**Physics.** — *"On the determination of quanta-conditions by means of adiabatic invariants."* By G. KRUTKOW. (Communicated by Prof. P. EHRENFEST.)

(Communicated at the meeting of September 25, 1920).

In a series of papers EHRENFEST has shown that only such functions of the general co-ordinates of a mechanical system may be quanticized as are *adiabatic invariants* [1]). These functions can always be found [2]). Moreover, as we shall see, theory may answer the question as to the number of *essential* adiabatic invariants, which in accordance with the quanta-hypothesis have to assume discontinuous values. If we suppose that the "density of probability" of the motion of the system, when not adiabatically acted upon, does not depend explicitly on the time, and if then by means of some hypothesis or some theorem which is derived from the properties of the system, we replace the *time-mean* of a phase-function by a *numerical mean*, it follows immediately that *the number of essential invariants is equal to the number of determining quantities of the system which is left after the numerical mean has been determined* (comp. equations (12) sqq. below). By the determination of the adiabatic invariants and the separation of the *essential* ones the uncertainty as to the choice of the forms of motion which are admissible on the quanta-hypothesis, becomes materially lessened. Still we must not expect that the adiabatic invariants which we have found are necessarily those which have to be quanticized: any arbitrary function of those quantities is again an adiabatic invariant and has thus equal claims to being selected. However, this liberty of choice can be somewhat restricted; there is a further condition to which we may subject the quanta-functions. This condition is of the nature of a hypothesis, but we may give it a simple statistical interpretation. In every case, where the theory of quanta has been applied with success [3]), the condition is fulfilled. It was introduced by PLANCK as a *fundamental theorem* for a complete determination of the quantities

[1]) P. EHRENFEST. These Proc. XIX N°. 3, p. 576. Ann. der Phys. 51 (1916) p. 327.

[2]) G. KRUTKOW. Proc. Amst XXI p. 1112. 1919.

[3]) My knowledge of the literature of the subject does not, however, extend beyond the beginning of 1917.



which have to be quanticized [1]). A new proof will be given by establishing a connection between the adiabatic invariants and the phase-space (below 18').

This connection, which will be found to arise in a natural way, with a concept derived from statistical mechanics, strengthens the bond between it and the theory of quanta, a bond which, as it seems to me, has gone into the background in the latest development of the theory or at least has not been sufficiently emphasized, although in my opinion it is of great importance. In view of this connection I think that the only justification of the expression "action-quantum" is the fact that it recalls to our mind the dimensions of the phase-extension.

Another conception of great importance to the theory of quanta which will find a place in our classification is PLANCK's [2]) *coherence of degrees of freedom*. To me it seems of fundamental importance. Its meaning will be found to appear very clearly by a juxtaposition of the properties of a conditionally periodic system and a BOLTZMANN "ergode".

This coherence of degrees of freedom must be very clearly distinguished from what is called *"degeneration"* [3]). For instance from our point of view an ergodic system is to the highest degree coherent, but could in no case be called degenerated. For a degenerated system the number of *essential* adiabatic invariants is *greater* than that of the degrees of freedom, for a coherent system it is *smaller*.

The question arises: must the *supernumerary* adiabatic invariants of a degenerated system be quanticised or, as suspected by SCHWARZSCHILD [4]), is the number of quanta-conditions smaller for such a system for the normal case without degeneration?

For the solution of these questions the three steps which have been taken viz. (1) establishment of the adiabatic invariants (2) selection of the *essential* ones and (3) "normalisation" of the latter are insufficient. In order to get nearer to the solution we must, I think, take into account, that *the quanta-functions must have a meaning which is independent of the system of co-ordinates.* We may undoubtedly postulate this: if the quanta-laws are really physical laws, they must necessarily satisfy this condition. The question is, how to formulate this new invariance of the quanta-functions? I shall not try to discuss it here in general; but only remark that

[1]) M. PLANCK. Ann. d. Phys. 50 (1916) p. 392.

[2]) M. PLANCK, l.c.

[3]) K. SCHWARZSCHILD. Sitzungsber. Berlin 1916. P. EPSTEIN Ann. d. Phys. 51 (1916). p. 168.

