Mathematics. "*Graphical Determination of the Moments of Transition of an Elastically Supported, Statically Indeterminate Beam*". II. By Prof. C. B. BIEZENO. (Communicated by Prof. J. CARDINAAL).

(Communicated at the meeting of December 29, 1917).

12. In quite the same way as it has been attempted in § 8 to prepare a transition from the case of three points of support to that of four, it might now be tried to use the construction just found for the treatment of the beam on five points of support, by cutting it above its last point of support but one and charging it there by a moment of 0, 1, 2, ... metretons. We are, however, arrested by two difficulties. In the first place the amount of drawing required becomes so extensive, that it is impossible to avoid mistakes. In the second place, however, an obstacle arises which has not yet been able to manifest itself in the case of the beam on three or four points of support.

When the beam has been cut above the fourth point of support, it is among others necessary to construct a link-polygon for the lefthand part on four points of support after applying a moment of transition of one metreton above the last point of support.

In the construction, however, of this link-polygon the beam on four points of support is again cut above the third point of support. Consequently this will be charged besides by the given forces acting on ABC, by a force of $\frac{1}{L}$ ton, directed upward and originating from the introduced unit couple M_D.

Of course a similar thing happened in the case of the beam on four points of support above the second point of support. But there the ascent $\frac{u}{L}$ of the second point of support, due to the extra force, was known, because the remaining lefthand part of the beam was only supported at two points. Here the ascent of the third point of support is not known, as the unaltered lefthand part of the beam is itself statically indeterminate.

And although it would of course be possible by the aid of §§ 3—7 to determine the ascent which the righthand end of the beam ABC supported at three points, would be subject to in consequence of a

force acting at this extremity, the execution of the required construction would only increase the difficulty mentioned at the beginning of this §.

Yet all the auxiliaries for a fit construction of the elastic link-polygon of a beam on five or more points of support have been provided, as will appear from the following.

13. *Beam on more than four points of support.*

Let the beam on n points of support $A, B, C, \ldots U, V, W$, be given, and let it for the present be required to determine the descent and the inclination at the last point of support W, when the beam is successively charged at W by a unit force and a unit moment. Then the experience gained in the preceding §§ leads to the expectation that the quantities in question only depend upon the corresponding ones for the beam $A, B, C, \ldots U, V$, i. e. upon the descent and the inclination which will appear at the last point of support of the beam $A, B, C, \ldots U, V$, when in V a unit force, resp. a unit moment, acts.

Let us suppose these latter quantities, which may be indicated by y_{n-2}, φ_{n-2}, $\bar{y}_{n-2}$, $\bar{\varphi}_{n-2}$, for a moment to be known, and let us attempt to derive from them the y_{n-1}, φ_{n-1}, $\bar{y}_{n-1}$, $\bar{\varphi}_{n-1}$ required. In determining each of these quantities we might again use the introduction of different moments of transition M_V above V.

For each moment of transition M_V it would be necessary to determine the situation of the point W in two ways:

1. by the aid of the equations of equilibrium of the field VW to the right, supposed to be free, by means of which a point W is found;

2. by means of an elastic link-polygon belonging to the beam $A, B, C, \ldots U, V, W$, which gives a point $\overline{W}$.

If then the series of points $\overline{W}$ and W should appear to be similar, it would be possible to construct their double point, i.e. the extreme point of the link-polygon determining the required quantities.

If the line of action described really causes y_{n-1}, φ_{n-1}, $\bar{y}_{n-1}$, $\bar{\varphi}_{n-1}$ to be found, we must accordingly attempt to determine the corresponding quantities of the beam $ABC \ldots UV$ on $(n-1)$ points of support.

But this would be possible if under the same conditions of charge the inclination and descent were known for the beam $ABC \ldots U\delta$ on $(n-2)$ points of support.

On arguing further in this way, we are driven back to the beam on two points of support, and it is therefore necessary first to find the quantities in question for this beam.

