_z \alpha_x^3 = \varrho A_x^3 ,$$

from which we deduce

$$\frac{z\alpha_0 + \alpha_1}{A_0} = \frac{z\alpha_1 + \alpha_2}{A_1} = \frac{z\alpha_2 + \alpha_3}{A_2} = \frac{z\alpha_3 + \alpha_4}{A_3}.$$

In the first place these equations determine z, moreover for the coefficients of α_x^4 two relations remain, expressed by any two of the four following equations

$$\begin{aligned}
6 A_2 \Delta_0 - 12 A_1 \Delta_1 + 6 A_0 \Delta_2 &= A_0 i, \\
2 A_3 \Delta_0 - \ 6 A_1 \Delta_2 + 4 A_0 \Delta_3 &= A_1 i, \\
4 A_3 \Delta_1 - \ 6 A_2 \Delta_2 + 2 A_0 \Delta_4 &= A_2 i, \\
6 A_3 \Delta_2 - 12 A_2 \Delta_3 + 6 A_1 \Delta_4 &= A_3 i.
\end{aligned}$$

There are still two other conditions for α_x^4, namely that the fourth and the fifth of the given quantities are roots of the covariant $i\,\Delta_x^4 + 2j\,\alpha_x^4$. By the four conditions together the quantic α_x^4 is entirely determined, and thereby the sixth branch-point c_3 of the integral.

We obtain the example treated by GOURSAT by putting

$$A_1 = 0 \ , \quad \alpha_1 = 0 \ , \quad \alpha_2 = 0 \, .$$

Then we have

$$\Delta_0 = 0, \quad \Delta_1 = \alpha_0 \alpha_3, \quad \Delta_2 = \frac{1}{3} \alpha_0 \alpha_4, \quad \Delta_3 = 0, \quad \Delta_4 = -2\alpha_3^2,$$

$$i = 2\alpha_0 \alpha_4 \ , \quad j = -6\alpha_3^2 \alpha_0 \ ,$$

the coefficients α satisfying the equation

$$A_3 \alpha_0 \alpha_3 - A_2 \alpha_0 \alpha_4 - A_0 \alpha_3^2 = 0.$$

Lastly,

$$i\,\Delta_x^4 + 2j\,\alpha_x^4 = 4\alpha_0 (3\alpha_3 x + \alpha_4)(-\alpha_0 \alpha_3 x^3 + \alpha_0 \alpha_4 x^2 - 4\alpha_3^2),$$

therefore c_1, c_2, c_3 are the roots of the cubic

$$C_0 x^3 + 3 C_1 x^2 + C_3 = -\alpha_0 \alpha_3 x^3 + \alpha_0 \alpha_4 x^2 - 4\alpha_3^2,$$

and c_4, associated with the root $b_4 = \infty$ of the Hessian Δ_x^4, is determined by $3\,\alpha_4 x + \alpha_4 = 0$.

Hence the integral

$$W = \int \frac{dx}{\sqrt{(A_0 x^3 + 3 A_2 x + A_3)(C_0 x^3 + 3 C_1 x^2 + C_3)}}$$

proves to be reducible under the condition that the coefficients A and C obey the relation

$$A_0 C_3 = 4 A_3 C_0 + 12 A_2 C_1 .$$

The formulae of transformation will be found to be

$$\frac{\bar{p} W - \varepsilon_1}{\varrho_1 (x - c_1)(x - b_1)^2} = \frac{\bar{p} W - \varepsilon_2}{\varrho_2 (x - c_2)(x - b_2)^2} = \frac{\bar{p} W - \varepsilon_3}{\varrho_3 (x - c_3)(x - b_3)^2} =$$

$$= \frac{1}{A_0 x^3 + 3 A_2 x + A_3},$$

from which we can also deduce

$$\lambda \bar{p} W + \mu = \frac{C_0 x - C_1}{A_0 x^3 + 3 A_2 x + A_3}.$$

We obtain the second reducible integral by regarding $C_0 x^3 + 3 C_1 x + C_3$ as first polar of the biquadratic $\alpha_x{}^4$. In this supposition the Hessian Δ_x^4 will admit the root $b_4 = 0$, the polar $(x - c_4)(x - b_4)^2$ takes the form $x^2 (A_2 x - A_3)$ and the required integral itself is

$$W = \int \frac{x\, dx}{\sqrt{(A_0 x^3 + 3 A_2 x + A_3)(C_0 x^3 + 3 C_1 x^2 + C_3)}}.$$

