**Mathematics.** — *"An Involutory Transformation of the Rays of Space which is defined by two Involutory Homologies."* By Prof. JAN DE VRIES.

(Communicated at the meeting of February 22, 1919).

1. In a plane $\alpha$ I consider the *involutory homology* (central collineation) which has $A$ for centre and $a$ for axis, in a plane $\beta$ a similar involution with centre $B$ and axis $b$. If $P, P'$ is a pair of the first involution, $Q, Q'$ a pair of the second, I associate the rays $t \equiv PQ$ and $t' \equiv P'Q'$. In this way arises an *involution in the rays of space*, which will be investigated in what follows.

When $PQ$ and $P'Q'$ intersect in a point $M$, the pair $Q, Q'$ is the central projection of $P, P'$ out of $M$ as centre. By means of this projection the pairs of the involution $[\alpha]$ lying on $p \equiv PP'$ are transformed into the pairs of an involution situated on $q \equiv QQ'$; the latter has one pair in common with the involution which is defined on $q$ by the homology $[\beta]$. Consequently through $M$ passes one pair of rays $t, t'$.

Along $AB$ two rays $t$ and $t'$ coincide. Also the straight lines through $A$ to the points of $b$, and through $B$ to the points of $a$ are *double rays* of the involution $(t, t')$. The rest of the *double rays* form the *bilinear congruence* which has $a$ and $b$ as directrices.

2. Let $t_\alpha$ be a straight line in $\alpha$; each of its points can be considered as its passage $P$, while its passage $Q$ lies on the straight line $c \equiv \alpha\beta$. If $C_\beta$ is the point that in $[\beta]$ corresponds to $C \equiv Q$ and $t'_\alpha$ the image of $t_\alpha$ in $[\alpha]$, the involution $(t, t')$ associates to $t_\alpha$ all the rays $t$ of the plain pencil which has $C_\beta$ as vertex and lies in the plane $(C_\beta t'_\alpha)$. *All the rays $t_\alpha$ are therefore singular.*

When $t_\alpha$ revolves round $C$, $t'_\alpha$ describes a plane pencil round the point $C_\alpha$ which in the homology $[\alpha]$ corresponds to $C$. The plane pencils $(t')$ corresponding to $t_\alpha$ belong to the sheaf $[C_\beta]$; their planes pass through the straight line $C_\alpha C_\beta$.

When $C$ describes the straight line $c$, $C_\beta$ describes the straight line $c_\beta$, which in $[\beta]$ is associated to $c$. Hence to the *singular rays* $t_\alpha$ are associated the rays $t'$ of the *axial linear complex* $|c_\beta|$ which has $c_\beta$ as a directrix.



Analogously the rays of the *axial complex* $|c_\alpha|$ are associated to the *singular rays* $t_\beta$; to each ray $t_\beta$ correspond the rays $t'$ of a plane pencil belonging to $|c_\alpha|$.

The intersection of the complexes $|c_\alpha|$ and $|c_\beta|$ is a bilinear congruence of which the rays are associated to the ray $t \equiv c$. The straight line $c$ is therefore a *principal ray*; indeed, we can consider two arbitrary points of $c$ as passages $P$ and $Q$.

All the rays $t$ through a point $P \equiv Q \equiv C$ of $c$ are associated to the ray $t'$ joining $P'$ $Q'$; hence also $t'$ is a principal ray. When $C$ moves along $c$, $P'$ and $Q'$ describe two projective ranges of points on $c_\alpha$ and $c_\beta$; $P'Q'$ describes a scroll $(c)^2$. *The quadratic scroll* $(c)^2$ consists therefore of *principal rays*, each of which is associated to the rays of a star $[C]$.

3. When $t_\alpha$ revolves round a point $T$, $C_\beta$ moves along $c_\beta$ and the plane pencil with $C_\beta$ as vertex of which the rays $t'$ cut the line $t'_\alpha$ in $[\alpha]$ associated to $t_\alpha$, defines a congruence. The range of points which $C_\beta$ describes on $c_\beta$, is projective to the plane pencil $T')$ described by $t'_\alpha$; when it is projected out of any point $M$ on $\alpha$, there will be two rays $t'_\alpha$ which pass through the projection of the corresponding point $C_\beta$. Through $M$ pass therefore two rays of the congruence. Any plane $\mu$ contains one point $C_\beta$ and also the passage of the corresponding ray $t'_\alpha$, hence one ray $t'$ of the congruence. The *plane pencil* $(t_\alpha)$ is accordingly represented by a *congruence* (2,1).

As the ray $T' C_\beta$ in each of its positions belongs to the (2,1), $(T' C_\beta)$ is one of the *singular planes* of the congruence. Also $\alpha$ is a *singular plane*, for it contains the plane pencil the vertex of which lies in the point of intersection $C \equiv C^\beta$ of $c$ and $b$.

4. If $t$ describes a plane pencil $(T, \tau)$ in the plane $\tau$, its passages $P$ and $Q$ describe projective ranges on the straight lines $p \equiv \alpha\tau$ and $q \equiv \beta\tau$. But then also the ranges of points which the homologous points $P'$ and $Q'$ describe on $p'$ and $q'$, are projective, so that $P'Q'$ describes a quadratic scroll. Accordingly in the transformation $(t, t')$ the *image* of a *plane pencil* is in general a *quadratic scroll.*

If $t$ describes a field of rays $\mu$, the passages $P$ and $Q$ remain on the straight lines $p \equiv \alpha\mu$ and $q \equiv \beta\mu$; $P'$ and $Q'$ lie in this case on the homologous straight lines $p'$ and $q'$. The *field of rays* is therefore represented by a *bilinear congruence.*

The ray $t'$ in $\mu$ joins the points $pp'$ and $qq'$; it is therefore a double ray of the involution.

