type. In the end, it is of the same nature as specific differentiation in general.

So in *Hepialus humuli* the white masculine form has evidently lost the primitive specific livery, which is still preserved by the female and by the Shetland-male.

Though in general my opinions on these subjects disagree with those of HAASE, I feel much satisfaction in making the following quotation from the concluding passage of his "Resumption" (p. 112): "The mimetic transformation was preceded in most cases by atavistic phenomena from the side of the females, which in the beginning reached back to the patterns of the nearest relatives, but as the process proceeded, passed over to those of more distanced forerunners and in this way procured the material for the mimetic adaptation".

So HAASE attributes the uniforms of mimetic females to hereditary influences, instead of considering them as the consequence of secondary deviations from the primitive specific type.

Groningen, Nov. 1920.

Physics. *"On the Equation of State for Arbitrary Temperatures and Volumes. Analogy with Planck's Formula."* II. By Dr. J. J. VAN LAAR. (Communicated by Prof. H. A. LORENTZ).

(Communicated at the meeting of November 27, 1920).

§ 7. Some Notes to § 1—6.

It will be soon two years ago that I wrote the first part of this Article[1]); studies of various kinds prevented me from continuing the subject, and not until now could I take it up again.

Before I proceed to the derivation of the equation of state, based on the found general expression (6) on p. 1194 loc. cit. for the *time*-average of the square of velocity u_t^2, expressed in u_0^2 (in which u_0 represents the velocity with which the considered molecule passes the neutral point in its motion to and fro between two neighbouring molecules), I will add a few remarks to elucidate and complete what was treated before.

1. In the first place a few words about the transition of some "linear" quantities to the corresponding "spatial" quantities.

If we have linear quantities, we can consider all our velocities as the components of the *relative* velocities directed *normally*; as we always imagine a molecule moving rectilinearly to and fro between two molecules *at rest.* We know that $\overline{u_r^2}=2\overline{u^2}$, and that the mean value of the component of $\overline{u_r^2}$, directed normally, in its turn is the third part of this, so that we have (cf. also p. 1195 loc. cit.):

$$\overline{(u^2)_n}=\frac{2}{3}\overline{u^2}.$$

Hence we may write:

$$\frac{1}{2}Nm\,\overline{(u_r^2)_n}=\frac{2}{3}\times\frac{1}{2}Nm\,\overline{u^2},$$

or also, denoting the *time*-average by the index t:

$$\frac{1}{2}Nm\,\{\overline{(u_r^2)_n}\}_t=\frac{2}{3}\times\frac{1}{2}Nm\,\overline{(u^2)_t}.$$

In this $^1/_2\,Nm\,\overline{(u^2)_t}$ ($= {}^3/_2\,pv$ in ideal gases) $= {}^3/_2\,RT$, so that we may henceforth write:

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[1]) These Proc., Vol. XXI, p. 1184.

$$\frac{1}{2} Nm \{\overline{(u_r^2)_n}\}_t = RT,$$

by which the transition in question has been accomplished. In what follows u^2 will, however, always simply be written instead of $\overline{(u_r^2)_n}$, with omission of the indices r and n and of the usual mean-value dash (the time-average is then denoted by u_t^2); the *real* mean velocity square $\overline{u^2}$, if it should occur, being expressed by (u^2). Hence we have:

$$\frac{1}{2} Nm\, u_t^2 = RT \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Starting from the relation (cf. equation (a) on p. 1189 loc. cit.)

$$\frac{1}{2} Nm\, u_\sigma^2 = \frac{1}{2} Nm\, u_0^2 + Nf(l-\sigma)^2,$$

in which σ represents *that* distance from the centre of the moving molecule to that of the molecule supposed stationary, towards which it moves, at which the work of the attractive forces reaches its *maximum* value (hence at which the attraction changes into repulsion) — we shall find, after multiplication by $^3/_2$, for the *real* mean squares of velocity:

$$\frac{1}{2} Nm\, (u_\sigma^2) = \frac{1}{2} Nm\, (u_0^2) + \frac{3}{2} Nf(l-\sigma)^2 .$$

In this $\frac{1}{2} Nm\,(u_\sigma^2) = E$ represents the total Energy of the system (the atom-energies within the molecule being left out of consideration). Further $^1/_2\, Nm\,(u_0^2) = L_0$ is the mean kinetic Energy at the neutral point halfway between the two molecules at rest (where the attractive forces neutralise each other), $^3/_2\, Nf(l-\sigma)^2 = \Delta$ representing the maximum work of the attractive forces. We have represented this last quantity by E_0 in our first paper, but as this way of representation can easily give rise to misunderstanding, we shall substitute Δ for E_0 in what follows. We have therefore:

$$E = L_0 + \Delta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

in which accordingly $E = {}^3/_2 \times {}^1/_2\, Nmu_\sigma^2$, $L_0 = {}^3/_2 \times {}^1/_2\, Nmu_0^2$. Hence in the joint neutral points $E = L_0 +$ the total potential energy of the attractive forces; and in the joint points σ in the immediate neighbourhood of the molecules, with which the moving molecule will impinge, E will be $= L_0 +$ the total increment of the kinetic energy in consequence of the attractive forces.

The quantity Δ, therefore, represents the fixed, invariable (potential or kinetic) energy of the attractive forces, which rise or fall of temperature cannot increase or decrease. Change of temperature can only modify L_0, and consequently also E. Henceforth $E-\Delta$ may always be written for L_0.