The reduction of the integral is obtained by putting

$$\lambda \bar{p} W + \mu = \frac{x^2 (A_2 x - A_3)}{C_0 x^3 + 3 C_1 x^2 + C_3}.$$

Another means of constructing reducible integrals is founded on a peculiar form which can always be given to the invariant relation between the six branch-points.

From the identical equation

$$(x-a_1)(x-a_2)(x-a_3)=\lambda_1(x-c_1)(x-b_1)^2+\lambda_2(x-c_2)(x-b_2)^2$$

we derive by the successive substitutions $x=a_1,\ a_2,\ a_3$

$$\frac{(a_1-c_1)(a_1-b_1)^2}{(a_1-c_2)(a_1-b_2)^2}=\frac{(a_2-c_1)(a_2-b_1)^2}{(a_2-c_2)(a_2-b_2)^2}=\frac{(a_3-c_1)(a_3-b_1)^2}{(a_3-c_2)(a_3-b_2)^2},$$

or

$$\frac{(a_1-b_1)\sqrt{\dfrac{a_1-c_1}{a_1-c_2}}}{a_1-b_2}=\frac{(a_2-b_1)\sqrt{\dfrac{a_2-c_1}{a_2-c_2}}}{a_2-b_2}=\frac{(a_3-b_1)\sqrt{\dfrac{a_3-c_1}{a_3-c_2}}}{a_3-b_2}.$$

After multiplying numerator and denominator of these three fractions successively by (a_2-a_3), (a_3-a_1) and (a_1-a_2), addition gives

$$0=(a_2-a_3)(a_1-b_1)\sqrt{\frac{a_1-c_1}{a_1-c_2}}+(a_3-a_1)(a_2-b_1)\sqrt{\frac{a_2-c_1}{a_2-c_2}}+$$

$$+(a_1-a_2)(a_3-b_1)\sqrt{\frac{a_3-c_1}{a_3-c_2}}.$$

Similarly by interchanging c_2 and c_3 we obtain

$$0=(a_2-a_3)(a_1-b_1)\sqrt{\frac{a_1-c_1}{a_1-c_3}}+(a_3-a_1)(a_2-b_1)\sqrt{\frac{a_2-c_1}{a_2-c_3}}+$$

$$+(a_1-a_2)(a_3-b_1)\sqrt{\frac{a_3-c_1}{a_3-c_3}},$$

and having moreover the identity

$$0=(a_2-a_3)(a_1-b_1)+(a_3-a_1)(a_2-b_1)+(a_1-a_2)(a_3-b_1),$$

the quantities (a_1-b_1), (a_2-b_1) and (a_3-b_1) can be eliminated and the invariant relation between the branch-points of a reducible integral takes the form

$$0 = \begin{vmatrix} \frac{1}{\sqrt{a_1-c_1}} & \frac{1}{\sqrt{a_2-c_1}} & \frac{1}{\sqrt{a_3-c_1}} \\ \frac{1}{\sqrt{a_1-c_2}} & \frac{1}{\sqrt{a_2-c_2}} & \frac{1}{\sqrt{a_3-c_2}} \\ \frac{1}{\sqrt{a_1-c_3}} & \frac{1}{\sqrt{a_2-c_3}} & \frac{1}{\sqrt{a_3-c_3}} \end{vmatrix},$$

the arguments of the nine surds being determined save as to a multiple of π. The symmetry between the a's and the c's shows, that, as soon as there exists the involution we started with, a similar involution can be found by interchanging the c's and the a's, so that reducible integrals always present themselves in pairs.