When $t$ belongs to the sheaf $[M]$, the passages $P$ and $Q$ form two projective fields. As in this case also $P'$ and $Q'$ correspond in projective fields, we find for the *image* of the *sheaf* a *congruence* (3,1).

Of the three rays which this congruence sends through an arbitrary point, two are associated to each other in the involution $(t, t')$, while the third is a double ray (§ 1). The ray $t'$ which it has in an arbitrary plane $\mu$, is the image of the ray $t$ which the (1,1) associated to $\mu$, sends through $M$.

As the sheaf $[M]$ contains the plane pencil of which the rays intersect the straight line $c$, the scroll $(c)^2$ belongs to the image (3,1) of the sheaf.

The sheaf $[M]$ contains a plane pencil of rays $t$ intersecting $c_\beta$. This defines on the intersection $m$ of the plane $(Mc_\beta)$ with $\alpha$ a range of points $(P')$. Any homologous point $P'$ defines with the point $C$ corresponding to $C_\beta$ one ray $t_\alpha$. Any plane pencil $(t_\alpha)$ with vertex $C$ contains therefore one ray corresponding to a ray of the axial complex $[c_\beta]$ belonging to $[M]$. But also the line $c$ belongs to the congruence (3,1), it being the image of the transversal through $M$ to $c_\alpha$ and $c_\beta$. Consequently the images $t_\alpha$ of the rays of the plane pencil in $(Mc_\beta)$ envelop a conic. From this appears that $\alpha$ and $\beta$ belong to the *singular planes* of the congruence (3,1); in other words, $\alpha$ and $\beta$ are osculating planes of the twisted cubics of which the axes (intersections of two osculating planes) form the (3,1).

5. The rays $t$ resting on the straight lines $d_1$ and $d_2$ and also on $c_\beta$, form a quadratic scroll; their passages $P$ lie therefore on a conic $\delta^2$. The corresponding points $P'$ form on a conic $\delta'^2$ a range of points projective to the range of the points $C_\beta$, hence also to the range of the points $C$. Consequently the ray $t'$ envelops a *curve of the third class*. Through a point $N'$ of $\alpha$ pass four lines $t'$, the images of rays $t$ of the bilinear congruence with directrices $d_1, d_2$, namely three rays $t_\alpha$ and besides the ray associated to the ray which the point $N$ sends to the (1,1).

The bilinear congruence representing the field of rays $[\mu]$, has two rays in common with the (1,1) mentioned above; the image of the latter has therefore two rays in the plane $\mu$. Consequently a *bilinear congruence* is represented by a *congruence* (4, 2).

The latter has $\alpha$ and $\beta$ as *singular planes* of the third class.

The rays sent by the (4, 2) through a point $M$, are the images of the rays which the (1, 1) has in common with the image (3, 1) of the sheaf $[M]$.

The images of two bilinear congruences have among others the



scroll $(c)^2$ in common; for any sheaf $[C]$ furnishes one ray for each of the two (1, 1).

6. The *axial complex* with axis $d$ is transformed by the transformation $(t, t')$ into a *quadratic complex* $\{t'\}^2$; indeed, to the two rays of the scroll $(t)^2$ representing the plane pencil $(t')$, correspond two rays of the image-complex lying in the plane pencil $(t')$.

As $[d]$ singles out one ray out of each plane pencil of singular rays, $\{t'\}^2$ contains the two fields of rays $[\alpha]$ and $[\beta]$. Two congruences $\{t'\}^2$ have besides those two congruences (0, 1) one more congruence (4, 2) in common; from this appears again that a bilinear congruence is transformed into a (4, 2).

The image (3, 1) of a sheaf $[M]$ has four rays in common with the image (1, 1) of the field $[\mu]$. One of them belongs to the scroll $(c)^2$ and is associated to any ray that the corresponding sheaf $[C]$ has in common with $[M]$ and $[\mu]$. Another coincides with $c$; for $[M]$ and $[\mu]$ send each one ray to $c_\alpha$ and $c_\beta$.

The straight line through $M$ and the point $C_\beta$ in $\mu$ belongs to a plane pencil that is associated to a definite ray $t_\alpha$; as $\mu$ also contains a ray of this plane pencil, the image-congruences (3, 1) and (1, 1) have this ray $(t_\alpha)$ in common. Analogously they have a ray $t$ in common.

The images of two fields of rays $[\mu]$ and $[\mu^*]$ have two rays in common. One of them is the image of the straight line $\mu\mu^*$, the other is the line $c$; this is associated to the two transversals of $c_\alpha$ and $c_\beta$ in $\mu$ and in $\mu^*$.

The image (1, 1) of the field $[\mu]$ has six rays in common with the image (4, 2) of a bilinear congruence with directrices $d_1, d_2$. To them belongs the ray of the scroll $(c)^2$ associated to the sheaf of which the vertex lies in the point $(c_\mu)$. They have twice the line $c$ in common, for two transversals of $c_\alpha$ and $c_\beta$ rest also on $d_1$ and $d_2$, while one straight line of $\mu$ rests on $c_\alpha, c_\beta$. The transversal through the point $(\mu c_\beta)$ to $d_1, d_2$ belongs to a plane pencil which has also one ray in $\mu$; to both of them corresponds the same line $t_\alpha$. Analogously the image-congruences have a straight line $t_\beta$ in common. The sixth common ray is the image of the transversal of $d_1$ and $d_2$ in $\mu$.