The work of the repulsive forces, which become active after the attractive forces in the above-indicated point σ have ceased to act, has been left out of consideration in what precedes, because Δ is entirely unaffected by it. For the diminished kinetic energy is simply converted into a corresponding increase of the potential energy — now of the repulsive forces — which reaches its culminating point when u has become $= 0$ (culminating point of the collision). We have, therefore, only to do with the maximum work of the attractive forces.

2. In the first paper it has been shown that the calculation of the *time*-average u_t^2 leads to the relation (equation (6) on p. 1194)

$$u_t^2 = \frac{1}{2} u_0^2 \frac{\left\{\varphi\sqrt{1+\varphi^2} + \log\left(\varphi + \sqrt{1+\varphi^2}\right)\right\}\sqrt{\frac{m}{2f}} + \frac{1}{2}\pi(1+\varphi^2)\sqrt{\frac{m}{2\varepsilon}}}{\log\left(\varphi + \sqrt{1+\varphi^2}\right)\sqrt{\frac{m}{2f}} + \frac{1}{2}\pi\sqrt{\frac{m}{2\varepsilon}}}.$$

This becomes after division of numerator and denominator by $\log\left(\varphi + \sqrt{1+\varphi^2}\right)\sqrt{\frac{m}{2f}}$:

$$u_t^2 = \frac{1}{2} u_0^2 \frac{\left(1 + \frac{\varphi\sqrt{1+\varphi^2}}{\log.}\right) + \frac{^1/_2\,\pi(1+\varphi^2)}{\log.}\sqrt{\frac{f}{\varepsilon}}}{1 + \frac{^1/_2\,\pi}{\log.}\sqrt{\frac{f}{\varepsilon}}}, \quad . \quad (c)$$

in which (loc. cit.) $\varphi = \frac{l-\sigma}{u_0}\sqrt{\frac{2f}{m}}$. The distance $\sigma - s'$, during which the repulsive forces will act, follows from

$$u_0^2 + \frac{2f}{m}(l-\sigma)^2 - \frac{2\varepsilon}{m}(\sigma-s')^2 = 0, \quad \text{or} \quad u_0^2(1+\varphi^2) = \frac{2\varepsilon}{m}(\sigma-s')^2$$

at the culminating point of the collision. Hence we have for $l-\sigma$ and $\sigma - s'$:

$$l-\sigma = u_0\varphi\sqrt{\frac{m}{2f}}; \quad \sigma-s' = u_0\sqrt{1+\varphi^2}\sqrt{\frac{m}{2\varepsilon}}; \quad \frac{\sigma-s'}{l-\sigma} = \frac{\sqrt{1+\varphi^2}}{\varphi}\sqrt{\frac{f}{\varepsilon}}, \quad (d)$$

whereas for the times t_1 and t_2 is found:

$$t_1 = \log\left(\varphi + \sqrt{1+\varphi^2}\right)\sqrt{\frac{m}{2f}}; \quad t_2 = \frac{1}{2}\pi\sqrt{\frac{m}{2\varepsilon}}; \quad \frac{t_2}{t_1} = \frac{^1/_2\,\pi}{\log}\sqrt{\frac{f}{\varepsilon}} \quad . \quad . \quad (e)$$

a. At *high temperatures* where, in consequence of the equation $\varphi = \frac{l-\sigma}{u_0}\sqrt{\frac{2f}{m}}$, φ becomes *small* when u_0 becomes large (supposing $l-\sigma$ always remains comparatively small, which is fulfilled here, because we always consider solid (at most liquid) systems), (c), (d) and (e) with $log\,(\varphi + \sqrt{1+\varphi^2}) = log\,(\varphi + 1) = \varphi$ pass into:

$$u_t^2 = \frac{1}{2}u_0^2\,\frac{2+\frac{{}^1\!/_2\pi}{\varphi}\sqrt{\frac{f}{\varepsilon}}}{1+\frac{{}^1\!/_2\pi}{\varphi}\sqrt{\frac{f}{\varepsilon}}};\qquad \frac{\sigma-s'}{l-\sigma} = \frac{1}{\varphi}\sqrt{\frac{f}{\varepsilon}};\qquad \frac{t_2}{t_1} = \frac{{}^1\!/_2\pi}{\varphi}\sqrt{\frac{f}{\varepsilon}}\ \text{high temp.)}\,,\quad (c_1)$$

so that in the case of weak collisions (in which ε, the constant of the repulsive force, is not *very* much greater than f, the constant of the attractive forces), in consequence of φ in the denominators of the second terms in numerator and denominator of the above fraction for u_t^2, these latter terms will prevail; hence u_t^2 will approach to ${}^1\!/_2\,u_0^2$ ($c_v = 6$). Whereas in case of *strong* collisions, when ε is supposed *very large* with respect to f, or when φ gradually increases somewhat on *decrease of temperature*, the first terms prevail, so that then u^2 will more and more approach to u_0^2 ($c_v = 3$).

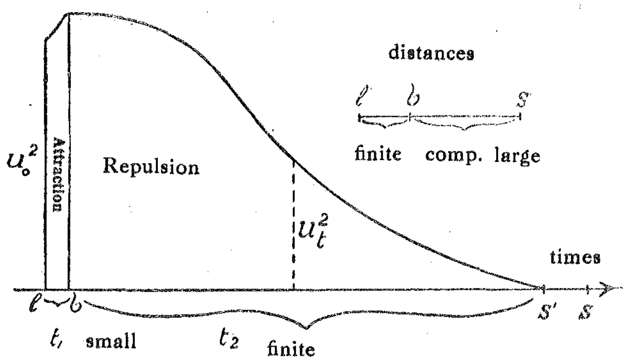
The ratios $(\sigma - s'):(l-\sigma)$ and $t_2:t_1$ will be *great* for φ small and $f:\varepsilon$ not very much smaller than 1; *smaller* on the other hand for somewhat larger φ, and ε much greater than f.