BURNSIDE stated incidentally that the curve

$$xy^2 = (x-1)(x-sn^2 u)\left(x-sn^2\left(u+\frac{2K}{3}\right)\right)\left(x-sn^2\left(u-\frac{2K}{3}\right)\right)$$

admits reducible integrals. The form given here to the invariant relation between the six branch-points readily provides a proof for this assertion.

In order to obtain this proof we introduce elliptic arguments instead of the a's and the c's by putting

$$a_1 = e_1, \qquad c_1 = pu_1 = p(u+v_1),$$

$$a_2 = e_2, \qquad c_2 = pu_2 = p(u+v_2),$$

$$a_3 = e_3, \qquad c_3 = pu_3 = p(u+v_3).$$

So the invariant relation becomes

$$0 = \begin{vmatrix} \frac{\sigma u_1}{\sigma_1 u_1} & \frac{\sigma u_2}{\sigma_1 u_2} & \frac{\sigma u_3}{\sigma_1 u_3} \\ \frac{\sigma u_1}{\sigma_2 u_1} & \frac{\sigma u_2}{\sigma_2 u_2} & \frac{\sigma u_3}{\sigma_2 u_3} \\ \frac{\sigma u_1}{\sigma_3 u_1} & \frac{\sigma u_2}{\sigma_3 u_2} & \frac{\sigma u_3}{\sigma_3 u_3} \end{vmatrix}.$$

As a function of u the determinant is doubly periodic; manifestly it has several zeros, e. g. $u = -v_1$, hence the equation is identically satisfied, if we choose v_1, v_2, v_3 in such a way that the function has no poles. There might be nine poles, since each of the denominators can be made to vanish. Consequently we have to find out whether it is possible to make the corresponding residues equal to zero. If we take e.g. the pole corresponding to $\sigma_\alpha u_1 = 0$, that is if we take $u = -v_1 + \omega_\alpha$, the residue is proportional to

$$\frac{\sigma_\gamma u_2 \sigma_\beta u_3 - \sigma_\beta u_2 \sigma_\gamma u_3}{\sigma_\gamma u_2 \sigma_\beta u_2 \sigma_\gamma u_3 \sigma_\beta u_3}.$$

The above expression however can be written

$$\frac{2(e_2 - e_3)\, \sigma \frac{u_2 + u_3}{2}\, \sigma \frac{u_2 - u_3}{2}\, \sigma_\alpha \frac{u_2 + u_3}{2}\, \sigma_\alpha \frac{u_2 - u_3}{2}}{\sigma_\gamma u_2 \sigma_\beta u_2 \sigma_\gamma u_3 \sigma_\beta u_3},$$

and this is zero, if only for $u = -v_1 + \omega_\alpha$ we have at the same time $\frac{1}{2}(u_2 + u_3) \equiv \omega_\alpha$.

This takes place if we have

$$\tfrac{1}{2}(v_2 + v_3 - 2v_1) \equiv 0,$$

and the residues of all nine poles will vanish if moreover we have simultaneously

$$\tfrac{1}{2}(v_3 + v_1 - 2v_2) \equiv 0,$$

$$\tfrac{1}{2}(v_1 + v_2 - 2v_3) \equiv 0.$$

As solution of these equations can be taken

$$v_1 = \frac{4\Omega_1}{3},\ v_2 = \frac{4\Omega_2}{3},\ v_3 = \frac{4\Omega_3}{3},$$

where $2\Omega_1$, $2\Omega_2$, $2\Omega_3$ denote periods connected by the relation

$$\Omega_1 + \Omega_2 + \Omega_3 = 0.$$

So we find the reducible integral

$$W = \int \frac{(x - b_4)\, dx}{\sqrt{(4x^3 - g_2 x - g_3)\left(x - p(u + \frac{4\Omega_1}{3})\right)\left(x - p(u + \frac{4\Omega_2}{3})\right)\left(x - p(u + \frac{4\Omega_3}{3})\right)}}.$$