[4]) K. SCHWARZSCHILD, l.c.

we may return from the canonical equations which are so convenient in the theory of quanta to the equations on cartesian co-ordinates. Here the invariance in question means: invariance with respect to the groups of rotations and translations; vector-analysis thus provides the means of testing hypothetical quanta-quantities for the new postulata [1]).

The above mentioned means enable us in special cases to separate the quanta-quantities without ambiguity, for instance for the mechanical systems considered by PLANCK in the paper quoted. In some cases, however, an ambiguity remains, which we may get rid of in the following manner: by putting all but one of the quanta-quantities equal to nought, a "*singular motion*" must be obtained. In this manner we are able to make a connection between the methods sketched out above and PLANCK's theory on the physical structure of the phase-space, PLANCK's singular motions forming the last step in the series. We may recapitulate as follows:

The quanta-quantities are ($\alpha$) functions of the integrals of the equations of motion ($\beta$) adiabatic invariants which ($\gamma$) must be "*normalized*" and ($\delta$) have a meaning which is independent of the system of co-ordinates and finally ($\varepsilon$) yield singular motions in PLANCK's sense of the expression.

§ 1. *The fundamental equation.*

Let a mechanical system of $n$ degrees of freedom be given by its canonical equations of motion

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \; ; \; \dot{q}_i = \frac{\partial H}{\partial p_i} \; (i = 1,2,\ldots n) \; . \; . \; . \; . \; . \; (1)$$

We shall consider a number of systems and introduce a function $\varrho(p_i,q_i,t)$ which may be called the *density of probability*: $\varrho$ must satisfy the fundamental equation of statistical mechanics [2]):

$$\frac{\partial \varrho}{\partial t} + \sum_{i=1}^{n}\left(\frac{\partial \varrho \dot{p}_i}{\partial p_i} + \frac{\partial \varrho \dot{q}_i}{\partial q_i}\right) = 0 \; . \; . \; . \; . \; . \; . \; (2)$$

or, using (1):

$$\frac{\partial \varrho}{\partial t} + \sum_{i=1}^{n}\left(\frac{\partial \varrho}{\partial p_i}\dot{p}_i + \frac{\partial \varrho}{\partial q_i}\dot{q}_i\right) = \frac{d\varrho}{dt} = 0 \; . \; . \; . \; . \; (2')$$

$\varrho$ is therefore a function of the integrals of the equation (1).

[1]) In the theory of the ZEEMAN-effect as given by SOMMERFELD and DEBIJE (Phys. Zschr. 17. (1916) a difficulty is met with here. This may, I think, be evaded in different ways, but I am not able to give a *uniquely* determined solution.

[2]) J. W. GIBBS. Scientific papers. II' p. 16; Statistical Mechanics. Chapter I.



If we suppose that the condition is stationary:

$$\frac{\partial \varrho}{\partial t} = 0 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (3)$$

it follows that: $\varrho$ is a function of those integrals which do not contain $t$ explicitly, i.e. of $(2n-1)$ integrals, if only, as we shall suppose all the time, $H$ does not depend on $t$ explicitly.

We are at liberty to understand by $\varrho$ the density of probability a posteriori or a priori. When applied to the theory of quanta our result expresses the fact, that *the quanta-quantities are functions of the* $(2n-1)$ *integrals of equations* (1) *which are independent of* $t$.

Replacing the $2n$-dimensional phase-space $(p_i, q_i)$ by the corresponding integral space $(c_i, t_i)$ [1]) the "path" of the system is a straight line parallel to the $t_1$-axis. We can describe these lines either by making $t$ increase, i.e. by following a definite system in its motion, or by keeping $t$ constant and varying $\tau$, i.e. considering together all the systems with given $c_1 \ldots c_n\, t_2 \ldots t_n$ and all possible values of $t_1$.