With regard to $l-\sigma$ and $\sigma - s'$ themselves, it may be observed that according to the supposition $l-\sigma$ always remains *finite*, so that $\sigma - s' = u_0\sqrt{\frac{m}{2\varepsilon}}$ can become large at increasing temperature and finite ε. But this increase is restricted first of all by this, that u_0 can never become *too* great, because then our suppositions (solid state with small values of $l-\sigma$) would not be fulfilled; and secondly by this that with comparatively large values of u_0, in consequence of which $\sigma - s'$ would become too large, ε will gradually greatly increase, so that the molecules can never approach each other more closely than to a certain minimum distance. Only in case of very strong collisions (ε very large with respect to f) $\sigma - s'$ can approach to 0 at not too large values of u_0.

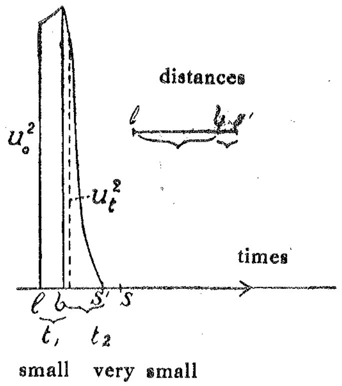
It holds for t_1 and t_2 themselves, that $t_1 = \varphi\sqrt{\frac{m}{2f}}$ will always approach 0 at high temperature, while $t_2 = \frac{1}{2}\pi\sqrt{\frac{m}{2\varepsilon}}$ remains finite — unless ε is very large, in which case t_2 can even become much smaller than t_1.

All these relations are graphically represented by Fig. 1a and Fig. 1b, in which the values of u are given in function of the *time*.

High temperatures (u_0 large, φ small).



Weak collisions.
(ε/f not very great; $u_t^2 = {}^1\!/_2\,u_0^2$; $c_v = 6$).
Fig. 1^a.



Strong collisions.
(ε/f very great, or T somewhat lower;
$u_t^2 = u_0^2$; $c_v = 3$).
Fig. 1^b.

In the so-called "weak" collisions the velocity of the colliding molecule will not diminish *suddenly*, but gradually. This is among others fulfilled when the attractive force is supposed to change into a repulsive one already before the molecules collide. It may then be further assumed that the repulsive force does not become infinite before the impact itself, so that in general — unless the velocity is infinitely great — the two molecules will never be in absolute contact. Hence there is always between σ and a value s' somewhat greater than s (the distance of the centres at contact) a certain space, in which the decrease of velocity in consequence of the repulsive forces can take place; and there always remains — even at $T=0$ — some distance, however slight, between the molecules, because of course l cannot become smaller than σ.

It is self-evident that this somewhat modified way of considering the matter is only of a formal nature. Theoretically there is nothing changed when s is displaced to σ, and s' from a point within s to a point outside it; now, however, we need not think the molecule greatly compressed in the weak collisions, as we had to do with the former way of considering the matter.

The two above figures also show clearly why in the case of Fig. 1a u_t^2 approaches to ${}^1\!/_2\,u_0^2$, and in the case of Fig. 1b to u_0^2. For as e.g. in the first case the time, during which the repulsive forces act, is so much greater than that under the influence of the attractive forces, the *time-average* will lie in the neighbourhood of

$^1/_2\, u_0^2$. In Fig. 1*b* on the other hand the *"action"* (energy × time) of the repulsive forces will be very much smaller than that of the attractive forces, with this result that now the time-average descends but little below u_0^2.

With decreasing values of u_0 (lower temperatures) the relations of Fig. 1*a* will more and more shift in the direction of Fig. 1*b* in consequence of the continual increase of φ, so that c_v will descend already to a smaller value from the limiting value 6, before the temperature has fallen to such a low value, that u_t^2 is in inverse logarithmic dependence to u_0^2 (see below) — in other words *before the region of quanta proper has been entered.*

b. At *low temperatures* φ will appear to be great; i.e. on the supposition that in $\varphi = \frac{l-\sigma}{u_0} \sqrt{\frac{2f}{m}}$ the quantity $l-\sigma$ does not approach 0 to the same degree as u_0, but much more slowly, so that $(l-\sigma) : u_0$ will approach ∞. It is even probable that $l-\sigma$ does not become $= 0$ even at $T = 0$, but approaches to a certain small limiting value. This is in agreement with the permanent decrease of the expansibility at very low temperatures, and with the remaining of a certain *finite* zero-point energy $\Delta = {}^3/_2\, Nf(l-\sigma)^2$ at $T = 0$.