Every reducible integral, with five assigned branch-points a_1, a_2, a_3, c_1, c_2, can be brought into this form by a linear substitution. For by means of the equations

$$\frac{a_1-a_2}{a_1-a_3} : \frac{c_1-a_2}{c_1-a_3} = \frac{e_1-e_2}{e_1-e_3} : \frac{p\,(u+\frac{4\Omega_1}{3})-e_2}{p\,(u+\frac{4\Omega_1}{3})-e_3},$$

$$\frac{a_1-a_2}{a_1-a_3} : \frac{c_2-a_2}{c_2-a_3} = \frac{e_1-e_2}{e_1-e_3} : \frac{p\,(u+\frac{4\Omega_2}{3})-e_2}{p\,(u+\frac{4\Omega_2}{3})-e_3}$$

we can determine the ratios $e_1 : e_2 : e_3$ and the argument u.

A somewhat laborious calculation gives the values of the quantities b and the formulae of substitution.

Putting

$$p\,(u_2-u_3) = p\,(u_3-u_1) = p\,(u_1-u_2) = pv,$$

the following results are obtained

$$b_1 = -\,2p\,v + p'v.\frac{pu_2-pu_3}{p'u_2+p'u_3}\ ,\ b_2 = -\,2p\,v + p'v.\frac{pu_3-pu_1}{p'u_3+p'u_1},$$

$$b_3 = -\,2p\,v + p'v.\frac{pu_1-pu_2}{p'u_1+p'u_2},$$

$$b_4 = \begin{vmatrix} b_1+2pu_1 & b_2+2pu_2 & b_3+2pu_3 \\ b_1pu_1 & b_2pu_2 & b_3pu_3 \\ 1 & 1 & 1 \end{vmatrix} : \begin{vmatrix} pu_1 & pu_2 & pu_3 \\ b_1 & b_2 & b_3 \\ 1 & 1 & 1 \end{vmatrix}.$$

Furthermore the formulae of substitution are

$$\bar{p}\,W-\varepsilon_1 = (pu_2+pu_3+pv)\,\frac{(x-pu_1)\,(x-b_1)^2}{4x^3-g_2x-g_3},$$

$$\bar{p}\,W-\varepsilon_2 = (pu_3+pu_1+pv)\,\frac{(x-pu_2)\,(x-b_2)^2}{4x^3-g_2x-g_3},$$

$$\bar{p}\,W-\varepsilon_3 = (pu_1+pu_2+pv)\,\frac{(x-pu_3)\,(x-b_3)^2}{4x^3-g_2x-g_3}$$

so that the roots of the $\bar{p}$-function are determined by

$$\varepsilon_2-\varepsilon_3=pu_2-pu_3\,,\ \varepsilon_3-\varepsilon_1=pu_3-pu_1\,,\ \varepsilon_1-\varepsilon_2=pu_1-pu_2\,.$$

By interchanging the a's and the c's we can obtain the second reducible integral.

Proceeding to consider reducible hyperelliptic integrals of deficiency $p=3$, it is clear that by the same methods also these can be constructed without great difficulty. If the curve f is given by the equation

$$(x-a_1)(x-a_2)(x-a_3)\,y^2=(x-a_4)(x-a_5)\ldots.(x-a_8)\,,$$

hat is: if the curve f is a quintic with a triple point, from which eight tangents can be drawn, each of the curves R_1, R_2, R_3, R_4 in the case $r=2$ is to be made up by a couple of these tangents. The twofold condition for the reducibility expresses that these four pairs of tangents are in involution and it is easily verified that when Ax^2+Bx+C defines the double elements of the involution, the reducible integral will have the form

$$\int\frac{Ax^2+Bx+C}{(x-a_1)(x-a_2)(x-a_3)y}\,dx\,.$$

The investigation of the next case $p=3$, $r=3$, is closely allied to that of the case $p=2$, $r=3$ treated before.