§ 2. *$H$ contains a variable parameter.*

If $H$ contains a parameter which may either have a constant value as in the case just considered or vary slowly [2]), the quantities $c_2$ and $t_1$ are no longer constant, but variable; they have to satisfy the following "equations of motion" [3]):

$$\dot{c}_i = -\frac{\partial K}{\partial t_i} \; ; \; \dot{t}_i = \frac{\partial K}{\partial c_i} \; . \; . \; . \; . \; . \; . \; . \; (3)$$

where:

$$K = c_1 + \left(\frac{\partial V}{\partial a}\dot{a}\right), \; . \; . \; . \; . \; . \; . \; . \; (4)$$

[1]) As in a previous communication we write the integrals of equations (1) in the form:

$$H = c_1, \quad H_2 = c_2; \ldots, \quad H_n = c_n.$$

and

$$\frac{\partial V}{\partial c_1} = t + \tau = t_1, \qquad \frac{\partial V}{\partial c_2} = t_2, \ldots, \qquad \frac{\partial V}{\partial c_n} = t_n,$$

where $c_1, \ldots, c_n, \tau, t_2, \ldots, t_n$ represent the $2n$ integration-constants and $V$ JACOBI's characteristic function. Comp. Proc. XXI, p. 1112, 1919.

[2]) The slowness is expressed in the fact, that $H$ contains only $a$, not the correspondiug momentum.

[3]) Proc. l.c.

or putting $\dot{a} = \text{const.}$ approximately and representing the derivatives with respect to $a$ by dashes:

$$\left.\begin{aligned} c'_i &= -\frac{\partial}{\partial t_i}\left(\frac{\partial V}{\partial a}\right) \quad ; \quad t'_x = \frac{\partial}{\partial c_x}\left(\frac{\partial V}{\partial a}\right) \\ t'_1 &= \frac{1}{\dot{a}} + \frac{\partial}{\partial c_1}\left(\frac{\partial V}{\partial a}\right) \end{aligned}\right\} \begin{pmatrix} i = 1, 2, \ldots, n \\ x = 2, 3, \ldots, n \end{pmatrix} \quad . \quad (3')$$

Since the equations have the canonical form, we have as the fundamental equation:

$$\frac{\partial \varrho}{\partial a} + \sum_{i=1}^{n}\left(\frac{\partial \varrho}{\partial c_i} c'_i + \frac{\partial \varrho}{\partial t_i} t'_i\right) = 0 \; . \quad . \quad . \quad . \quad . \quad (5)$$

Here we may not as before take $\frac{\partial \varrho}{\partial a} = 0$. A further difficulty presents itself: starting from a special line parallel to the $t_1$-axis in the $(c_i, t_i)$ space — a special "stream-line" — if we now vary $a$, as equation (3) or (3′) show, the stream-line becomes broken up. If we then keep $a$ constant again and take together the points, that lie on a straight line, $\varrho$ will vary along this stream-line, since it contains points of different origin. Thus on the new line $\varrho$ is not stationary, but explicitly dependent on $t_1$.

We now form [1]) the time-mean of $\varrho$, which we shall call $\bar{\varrho}$ and the difference $\varrho - \bar{\varrho}$. Since $\int dt_1 \, (\varrho - \bar{\varrho}) = 0$, the quantity $\varrho - \bar{\varrho}$ in its dependence on $t_1$ shows elevations and depressions round about $\bar{\varrho}$, the sum of the surfaces of the former being equal to that of the latter. Each point carries its $\varrho - \bar{\varrho}$ value along with it and hence the curve shifts regularly with the time $t_1$. A stationary curve represents the tendency towards condensation (in an elevation) or rarefaction (in a depression) for the points of the stream-line, on the supposition of the change of $a$ being sufficiently small. If we make our moving curve slide along the stationary one, in the course of time elevations will cover depressions and vice versa. A further small change of $a$ may therefore produce a diminution of the difference $\varrho - \bar{\varrho}$. By this reasoning it becomes clear that starting from a stationary density a *sufficiently slow* change of $a$ will to a corresponding degree of approximation produce a stationary density [2]).

[1]) For the method now following comp. J. W. GIBBS. Statistical Mechanics. Chapter XIII.

[2]) Comp. J. M. BURGERS. Proc. Amst. XX (1916) 149, Ann. d. Phys. 1917 (2) and my paper in the Proc. Amst. l. c.