Our equation (*c*) now becomes:

$$u_t^2 = \frac{1}{2} u_0^2 \, \frac{\dfrac{\varphi^2 (1 + {}^1/_2 \pi \sqrt{f/\varepsilon})}{\log (2\varphi + {}^1/_{2\varphi})}}{1 + \dfrac{{}^1/_2 \pi}{\log 2\varphi} \sqrt{\dfrac{f}{\varepsilon}}} = u_0^2 \, \frac{\varphi^2}{\log (4\varphi^2 + 2)} \times \frac{1 + \dfrac{1}{2} \pi \sqrt{\dfrac{f}{\varepsilon}}}{1 + \dfrac{{}^1/_2 \pi}{\log 2\varphi} \sqrt{\dfrac{f}{\varepsilon}}} \; (\textit{low temp.}), \quad (c_2)$$

which at *very low* temperatures, at which φ approaches to 0, will become nearer and nearer to

$$u_t^2 = u_0^2 \, \frac{\varphi^2}{\log (2\varphi^2 + 1)} \times \left(1 + \frac{1}{2} \pi \sqrt{\frac{f}{\varepsilon}}\right) \; (\textit{very low temp.}), \quad . \quad (c'_2)$$

because the finite term *log* 2 can then also be omitted by the side of $\log (2\varphi^2 + 1)$ in $\log (4\varphi^2 + 2) = \log (2\varphi^2 + 1) - \log 2$. But the factor $1 + \frac{1}{2}\pi \sqrt{\frac{f}{\varepsilon}}$ can be omitted only when ε is very large with respect to f (strong collisions), which is, however, not very probable in view of what was found at high temperatures — unless at high temperatures φ is so small, that notwithstanding ε is very much greater than f, the quantity $\frac{{}^1/_2 \pi}{\varphi} \sqrt{\frac{f}{\varepsilon}}$ would yet remain comparatively great.

But at all events in the case (c_2) or (c_2') u_t^2 (proportional to the temperature) will be very much greater than u_0^2 (proportional to $L_0 = E - \Delta$). Both — temperature and kinetic energy in the neutral point — approach to 0, but the energy *very much more rapidly* to Δ (the constant zero-point energy of the attractive forces that finally remains) than the temperature to 0.

The relations (*d*) and (*e*) now become:

$$\frac{\sigma - s'}{l - \sigma} = \sqrt{\frac{f}{\varepsilon}} \quad ; \quad t_1 = \log 2\varphi \, . \sqrt{\frac{m}{2f}} \quad ; \quad \frac{t_2}{t_1} = \frac{{}^1/_2 \pi}{\log 2\varphi} \sqrt{\frac{f}{\varepsilon}},$$

so that for a value of $l - \sigma$ remaining finite, the distance $\sigma - s'$ will not be very *much* smaller than $l - \sigma$, unless again ε is very much larger than f. The time t_1 approaches (logarithmically) to ∞, while at finite $t_2 \left(= \frac{1}{2} \pi \sqrt{\frac{m}{2\varepsilon}}\right)$ the ratio $t_2 : t_1$ will approach logarithmically to 0. These relations are represented by the subjoined figure.

Low Temperatures (u_0 small, φ large).

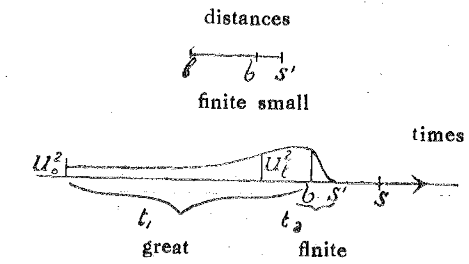


Fig. 2.

As has been said both u_t^2 and u_0^2 approach to 0, and the reason that u_t^2 (i.e. the temperature) does not remain finite at $u_0^2 = 0$ — since there is a *finite* increase of the square of velocity (originally $= 0$) in consequence of the attraction forces — but *likewise* approaches zero, lies in this that the *time* during which this increase takes place, approaches ∞ (though it be logarithmically). In the neutral point the attraction is $= 0$; when the moving point has got somewhat outside the neutral point, there will therefore be only very slowly question of any action of a force (which then increases further linearly with the deviation x, see p. 1188 loc. cit.), hence of acceleration.

3. When we now proceed from u_t^2 to T, and from u_0^2 to $L_0 = E - \Delta$, we have therefore in the case of *high* temperatures from $u_t^2 = {}^1/_2\, u_0^2$:

$$\frac{1}{2}\,N m u_t^2 = \frac{1}{2} \times \frac{1}{2}\,N m u_0^2,$$

i. e. (cf. Note 1)

$$RT = \frac{1}{2} \times \frac{2}{3}\,L_0 = \frac{1}{3}\,L_0 = \frac{1}{3}\,(E - \Delta).$$

Hence also

$$E = \Delta + 3\,RT, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (f)$$

or

$$c_v = \left(\frac{dE}{dT}\right)_v = 3R = 6 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (f')$$

If u_t^2 were $= u_0^2$ instead of $= {}^1/_2\, u_0^2$ (strong collisions, cf. Fig. 1*b*), then E would have become $= \Delta + {}^3/_2 RT$, $c_v = {}^3/_2\, R = 3$.

All this applies to monatomic substances. In the case of multi-atomic (n-atomic) substances it is necessary to take besides the energy of the attractive forces Δ also the atomic energy Δ' within the molecule into consideration, so that E becomes $= L_0 + \Delta + \Delta'$. Now $L_0 = 3RT$, while $\Delta' = 3(n-1)RT$ may be put, when $3(n-1)$ represents the number of supplementary degrees of freedom. We then find $E = \Delta + 3nRT$, i. e. $c_v = 3nR = 6n$ (NEUMANN's law) [1]).