Suppose the equation of the curve to be

$$(x-a_1)(x-a_2)(x-a_3)\,y^2=(x-a'_1)(x-a'_2)(x-a'_3)(x-c_1)(x-c_2).$$

We regard the product of three tangents $(x-a_1)(x-a_2)(x-a_3)$ as the curve R_1, in the same way we join the second three $(x-a'_1)(x-a'_2)(x-a'_3)$ to a curve R_2; the third curve R_3 will be the product of the next tangent $(x-c_1)$ and of a line through the node $(x-b_2)$, counted twice, similarly the curve R_4 is furnished by the tangent $(x-c_2)$ and the double line $(x-b_2)$.

The twofold condition for the reducibility is that the four binary cubics

$$R_1=(x-a_1)(x-a_2)(x-a_3)\,,\qquad R_2=(x-a'_1)(x-a'_2)(x-a'_3)\,,$$

$$R_3=(x-c_1)(x-b_1)^2\,,\qquad R_4=(x-c_2)(x-b_2)^2$$

are in involution. This will evidently take place when c_1 and c_2 are two roots of the covariant $i\,\Delta_x^4 + 2j\,\alpha_x^4$, where α_x^4 is the biquadratic which has R_1 and R_2 as first polars. The quantities b_1 and b_2 are two roots of Δ_x^4; if we call the other two b_3 and b_4,

$$W = \int \frac{(x-b_3)\,(x-b_4)\,dx}{\sqrt{\alpha_z\,\alpha_x^3\;.\;\alpha_t\,\alpha_x^3\;.\;\dfrac{1}{(x-c_1)\,(x-c_2)}(i\,\Delta_x^4 + 2\,j\,\alpha_x^4)}}$$

is a reducible integral,

As an example we may point to the following case

$$\alpha_x^4 = k_1^2\,x^4 + 4\,k_1\,x^3 + 6\,x^2 + 4\,k_2 x + k_2^2,$$

$$i = 2\,(k_1\,k_2 - 1)\,(k_1\,k_2 - 3),\; j = -\,6\,(k_1\,k_2 - 1)^2,$$

$$\Delta_x^4 = 2\,(k_1\,k_2 - 1)\,(2\,k_1\,x^3 + (k_1\,k_2 + 3)\,x^2 + 2\,k_2\,x),$$

$$\alpha_z\,\alpha_x^3 = k_1^2 x^3 + 3\,k_1 x^2 + 3\,x + k_2,\;\; \alpha_t\,\alpha_x^3 = k_1 x^3 + 3\,x^2 + 3\,k_2 x + k_2^2,$$

$$i\Delta_x^4 + 2\,j\,\alpha_x^4 = \varrho\,(3\,x + k_2)\,(x\,k_1 + 3)\,(k_1\,x^2 + (3 - k_1\,k_2)\,x + k_2).$$

The Hessian Δ_x^4 has two special roots $b_3 = 0$, $b_4 = \infty$, the corresponding quantities c_3 and c_4 are given by $3\,x + k_2 = 0$, $x\,k_1 + 3 = 0$, c_1 and c_2 are the roots of $k_1\,x^2 + (3 - k_1\,k_2)\,x + k_2 = 0$ and the resulting reducible integral is

$$W = \int \frac{x\,dx}{\sqrt{(k_1^2 x^3 + 3k_1 x + 3x + k_2)(k_1 x^3 + 3x^2 + 3k_2 x + k_2^2)(k_1 x^2 + (3 - k_1 k_2)x + k_2)}},$$

where under the radical sign we may replace each of the two cubic factors by any one of their linear combinations. The reduction of the integral will be effected by the substitution

$$\frac{A\,\bar{p}\,W + B}{C\,\bar{p}\,W + D} = \xi = \frac{x^3\,k_1 + 3\,x^2}{3\,x + k_2}.$$

The case $p = 4$, $r = 3$ evidently allows quite a similar treatment. Seven branch-points of the integral can be assigned arbitrarily; the triple condition determining the three remaining branch-points is again readily obtained by the consideration of the cubic involution.