We will therefore suppose that $a$ changes slowly in the sense of the theory of adiabatic invariants. Let $Da$ be the total change of $a$, i.e.

$$Da = \dot{a} \int dt$$

and let $Dc_i$ and $Dt_x$ represent the corresponding changes of $c_i$ and $t_x$; considering further that

$$\dot{a} = const.,$$

and hence

$$\dot{a} = \frac{Da}{Dt} \quad \text{with} \quad (Dt = \int dt):$$

we find

$$\frac{Dc_i}{Da} = -\overline{\frac{\partial}{\partial t^i}\left(\frac{\partial V}{\partial a}\right)} \quad ; \quad \frac{Dt_x}{Da} = \overline{\frac{\partial}{\partial c_x}\left(\frac{\partial V}{\partial a}\right)}, \quad . \quad . \quad . \quad . \quad (6)$$

where the horizontal line indicates the time-mean. If in equation (5) we take $c'_i$ and $t'_i$ to mean these time-means, we obtain

$$\frac{\partial \varrho}{\partial a} + \sum_{i=1}^{n}\left(\frac{\partial \varrho}{\partial c_i}\frac{Dc_i}{Da} + \frac{\partial \varrho}{\partial t_i}\frac{Dt_i}{Da}\right) = 0 \; . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Since $\varrho$ is independent of $t$, the corresponding term under the summation-sign in (7) must be omitted.

§ 3. *Phase-space and adiabatic invariants.*

The stationary density $\varrho$ need not depend on all the variables

$$c_1, \ldots, c_n \quad ; \quad t_2, \ldots, t_n$$

For example in a *conditionally-periodic system* without commensurable relations between the periodicity-moduli $\varrho$ depends on the quantities $c_i$ only. This follows from the theorem which allows us to replace the time-average by an averaging over a $\Omega$-cell [1]). For an *ergodic* (or quasi ergodic) system in consequence of the *ergode-hypothesis* $\varrho$ depends on the energy $c_1$ only. We shall here suppose, that $\varrho$ depends on $k$ quantities ($k \leqq n$), which we shall indicate by

$$c_1, c_2, \ldots, c_k$$

These integrals may be called *essential* integrals. Our supposition with regard to $\varrho$ comes to the same as assuming that for our system the time-mean may be replaced by a definite numerical mean. To compute this we proceed as follows.

Suppose the system of equations

$$H_1 = c_1, H_2 = c_2, \ldots, H_k = c_k \quad . \quad . \quad . \quad . \quad . \quad (8)$$

to be soluble with respect to $p_1, p_2, \ldots, p_k$, thus

[1]) J. M. BURGERS. l.c. and my paper l.c.

$$p_\lambda = k_\lambda (q_1, \ldots, q_n, p_{k+1} \ldots p_n; c_1, \ldots c_k) \ (\lambda = 1,2, \ldots, k) \quad . \quad (8')$$

Introducing the differentials $dc_1 \ldots . dc_k$ instead of the differentials $dp_1 \ldots . dp_k$ into the phase-integral

$$I = \iint \ldots \iint dp_1 \ldots . dp_n \, dq_1 \ldots . dq_n \quad . \quad . \quad . \quad . \quad (9)$$

we find

$$I = \int \ldots \int dc_1 \ldots dc_k \int \ldots \int dp_{k+1} \ldots dp_n \, dq_1 \ldots dq_n \frac{\partial (p_1, \ldots p_k)}{\partial (c_1, \ldots c_k)} \quad (10)$$

or

$$I = \int \ldots \int dc_1 \ldots dc_k \, \omega (c_1, \ldots c_k) \quad . \quad . \quad . \quad . \quad (10')$$

where

$$\omega (c_1, \ldots, c_k) = \int \ldots \int dp_{k+1} \ldots dq_n \frac{\partial (p_1, \ldots, p_k)}{\partial (c_1, \ldots, c_k)} \quad . \quad . \quad (11)$$

In (11) the integration has to be carried out, the limits being determined by (8). From the $(p_i, q_i)$-space or the $(c_i, t_i)$-space we may pass to the $k$-dimensional $(c_i, \ldots c_k)$-space. A streamline of the former space corresponds to a fixed point in the latter. The density $\varrho$ is replaced by $\varrho\omega$ in the $c$-space. Its elements therefore have the *weight* $\omega$. For the iso-parametric motion ($a = \text{const.}$) the $c$-space is *static* i.e. each point is fixed. The integral (11) gives us the numerical mean looked for, namely, if $f$ is a phase-function, we have:

$$F = \int \ldots \int f \frac{\partial (p_1 \ldots p_k)}{\partial (c_1 \ldots c_k)} dp_{k+1} \ldots dq_n / \omega. \quad . \quad . \quad . \quad . \quad (11')$$