At low temperatures we have:

$$\frac{1}{2}\,Nm\,u_t^2 = \frac{{}^1/_2\,Nm\,u_0^2\,\varphi^2\,.\,\theta}{log\,(4\,\varphi^2 + 2)},$$

according to (c_2), when we denote the factor

$$\left(1 + \frac{1}{2}\,\pi \sqrt{\frac{f}{\varepsilon}}\right) : \left(1 + \frac{{}^1/_2\,\pi}{log\,2\varphi} \sqrt{\frac{f}{\varepsilon}}\right)$$

by θ; hence because $u_0^2\,\varphi^2 = (l-\sigma)^2\,\dfrac{2f}{m}$, and thus ${}^1/_2\,Nmu_0^2\,\varphi^2 = Nf(l-\sigma)^2 = {}^2/_3\Delta$:

$$RT = \frac{{}^2/_3\,\theta\,\Delta}{log\left(4\,\dfrac{\Delta}{E-\Delta} + 2\right)},$$

because $\varphi^2 = \dfrac{(l-\sigma)^2\,\dfrac{2f}{m}}{u_0^2} = \dfrac{Nf\,(l-\sigma)^2}{{}^1/_2\,Nm\,u_0^2} = \dfrac{{}^2/_3\,\Delta}{{}^2/_3\,L_0} = \dfrac{\Delta}{L_0}$.

When we reverse the relation found for RT, omitting $log\,2$ by the side of the so much larger term $log\left(\dfrac{2\Delta}{E-\Delta} + 1\right)$, where $E - \Delta = L_0$ is small compared with Δ, and putting also $\theta = 1$ (which is fulfilled for large values of $\varepsilon : f$), we get:

$$E = \Delta + \frac{2\,\Delta}{e^{\frac{{}^2/_3\Delta}{RT}} - 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (g)$$

As we already remarked in our first paper, it is indeed exceedingly remarkable that (with the exception of a few numerical factors) *exactly the same* relation between E and T appears here as was derived by PLANCK on the ground of the hypothesis of "quanta" drawn up by him. For this it was only required to take into account the *time averages* in the ordinary dynamic relations, which gives rise especially at low temperatures to a *considerable difference* between u_t^2 (the time average of the value of u_0^2, which has greatly increased under the influence of the attractive forces) and u_0^2, both being very slight and approaching to 0.

From (9) follows with ${}^2/_3\Delta : R = \alpha$:

$$E = \frac{3}{2}\,R\alpha + \frac{3\,R\alpha}{e^{\alpha/T} - 1} \quad ; \quad c_v = \left(\frac{dE}{dT}\right)_v = 3R\,\frac{(\alpha/T)^2\,e^{\alpha/T}}{(e^{\alpha/T} - 1)^2}, \quad . \quad (g')$$

which exponentially approaches to 0 $\left(\text{viz. to } 3R\,\dfrac{\alpha^2}{T^2}\,e^{-\alpha/T}\right)$, when T approaches to 0.

There is, however, one great difference with PLANCK's formula. Apart from this that in PLANCK's work the well-known quantity $Nh\nu$ appears instead of ${}^2/_3\Delta = Nf\,(l-\sigma)^2$, so that $h\nu$ would have to be $h\nu = f\,(l-\sigma)^2$ [1]), our formula (g) is *only* valid for very low temperatures, and (f) *only* for very high temperatures. This is of course only owing to this that (c_1) only ensues from the general formula (c), when φ is supposed to be small, whereas with large values of φ the relation (c_2) results from it. Accordingly our (g) may,

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[1]) It should be remembered that for gases $E = \Delta + {}^1/_2 R\,(3 + \mu)\,T$ may be put, in which μ also represents the number of supplementary degrees of freedom (see among others BOLTZMANN, Gastheorie II, p. 124—125 and 128). But here μ is simply $= n$ for multi-atomic molecules, so that for mon-atomic molecules n is still $= 0$, for di-atomic molecules however $n = 2$, for tri-atomic ones $n = 3$, etc. Hence when the term Δ, which approaches 0, is neglected, and also the quantity ε introduced by BOLTZMANN, referring to the potential energy of the intramolecular movements, E becomes $= {}^1/_2 RT\,(3 + n)$ for gases, leading to $c_v = {}^1/_2 R\,(3 + n)$, hence (with $c_p - c_v = R$) to $\dfrac{c_p}{c_v} = 1 + \dfrac{2}{3 + n}$. (BOLTZMANN adds the above mentioned correction quantity ε to $3 + n$).

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[1]) Cf. what has been said concerning $l - \sigma$ under *b*) of Note 2.

therefore, not be applied in case of high temperatures, whereas this *may* be done with PLANCK's formula: the latter holds (at small volumes) both for low and for high temperatures.

It is, however, remarkable that *if* (g) were valid for high temperatures (which is not the case according to us), $E = \Delta + 3RT$ is duly obtained as limiting value for E, identical to (f). Our formula, from which (f) ensues for high temperatures and (g) for low temperatures, seems to be more general, and the approach to (f) takes place in a somewhat different way than with PLANCK's formula.

At any rate it will have to be assumed — if φ is to be small at high temperature, and large at low temperature, and if Δ is not to become $= 0$ at $T = 0$ — that with condensed (solid) systems $(l - \sigma)^2$ changes only comparatively slightly; and that it does so in the same degree as the frequency ν. Then PLANCK's quantity h would be related in a definite way with the constant of the attractive forces f (being in its turn again in relation to e^2, when e represents the electric elementary quantum), and in consequence of this also with $^{a_0}/_{v_0}$ at the absolute zero. There are very strong indications for this: particularly the undeniable connection between the so-called *chemical* constant and also the constant of the *vapour-pressure* on one side, and the quantity $^{a_0}/_{v_0{}^2}$ on the other side, as I demonstrated shortly ago in a Paper in the Recueil des Tr. Ch. of March and May 1920 — while it is known that this chemical constant in its turn is again in relation with h.