14*a*. With a view to this let us consider the freely supported beam AB charged at its right end by a force of one ton (fig. 3*a*). The descent $\underset{1T}{B\,B}$[1]) $= y_1$ as well as the angle of inclination φ_1 is

Fig. 3*a*.

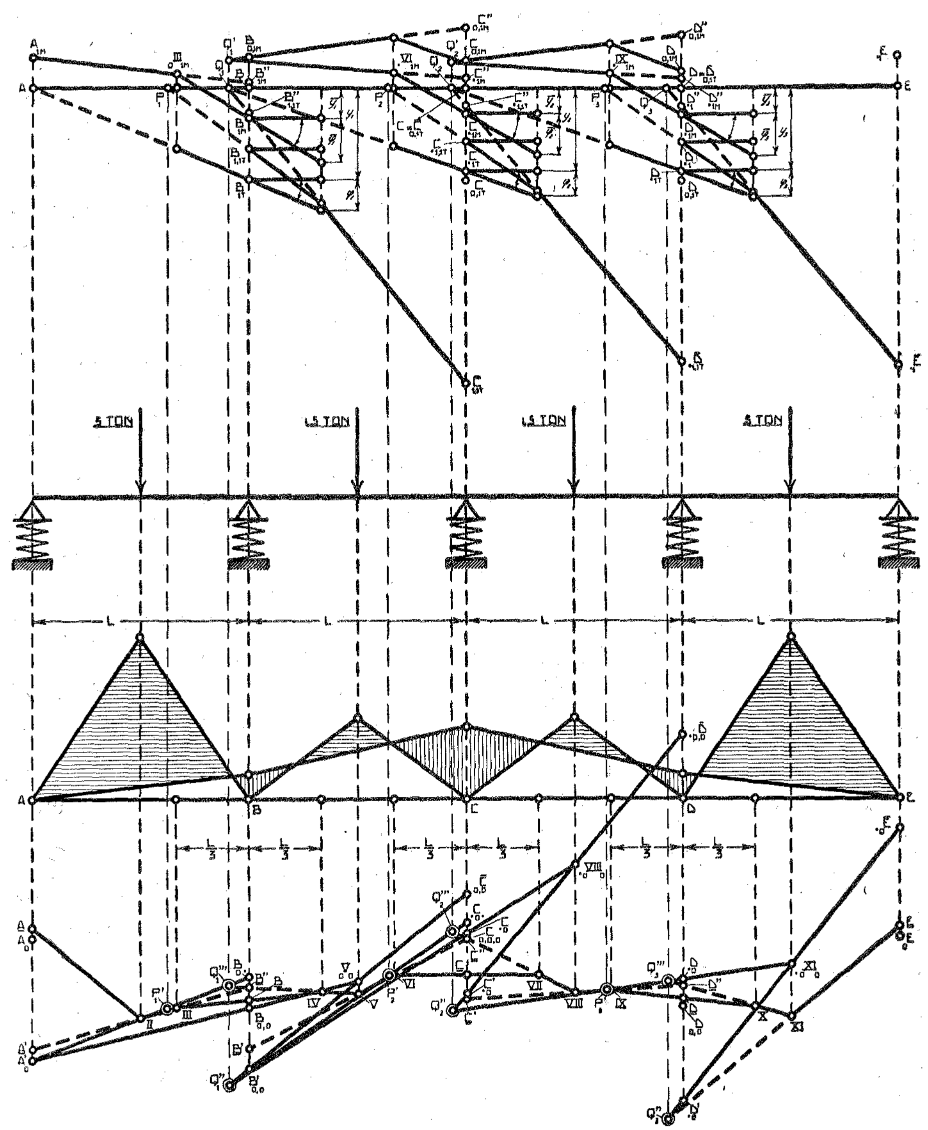


Fig. 3*b*.