Returning to equation (7) we now have:

$$\frac{\partial \varrho}{\partial a} + \sum_{i=1}^{k} \frac{\partial \varrho}{\partial c_i} \frac{Dc_i}{Da} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

since $\varrho$ is a function of $c_1, \ldots, c_k$ only. Similarly the quantities $Dc_i/_{Da}$ only depend on $c_1, \ldots, c_k$, as is easily seen from (6), if on the right hand side we replace the time-mean by the numerical mean (11'). Therefore $\varrho$ retains its property $\varrho = \varrho (c_1, \ldots, c_k)$ when $a$ changes. Equation (12) expresses, that $\varrho$ is a function of those $k$ integrals of the differential equations (6) which only contain the quantities $c_1, \ldots, c_k$. These integrals are obtained by integrating the set of $k$ differential equations which on the left side contain the quantities $Dc_i/_{Da}$ $(i = 1, 2, \ldots, k)$. They are the *essential adiabatic invariants*, and we have thus proved that $\varrho$ *is a function of the essential adiabatic invariants.*



Let us further consider the $c$-space. If $a$ varies slowly, the fixed points in it begin to move. Since in this motion the points do not disappear nor new points are formed, the density $\varrho\omega$ must satisfy the equation of continuity, i. e. our fundamental equation. As $\varrho = \text{const.}$ is certainly a possible solution, $\omega$ itself must satisfy the equation

$$\frac{\partial \omega}{\partial a} + \sum_{i=1}^{k} \frac{\partial \omega \bar{c'}_i}{\partial c_i} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where

$$\bar{c'}_i = \frac{Dc_i}{Da}$$

Or with the notation

$$\frac{D\omega}{Da} = \frac{\partial \omega}{\partial a} + \sum_{i=1}^{k} \frac{\partial \omega}{\partial c_i} \frac{Dc_i}{Da},$$

in the equivalent forms

$$\frac{D\omega}{Da} + \omega \sum_{i=1}^{k} \frac{\partial \bar{c'}_i}{\partial c_i} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13')$$

or

$$\sum_{i=1}^{k} \frac{\partial \bar{c'}_i}{\partial c_i} = -\frac{1}{\omega} \frac{D\omega}{Da} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13'')$$

For the quantity on the left side — the "divergence" — we shall deduce another expression.

The essential adiabatic invariants — $k$ in number — satisfy the equations

$$\frac{Dv_i}{Da_i} = \frac{\partial v_i}{\partial a} + \sum_{i=1}^{k} \frac{\partial v_i}{\partial c_i} \bar{c'}_i \ (i = 1,2, \ldots, k) \quad . \quad . \quad . \quad . \quad (14)$$

We shall suppose that the quantities $v_i$ can be expressed in the quantities $c_\lambda$ $(\lambda = 1, 2 \ldots k)$ or

$$\frac{\partial (v_1, \ldots, v_k)}{\partial (c_1, \ldots, c_k)} = \Upsilon \neq 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The properties of our system can be equally well described by the quantities $v_i$ as by the quantities $c_i$. The $(v_1 \ldots v_k)$-space has the advantage over the $c$-space of being *static*, also with respect to the action of adiabatic influences. Let us now examine the mutual relation of the two spaces.

To this end we shall consider the $D$-derivative of the determinant $\Upsilon$:

$$\frac{D\Upsilon}{Da} = \sum_{i\mu} V_{i\mu} \frac{Dv_{i\mu}}{Da}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where $v_{i\mu} = \frac{\partial v_i}{\partial c_\mu}$ and $V_{i\mu}$ represents the corresponding sub-determinant. We have identically

$$\frac{Dv_i}{Da} = \frac{\partial v_i}{\partial a} + \sum_\lambda \frac{\partial v_i}{\partial c_\lambda} \overline{c'_\lambda} = 0,$$