I hope to return to this special subject later on.

4. We will now discuss somewhat more fully the nature and the way of acting of the forces assumed by us between the molecules.

In connection with what was already observed above, we might assume that the attractive action of M_1 e.g. rapidly decreases at a certain small distance from M_1, and disappears at a certain very small distance σ, being replaced by a rapidly increasing repulsive force, which for $x = s$, when the moving molecule P would touch the molecule M_1, would become infinitely great. (Cf. further what was already said on this head under a) of Note 2).

Thus no two separate forces are required, nor two separate Virial-parts — an Attractive-Virial part and a Repulsive-Virial part — but only one; which point of view was already set forth by me some twenty years ago.[1] The difference with the assumption in the first part of this paper lies, therefore, chiefly in this, that then the attractive force continued to increase up to σ, after which it suddenly (hence discontinuously) changed into a repulsive force, with another constant of intensity ε than that of the attractive force f; whereas now we suppose a *continuous* change of force at σ with a single constant f.

Total force F.

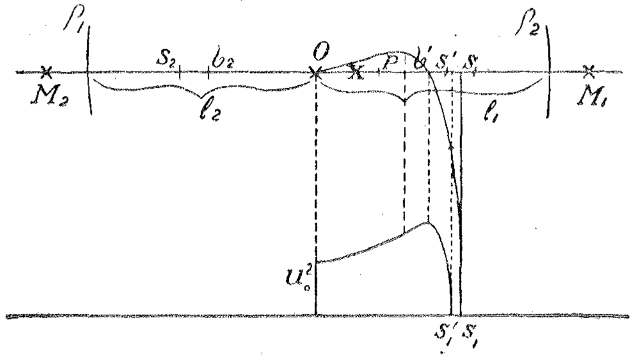

Fig. 3.

Analytically this may be expressed — as far as e.g. the action of a force, exerted on P by M_1 is concerned — by a formula of the form

$$F_1 = f(\varrho_1 - l_1 + x)\,\frac{l_1 - \sigma_1 - x}{l_1 - s_1 - x}, \quad \ldots \ldots \quad (1a)$$

in which $x = OP$, and the indices 1 all refer to distances from M_1, measured towards the left; and this instead of simply $F_1 = f(\varrho_1 - (l_1 - x_1))$ as we put formerly (loc. cit. p. 1188 et seq.), i.e. the attractive force proportional to the distance from the moving point P to the boundary of the sphere of attraction ϱ_1 (of M_1), so that $F_1 = 0$, when P lies on the boundary of this sphere or outside it. In consequence of this the attraction, after having reached a maximum, again becomes $= 0$ at σ_1 (hence $(x = l_1 - \sigma_1)$, reverses its sign, and again changes into a repulsion, which would become infinite at s_1 $(x = l_1 - s_1)$. From the other molecule M_2 P is subjected to an attraction

$$F_2 = f(\varrho_2 - l_2 - x)\,\frac{l_2 - \sigma_2 + x}{l_2 - s_2 + x}, \quad \ldots \ldots \quad (1b)$$

in which x is again $= OP$, and the indices now refer to distances from M_2, measured to the right.

Hence after some reductions the *total* action exerted on P to the right (see Fig. 3), (with omission of the indices, because $l_1 = l_2 = l$, etc.) is now found to be:

$$F = F_1 - F_2 = f\,.\,2x\left[1 - \frac{(\varrho - s)(\sigma - s)}{(l - s)^2 - x^2}\right], \quad \ldots \quad (2)$$

instead of simply $F = F - F_2 = f\,.\,2x$, as before (p. 1188).

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[1] See Arch. Teyler (2) T. VII, 3ième Partie, p. 1—34 (1901): „Sur l'influence des corrections etc." (particularly p. 28 et seq.).

It is self-evident that the *total* attractive force will become $= 0$ somewhat earlier, when P moves towards M_1, than at σ_1 (when F_1 acts *alone*) — it does so at σ'_1 or simply σ' — because the attractive force of M_2 acts in opposite sense. In fact the above quantity becomes $= 0$, when

$$x_{\sigma'} = \sqrt{(l-s)^2 - (\varrho-s)(\sigma-s)} = \sqrt{(l-\sigma)^2 - (\varrho+\sigma-2l)(\sigma-s)} = l-\sigma'.$$

As in the case under consideration the molecule P will always be within the spheres of attraction of the two molecules M_1 and M_2, $2l$ is always $< \varrho + s$, hence a fortiori $2l < \varrho + \sigma$ (cf. p. 1187 l.c.). The value of $x_{\sigma'}$ is therefore $< l - \sigma$, i.e. F becomes $= 0$ in σ', on the left side of σ_1 [1].

However, all such functions have the drawback, that the further integrations become impossible to carry out by means of closed forms; for both at high and at low temperatures (u_0 large or small) the term of work ${}^2/_m\int$ in $\sqrt{u^2_0 + \frac{2}{m}\int\limits_0^x F\,dx}$ can never be considered as permanently small with respect to u_0^2 between the limits $x = 0$ and $x = s'$. For in the end (at the culmination point of the collision) the quantity under the sign of the root becomes $= 0$ in both cases (high and low temperature), hence the term of work under consideration of the same order of magnitude as u_0^2. And at low temperatures, which is justly the most important case in our considerations, that term is almost everywhere of the order of magnitude u_0^2 — except in the neighbourhood of the points O and somewhere between σ' and s', where this term becomes $= 0$ (Cf. Fig. 3).