[1]) In order to distinguish it from a moment of transition, a unit charge acting in a point of support $B, C \ldots$ is indicated by indices $1T$ or $1M$ placed under the letters $B, C \ldots$ according as it represents a force or a couple.

immediately known, as the point A remains in its place and AB remains straight. [1])

14*b*. If the end B is charged by a moment of 1 metreton, the point A rises by an amount AA_{1M}, while the point B descends by an amount $\underset{1M}{BB}$. As the sides $A_{1M}\,3_{1M}$ and $3_{1M}\,\underset{1M}{B}$ must cut a segment of known length from $\underset{B}{l}$, the former, hence also the latter, is known.

Consequently the angle of inclination $(\bar{\varphi}_1)$ and the descent $(\bar{y}_1)$ at B can also for this charge be found in a very simple way.

15*a*. We can now proceed to the treatment of the beam ABC charged at its extremity C by 1 ton.

If the beam is cut above B, the point C descends by an amount $\underset{0\,1T}{C\,C} = \mu$. The beam AB remains uncharged. The construction of the elastic link-polygon furnishes therefore the straight line ABC; the point $\underset{0\,1T}{\bar{C}}$ coincides with C.

If then a moment of transition of 1 metreton is introduced at B, the point $\underset{0,1T}{C}$ rises by an amount $\underset{0,1T\;1,1T}{C\quad C} = \frac{\mu}{L}$, so that $\underset{1,1T}{C}$ is known.

No more does the construction of $\underset{1,1T}{\bar{C}}$ by the aid of the elastic link-polygon give rise to difficulties. The introduction of the moment of 1 metreton above B will cause the point of support B to descend by an amount $\underset{1,1T}{B\,B} = \bar{y}_1 + \frac{1}{L}y_1$, while the side III $\underset{1,1T}{B}$ assumes an angle of inclination $\bar{\varphi}_1 + \frac{1}{L}\varphi_1$.

The side $\mathrm{III}_1\,\underset{1,1T}{B}\,{}_1\mathrm{IV}_{1T}$ (indicated in fig. 3*a* by the line $P_1\,\underset{1,1T}{B}$) can therefore be drawn, hence, also the side ${}_1\mathrm{IV}_{1T}\,{}_1\underset{1T}{\bar{C}}$, since this side together with $\mathrm{III}_1\;{}_1\mathrm{IV}_{1T}$ must cut a segment of known length from l_B.

It is clear that the sides ${}_0\mathrm{III}_x\;{}_x\mathrm{IV}_{1T}$ belonging to the different moments of transition $M_B = x$ metretons, pass through one fixed point P_1 on AB, because the descents of the point $\underset{x1T}{B}$ as well as the angles of inclination of the side ${}_0\mathrm{III}_x\;{}_x\mathrm{IV}_{1T}$ increase in propor-

[1]) The angles of inclination φ are replaced in the usual way by their tangents. These tangents are read in fig. 3*a* on perpendiculars drawn at distance one to the right of the different points of support.

tion to x. The series of points B''_{x1T} and ${}_x\mathrm{IV}_{1T}$ being moreover similar, also the joins $B''_{x1T}\ {}_x\mathrm{IV}_{1T}\ {}_x\overline{C}$ of corresponding points of these series pass through one fixed point Q_1, likewise situated on the line ABC.

The series of points $\overline{C}_{x.1T}$ and $C_{x,1T}$ are therefore similar.

Their double point at finite distance supplies the point $C_{.1T}$, while the line $C_{.1T}\ Q_1$ determines the angle of inclination in question.

15*b*. Also in the case of a charge of 1 metreton at C the beam is cut above B. In this case, however, the elastic support of B is charged by a force of $\frac{1}{L}$ ton. Consequently the point B rises by an amount $\frac{1}{L} y_1$ and the beam AB assumes at B an angle of inclination $\frac{1}{L}\varphi_1$. If for ABC a link-polygon is drawn on the supposition $M_B = 0$, the side $\mathrm{III}_0\ {}_0\mathrm{IV}_{1M}\ {}_0\mathrm{VI}_{1M}$ (indicated in the diagram by $B_{0,1M}\ C'''_{0,1M}$); hence also the side ${}_0\mathrm{VI}_{1M}\ C_{0,1M}$ is fixed.