hence

$$\frac{\partial}{\partial c_\mu}\frac{Dv_i}{Da} = \frac{\partial^2 v_i}{\partial a \partial c_\mu} + \sum_\lambda \frac{\partial^2 v_i}{\partial c_\lambda \partial c_\mu}\overline{c'_\lambda} + \sum_\lambda \frac{\partial v_i}{\partial c_\lambda}\frac{\partial \overline{c'_\lambda}}{\partial c_\mu} = 0 \quad . \quad . \quad . \quad (a)$$

and on the other hand

$$\frac{D}{Da}\frac{\partial v_i}{\partial c_\mu} = \frac{\partial^2 v_i}{\partial a \partial c_\mu} + \sum_\lambda \frac{\partial^2 v_i}{\partial c_\mu \partial c_\lambda}\overline{c'_\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

From $(a)$ and $(b)$ it follows that

$$\frac{D}{Da}\frac{\partial v_i}{\partial c_\mu} = \frac{Dv_{i\mu}}{Da} = -\sum_\lambda \frac{\partial v_i}{\partial c_\lambda}\frac{\partial \overline{c'_\lambda}}{\partial c_\mu} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

or

$$\frac{Dv_{i\mu}}{Da} = -\sum_{\lambda=1}^{k} v_{i\lambda}\frac{\partial \overline{c'_\lambda}}{\partial c_\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17')$$

Substituting in (16) we obtain

$$\frac{D\Upsilon}{Da} = -\Sigma v_{i\lambda} V_{i\mu}\frac{\partial \overline{c'_\lambda}}{\partial c_\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16')$$

Since $\Sigma v_{i\lambda} V_{i\mu} = 0$ for $\lambda \neq \mu$ and equal to $\Upsilon$ for $\lambda = \mu$, (16') becomes

$$\frac{D\Upsilon}{Da} = -\Upsilon \sum \frac{\partial \overline{c'_\lambda}}{\partial c_\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16'')$$

Hence

$$\sum_\lambda \frac{\partial \overline{c'_\lambda}}{\partial c_\lambda} = -\frac{1}{\Upsilon}\frac{D\Upsilon}{Da} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16''')$$

Comparing this result with (13'') it follows that

$$\frac{1}{\Upsilon}\frac{D\Upsilon}{Da} = \frac{1}{\omega}\frac{D\omega}{Da} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

or

$$\frac{D}{Da}\log\frac{\Upsilon}{\omega} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18')$$

In other words: $\Upsilon/\omega$ *is an adiabatic invariant* or

$$\Upsilon = \omega f(v_1, \ldots, v_k), \quad \omega = \Upsilon F(v_1, \ldots, v_k) \quad . \quad . \quad . \quad (18'')$$

Substituting this value of $\omega$ in the integral (10') we find

$$I = \int \ldots \int dc_1 \ldots dc_k \, \Upsilon F(v_1, \ldots, v_k) = \int \ldots \int dc_1, \ldots, dc_k \frac{\partial(v_1 \ldots v_k)}{\partial(c_1 \ldots c_k)} F. \quad (19)$$



or

$$I = \int \ldots \int dv_1 \ldots dv_k \, F(v_1, \ldots, v_k) \quad . \quad . \quad . \quad . \quad (19')$$

Now we can always arrange, that $F$ becomes equal to 1. We have only to introduce, instead of one of the $v_i$, the adiabatic invariant

$$r_1 = \text{funct.}(v_1, \ldots, v_k)$$

and submit it to the condition

$$\frac{\partial r_1}{\partial v_1} = F(v_1, \ldots, v_k) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

We then find

$$\frac{\partial r_1}{\partial c_\lambda} = \sum_x \frac{\partial r_1}{\partial v_x}\frac{\partial v_x}{\partial c_\lambda} = \sum_x \frac{\partial r_1}{\partial v_x} v_{x\lambda}$$

$$\Upsilon^* = \frac{\partial(r_1, v_2, \ldots, v_k)}{\partial(c_1, c_2, \ldots, c_k)} = \sum_\lambda \frac{\partial r_1}{\partial c_\lambda} V_{1\lambda} = \sum \frac{\partial r_1}{\partial v_x} V_{i\lambda} v_{x\lambda} = \frac{\partial r_1}{\partial v_1}\Upsilon,$$