For this reason we were obliged in our first paper to consider the attraction and the repulsion *separately*, and to assume, instead of the course of F drawn in Fig. 3, a force which continues to increase in direct ratio to x as far as σ_1, after which it suddenly changes into a repulsive force, which likewise increases linearly as the distance from P to σ_1 (P now thought on the righthand side of σ_1). This renders the integrations easy to carry out, and does not touch the nature of the matter.

If it is thought desirable to avoid the introduction of a so-called "sphere of attraction" — which at the same time offers the advantage that the assumption of a transition case (see p. 1186) becomes unnecessary (viz. the case in which the moving molecule is always within the sphere of attraction of M_1, but not always within that of M_2) — a plausible law of attraction must be substituted for $\varrho_1 - l_1 + x$ and $\varrho_2 - l_2 - x$ in the expressions for F_1 and F_2, so that we have e.g.:

$$F_1 = \frac{f}{(l-x)^n}\,\frac{l-\sigma-x}{ls-x} \quad ; \quad F_2 = \frac{f}{(l+x)^n}\,\frac{l-\sigma+x}{l-s+x},$$

through which we obtain with *small* values of x:

$$F = \frac{f}{l^n}\left(1+n\frac{x}{l}\right)\frac{l-\sigma}{l-s}\left(1-\frac{\sigma-s}{(l-\sigma)(l-s)}\,x\right) -$$

$$- \frac{f}{l^n}\left(1-n\frac{l}{x}\right)\frac{l-\sigma}{l-s}\left(1+\frac{\sigma-s}{(l-\sigma)(l-s)}\,x\right),$$

i.e.

$$F = \frac{f}{l^n}\frac{l-\sigma}{l-s}\left[\frac{n}{l} - \frac{\sigma-s}{(l-\sigma)(l-s)}\right]2x = \frac{f}{l^{n+1}}\,\frac{l-\sigma}{l-s}\left[n - \frac{l(\sigma-s)}{(l-\sigma)(l-s)}\right]2x,$$

i.e. again proportional to x. This first proportionality and the corresponding quadratic form of the term of work $\int F dx$ continues to exist whatever form is given to the expressions of the action of the force.

According to DEBYE [1]) the exponent n would have the value 9 (for anomalous "Dipol" gases n would be $= 7$ on the other hand; cf. the note on p. 183 loc. cit.).

But also the above forms of F_1 and F_2 are in a still greater degree subject to the drawback, that they lead to integrations which it is impossible to carry out in the further calculations.

5. However — without having recourse to the dualistic law of force (one for the attraction and another for the repulsion) which we have chosen for practical reasons — also (2) might be used for the calculation of

$$t = \int\limits_0^{l-s'} \frac{dx}{\sqrt{u_0^2 + \omega}} \quad ; \quad u_t^2 = \frac{1}{t}\int\limits_0^{l-s'} \sqrt{u_0^2 + \omega_x}\, dx, \quad . \quad . \quad . \quad (3)$$

in which $\omega_x = \frac{2}{m}\int\limits_0^x F dx$; provided one is satisfied with a certain approximation in the logarithmic expression which is then obtained for ω_x. We find namely:

$$\omega_x = \frac{2f}{m}\left[x^2 + (\varrho-s)(\sigma-s)\log\frac{(l-s)^2 - x^2}{(l-s)^2}\right], \quad . \quad . \quad . \quad (4)$$

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[1] When the distance l of the molecules becomes *too* small, F will not first become positive on the righthand side of O, become $= 0$ in σ', and then negative — but at once negative, i.e. there is already immediately repulsion on the righthand side of O. The same thing applies of course to the left side. We shall return to all these different cases in our next paper on the calculation of the Virial.

[1] DEBYE, Die v. D. WAALS'schen Kohäsionskräfte, Physik. Zeitschr. **21**, p. 178—187 (1920).

in which x changes in the above integrations from $l-s'$ to 0. And as s' is always $> s$ (only for an infinitely large value of u_0 could s be reached at the culmination of a collision), $l-s'$ is always $< l-s$. Especially at lower temperatures, at which s' remains comparatively far from s, $(l-s'):(l-s)$ can remain considerably smaller than 1 even at the extreme value of x. (If e.g. $l=1{,}2\,s$, $s'=1{,}1\,s$, this ratio becomes already $^1/_2$). We may therefore write in approximation for the logarithmic term:

$$log\left(1-\frac{x^2}{(l-s)^2}\right)=-\frac{x^2}{(l-s)^2}-\frac{x^4}{2\,(l-s)^4},$$

so that with

$$\frac{(\varrho-s)\,(\sigma-s)}{(l-s)^2}=\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

the following form is obtained for ω_x:

$$\omega_x=\frac{2f}{m}(l-s)^2\left[(1-\alpha)\frac{x^2}{(l-s)^2}-\frac{\alpha}{2}\frac{x^4}{(l-s)^4}\right]. \quad . \quad . \quad . \quad (4^a)$$