The point $C_{0,1M}$ conjugated to $\overline{C}_{0,1M}$ lies $\frac{1}{L}$ below C.

Now a second construction would be necessary for a moment of transition $M_B = 1$ metretons in order to construct, in addition to the pair of points $C_{0,1M}$, $\overline{C}_{0,1M}$ just found, a second, which would make possible the determination of the double point of the series C_{x1M} and $\overline{C}_{x1M}$. The situation of this double point, however, depends exclusively upon the ratio $\dfrac{\overline{C}_{01M}\ \overline{C}_{11M}}{C_{01M}\ C_{11M}}$, which in its turn only depends upon the situation of the centres of rotation that appear to exist for the sides III_x, ${}_x\mathrm{IV}_{1M}$ and ${}_x\mathrm{IV}_{1M}$, ${}_x\mathrm{VI}_{1M}$ and of which in the diagram only that of the sides ${}_x\mathrm{IV}_{1M}$, ${}_x\mathrm{VI}_{1M}$ has been indicated as Q'_1.

According to the reasoning of § 7 however, these points must lie perpendicularly above the points P_1 and Q_1.

The ratio in question has therefore already been found (fig. 3*a*) in the ratio $\dfrac{\overline{C}_{11T}\ \overline{C}_{01T}}{C_{11T}\ C_{01T}}$. Hence the double point $C_{.1M}$, when once $\overline{C}_{01M}$ and C_{01M} are known, can immediately be determined as well as the corresponding point $C'''_{.1M}$.

The side ${}_.\mathrm{IV}_{1M}\ {}_.\mathrm{VI}_{1M}$ must lie along the line joining this point to Q'_1, so that the point ${}_.\mathrm{VI}_{1M}$ can be constructed. Then ${}_.\mathrm{VI}_{1M}\ C_{.1M}$ can be drawn, by means of which $\overline{y_2}$ and $\overline{\varphi_2}$ are found.

16. In quite the same way as $y_2, \overline{y_2}, \varphi_2, \overline{\varphi_2}$ have been determined out of $y_1, \overline{y_1}, \varphi_1, \overline{\varphi_1}$ also $y_3, \overline{y_3}, \varphi_3, \overline{\varphi_3}$ can be found out of $y_2, \overline{y_2}, \varphi_2, \overline{\varphi_2}$ and in general $y_n, \overline{y_n}, \varphi_n, \overline{\varphi_n}$ out of $y_{n-1}, \overline{y_{n-1}}, \varphi_{n-1}, \overline{\varphi_{n-1}}$.

Any new quadruple of unknowns belonging to a following point of support, requires the drawing after a fixed precept of only six lines.

Although for completeness' sake the construction of the point $C_{.1M}$ has been discussed, it will not be necessary to execute it in reality. For the theorem of MAXWELL teaches that $\overline{y_2} = \varphi_2$, so that $C_{.1M}$ is directly determined.

17. By the aid of the quantities y, φ, $\overline{y}$ and $\overline{\varphi}$, to be found according to the §§ 14 and 15, the construction of the elastic link-polygon of the arbitrarily charged beam can now be executed in the way as has been indicated in fig. 3*b* for a beam on 5 points of support.

The charge is again applied in the middle of the fields, and amounts for the successive fields resp. to 3, $1^1/_2$, $1^1/_2$ and 3 tons.

First the descents $A\,A_{00}$, $B\,B_{000}$, $C\,C_{00}$ ... of the points of support are determined which appear when the beam is cut above all the points of support.

Then the point $C_{.0}$ is determined over which the beam ABC considered as a whole, must pass.