or substituting for $\Upsilon$ its value $\omega/F$:

$$\Upsilon^* = \omega \frac{\partial r_1}{\partial v_1}\frac{F}{1} = \omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Calling the thus *normalized set of essential adiabatic invariants* $v_1, \ldots, v_k$, we find

$$I = \int \ldots \int dv_1 \ldots dv_k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The *v-space* which is static with respect to adiabatic action is "*weightless*": its density $\varrho$ is simply equal to $\varrho(v_1, \ldots, v_k)$. The quantities $v_1, \ldots, v_k$ *may* be quanticized. Its property which is expressed by eq. (22) is nothing but the fundamental law which according to PLANCK's hypothesis the quanta-quantities have to obey [1]). By our theorem (18'') this hypothesis is connected with the adiabatic invariants and thus finds a new confirmation. The property of the $v$-space being "weightless" displays the character of this fundamental law as a natural generalisation of the old quanta-hypothesis.

§ 4. *On the coherence of degrees of freedom.*

From the point of view now attained this very important conception appears as a natural consequence of our suppositions. If the number $k$ — that of the essential integrals and adiabatic invariants — is smaller than the number of degrees of freedom $n$, as appears from (22), some of the quantities $v_i$ must necessarily be of the dimen-

[1]) M. PLANCK. l.c.

sion $h^p$ $(p > 1)$, since the dimension of $I$ is $h^n$. In order to illustrate this and the previous results we shall contrast the properties of a BOLTZMANN *ergode* and a *conditionally periodic* system without commensurable relations:

| *ergode* | *conditional periodic system* |
|---|---|
| essential integrals | |
| $H = c_1$ | $H_1 = c_1,\ H_2 = c_2, \ldots,\ H_n = c_n.$ |
| numerical mean | |
| $\int\ldots\int dp_2 \ldots dp_n\, dq_1 \ldots dq_n \frac{\partial p_1}{\partial c_1}$ . (23) | $\int\ldots\int dq_1 \ldots dq_n \frac{\partial (p_1, \ldots, p_n)}{\partial (c_1, \ldots, c_n)}$ (23') |
| density | |
| $\varrho = \varrho\,(c_1, a)$ | $\varrho = \varrho\,(c_1, \ldots, c_n; a)$ |
| essential adiabatic invariants | |
| $V = \int\limits_{H \leq c_1}\ldots\int dp_1 \ldots dq_n$ | $v_i = \int\limits_0 p_i\, dq_i\ (i = 1, 2, \ldots, n).$ |
| density | |
| $\varrho = \varrho\,(V)$ | $\varrho = \varrho\,(v_1, v_2, \ldots, v_n)$ |

The conditionally periodic system is what BOLTZMANN calls a *subergode*. On the other hand the ergode appears as a *coherent* system with a smallest value of $k$, viz. $k = 1$. These short indications may suffice for the present.

§ 5. *Degeneration.*

A conditionally periodic system is called degenerated, if there are commensurable relations between the periodicity moduli. It is evident, that our system covers a *lower* set of points with its orbital curve everywhere densely, than when there are no such relations. Accordingly the numerical mean will be of a lower dimension and more quantities will remain free after the averaging process. Thus besides the quantities $c$ the quantities $t$ will play a part: the number of essential adiabatic invariants becomes larger than the number of degrees of freedom. The question, whether these supernumerary quantities have to be quanticized, we shall not discuss here. A good instance for the discussion of the questions which may arise here is afforded by the quanta-quantities in EPSTEIN's theory [1]) of the STARK-effect for an infinitely weak external electric field; the "parabolic" quanta-quantities which are found in this case cannot

[1]) P. EPSTEIN. Ann. d. Phys. 50. p. 490.



be represented as function of SOMMERFELD's "spherical" quanta-quantities alone; other adiabatic invariants containing the quantities $t_2$ and $t_3$ are essential in this case.

I am fully conscious of the fact, that by the above considerations the difficulties which still beset the theory of quanta are in no way removed, but only shifted. Still it seems to me that even the possibility of such displacement deserves attention. Moreover I expect that in *special* cases the general theory tentatively sketched out here may be found useful.

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