This point is determined exclusively by the aid of the link-polygon A', $\overline{C}_{000}$ conjugating the point $\overline{C}_{00}$ to the point C_{000}. For we can omit the construction of the link-polygon $A'_{01} \ldots B_{10} \ldots C_{10}$ because it is only used to determine two points $\overline{C}_{100}$ and C_{100} the situation of which gives the ratio of the pieces into which the distance $\overline{C}_{00}\ C_{000}$ is divided by the point in question $C_{.0}$. This ratio, however, is already fixed in fig. $3a$ by $\frac{\overline{C}_{01T}\,\overline{C}_{11T}}{C_{01T}\,C_{11T}}$, for a reason mentioned before.

In an analogous way by the aid of the link-polygon $A_{.0} \ldots C_{.0}\,\mathrm{VIII}_{0}\,\overline{D}_{00}$ the endpoint $D_{.0}$ is determined for the beam $ABCD$ considered as a whole. Of this polygon, belonging to the moment of transition $M_C = 0$, the side $Q''_1\,C_{.0}\,\mathrm{VIII}_{0}$ and therewith $\mathrm{VIII}_{.0}\,\overline{D}_{.00}$ is determined. The situation of the double point $D_{.0}$ of the series of points $\overline{D}_{.y0}$ and $D_{.y0}$, which appear through the introduction of various moments of transition M_C and of which only the points $\overline{D}_{.00}$ and $D_{.00}$ are known, is again found with the ratio $\frac{\overline{D}_{01T}\,\overline{D}_{11T}}{D_{01T}\,D_{11T}}$, which appeared already in fig. $3a$.

Finally $\overline{E}_{.0}$ is determined by the aid of the link-polygon $Q''_2\,D_{.0}\,\mathrm{XI}_{.0\,0},\,\overline{E}_{.0}$ of which the extreme point $\overline{E}_{.0}$ forms together with E_0 a pair of points of the series $\overline{E}_{.z}$ and $E_{.z}$. The ratio in which the distance $\overline{E}_{.0}\,E_{.0}$ must be divided by the required point E, has also been found already in fig. $3a$ by the ratio $\frac{E,\,_{.1}\overline{E}}{E,\,_{.1}E}$ by the introduction of a moment of 1 metreton above D.

The endpoint $\underline{E}$ of the link-polygon in question is therefore fixed, so that no more is necessary than to construct the polygon itself.

In the first place the side $\underline{E}\,\mathrm{XI}$ can be drawn through Q'''_3, then XI X $\underline{D}''$ through Q'''_3, X IX through P'_3 and IX VIII through D''. The last side cuts with VIII VII from l_B a piece of known length; hence VIII VII can also be drawn. This side cuts $l_{Q''_2}$ in a point Q'''_2, which enables us to determine the point P'_2 through which the side VII VI is to be drawn.

While the supposition was made that no moment of transition existed in D, the sides VI VII and VII VIII on the introduction of various moments of transition M_C had to rotate round fixed points P and Q, which were determined as the intersections of the lines l_{P_2} and l_{Q_2} with $Q''_1\,C_{.0}$. But the moment of transition in D influences the situation of the point $C_{.0}$. According to § 15 this point rises by an amount $C_{.0}\,C_{0.} = \frac{M_D}{L}\,y_2$, in consequence of which the line $Q''_1\,\mathrm{VIII}_{.0\,.0}$, which in absence of M_D contained the centres of rotation of the sides VI VII and VII VIII, must be replaced by the line $Q''_1\,C_{.0.}$ Now the point Q'''_2 of this line is already determined as the intersection of $l_{Q''_2}$ with the side VIII VII, so that P'_2, through which VII VI must pass, can be found as the intersection of $Q''_1\,Q'''_2$ with l_{P_2}.

After VII VI we can draw VI V through C'', V VI being again determined by the known segment which $\overline{\mathrm{VI}}$ V and V IV must cut from l_B.

Now V IV determines again on $l_{Q''_1}$ a point Q'''_1 through which the line $A'_0\;Q'''_1$ can be drawn, which intersects l_{P_1} in the point P'_1 of the side IV III. The completion of the link-polygon does not present any difficulties.

18. It appears from the preceding considerations, as was indeed already noted in § 2, that with the beam on five points of support the subject mentioned in the title of this paper has been treated generally, save for the restrictions made, that the fieldlengths of the beam as well as the coefficients of stiffness of the elastic supports are supposed to be equal.

These restrictions, however, do not affect the general soundness of the construction.

If for instance the fieldlengths AB and BC (fig. 1) were unequal, the sides II III and IV V would indeed not meet on l_B, but at any rate on a perpendicular dividing the distance $l_{III}\,l_{IV}$ into pieces which are inversely proportional to the fieldlengths AB and BC.

Neither is the inequality of the coefficients of stiffness of the springs essential. If e. g. for the beam ABC with the fieldlengths $AB = L_1$ and $BC = L_2$ the coefficients of stiffness of the springs are α, β, γ, the extra descents of the springs which appear when the beam is cut above B in consequence of the introduction of a moment of transition of 1 metreton, will be $-\frac{\alpha}{L_1}$, $\frac{\beta}{L_1} + \frac{\beta}{L_2}$, $-\frac{\gamma}{L_2}$ instead of $-\frac{\mu}{L}$ $2\frac{\mu}{L}$, $-\frac{\mu}{L}$. This alters indeed the values of the segments $A'_0\,A'_1$, $\underset{0}{B}\,\underset{1}{B}$, $_0C\,_1C$, but not essentially the construction itself.

When the beam is not prismatic as was assumed in § 1, but has a variable cross-section, the diagrams of bending moment can be reduced in the way already indicated by MOHR.

19. Lines of influence.

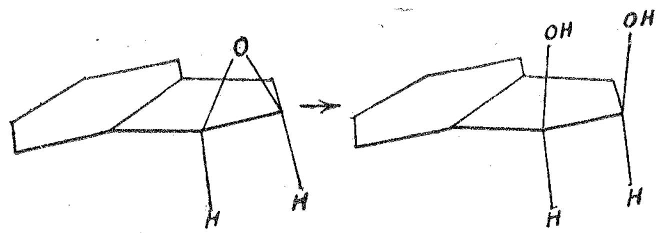
By the aid of the construction given in the preceding §§ the lines of influence for the moments of transition of a statically indeterminate elastically supported beam can now also be constructed graphically. In order to determine for instance the line of influence for the moment of transition above the p^{th} point of support, we need only construct the elastic line giving the position of the beam on being cut above the p^{th} point of support and charged in the arising cross-section by a moment of transition of 1 metreton. For each of the pieces into which the beam is divided, the moments of transition — hence the diagram of bending moment — can be determined as described, after which the elastic line can be drawn in the known way.

Chemistry. — *On the determination of the configuration of cyclic cis and trans diols and the rearrangements of atoms and groups of atoms during chemical reactions*". By Prof. J. BÖESEKEN and CHR. VAN LOON.

(Communicated at the meeting of June 28, 1919)

In former communications in which the configuration of the hydrindene diols was discussed, we called attention to the fact that *indene oxide*, when being hydrated, may yield the cis diol as well as the trans diol. In the meantime we ascertained that the quantities formed of these diols depend on the reaction of the medium, the formation of the trans compound is favoured by alkaline media.

Considering the probable situation of the atoms in indene oxide, the generation of trans diol deserves close attention; one might expect the cis compound:



Now it was of high importance to determine the configuration of the two diols with full certainty. Unfortunately the classic method, by testing the resolvability into optical antipodes, is of no avail here as both diols are asymmetric and consequently resolvable, so that another method had to be looked for.

We have already proved that of the two diols only one increases the conductivity of boric acid and we have deemed ourselves justified in assigning to this diol the cis configuration — rightly, as will be proved below.

[Here I may take it to be known, that the increase of the acid properties of boric acid by a number of substances, is due to the formation of complex dissociable compounds; these are formed especially if two hydroxyl groups are situated "favourably" in regard to boric acid.]

As this sole argument depends on the efficiency of the boric acid method, it called for a confirmation which could be given by our investigation on the *cyclapentane diols.*