For the form under the sign of the root may therefore be written:

$$u_0{}^2\left[1+\frac{2f}{m}\frac{(l-s)^2}{u_0{}^2}\left\{(1-\alpha)\,y^2-{}^1/_2\,\alpha\,y^2\right\}\right]=u_0{}^2\left[1+\varphi^2\{(1-\alpha)\,y^2-{}^1/_2\alpha y^4\}\right],$$

when again, as in the first paper, $\dfrac{l-s}{u_0}\sqrt{\dfrac{2f}{m}}=\varphi$ is put, and further y is substituted for $\dfrac{x}{l-s}$. Then:

$$t=\frac{l-s}{u_0}\int\limits_0^{\frac{l-s'}{l-s}}\frac{dy}{\sqrt{1+\varphi^2\{(1-\alpha)\,y^2-{}^1/_2\,\alpha\,y^4\}}} \quad ; \quad u^2=(l-s)\frac{u_0}{t}\int\limits_0^{\frac{l-s'}{l-s}}\sqrt{ibid.}\,dy.$$

For the form under the sign of the root $(1-w_1y^2)\,(1+w_2y^2)$ may be written in this, when

$$w_2-w_1=\varphi^2\,(1-\alpha) \quad ; \quad w_1\,w_2={}^1/_2\,\varphi^2\,\alpha,$$

so that this form with $y\sqrt{w_1}=z$ passes into $(1-z^2)\left(\quad+\dfrac{w_2}{w_1}\,z^2\right)$, which becomes with $z=cos\,\psi$:

$$sin^2\,\psi\left(1+\frac{w_2}{w_1}(1-sin^2\,\psi)\right)=\frac{w_1+w_2}{w_1}\,sin^2\,\psi\left(1-\frac{w_2}{w_1+w_2}\,sin^2\,\psi\right).$$

For dy we have further $\dfrac{dz}{\sqrt{w_1}}=-\dfrac{1}{\sqrt{w_1}}\,sin\,\psi\,d\psi$, so that the above integrals pass into

$$t=-\frac{l-s}{u_0\sqrt{w_1}}\sqrt{\frac{w_1}{w_1+w_2}}\int\limits_{^1/_2\pi}^{0}\frac{d\psi}{\sqrt{1-k^2\,sin^2\,\psi}};$$

$$u_t{}^2=-\frac{l-s}{\sqrt{w_1}}\sqrt{\frac{w_1}{w_1+w_2}}\,\frac{u_0}{t}\int\limits_{^1/_2\pi}^{0}sin^2\,\psi\sqrt{1-k^2\,sin^2\,\psi}\,d\psi,$$

when $\dfrac{w_2}{w_1+w_2}=k^2$ is put (hence k^2 is always <1). With regard to the limits of the integrals evidently z is also $=0$ for $y=0$, hence $\psi={}^1/_2\pi$. And as at the upper limit $u_0{}^2+\omega_x$ becomes $=0$ (culmination point of the collision), also $(1-z^2)\left(1+\dfrac{w_2}{w_1}z^2\right)=0$, hence $z=1$, $\psi=0$. Thus we have, after reversal of the limits, in consequence of which the minus signs drop out:

$$t=\frac{l-s}{u_0}\,\frac{1}{\sqrt{w_1+w_2}}\int\limits_0^{^1/_2\pi}\frac{d\psi}{\Delta\psi} \quad ; \qquad u_t{}^2=u_0{}^2\,\frac{w_1+w_2}{w_1}\,\frac{\int\limits_0^{^1/_2\pi}sin^2\,\psi\,\Delta\,\psi\,.\,d\psi}{\int\limits_0^{^1/_2\pi}\dfrac{d\psi}{\Delta\psi}},$$

when for t its value is substituted in the expression for $u_t{}^2$. Following LEGENDRE and denoting the complete elliptical integral of the 1st kind, viz. $\int\limits_0^{^1/_2\pi}\dfrac{d\psi}{\Delta\psi}$, by F_1 or singly F, we have also:

$$t=\frac{\varphi}{\sqrt{(w_1+w_2)}}\sqrt{\frac{m}{2f}}\,.\,F \quad ; \quad u_t{}^2=\frac{w_1+w_2}{w_1}\,\frac{u_0{}^2}{F}\int\limits_0^{^1/_2\pi}sin^2\,\psi\,\Delta\psi.d\psi,$$

when $\varphi\sqrt{\dfrac{m}{2f}}$ is substituted for $\dfrac{l-s}{u_0}$ (see above). We have for the modulus k:

$$k^2=\frac{w_2}{w_1+w_2}=\frac{{}^1/_2\,\varphi\left[\varphi(1-\alpha)+\sqrt{\varphi^2(1-\alpha)^2+2\alpha}\right]}{\varphi\sqrt{\varphi^2(1-\alpha)^2+2\alpha}},$$

when we calculate the quantities w_2 and w_1+w_2 from the above expressions for w_2-w_1 and w_1w_2. Hence we may write:

$$k^2=\frac{1}{2}\left[1\pm\frac{1}{\sqrt{1\pm\dfrac{2\alpha}{\varphi^2\,(1-\alpha)^2}}}\right], \quad . \quad . \quad . \quad . \quad (6)$$

so that at low temperatures (large values of φ) k^2 is always near 1 (provided $\alpha<1$, in which case the $+$ sign is valid).

We must now still reduce the last elliptical integral (the one with

$sin^2\psi$) to that of the 1st and the 2nd kind. According to known formulae of reduction[1]) we have:

$$\int_0^{\psi} sin^2\psi\,\Delta\psi\,.\,d\psi = \frac{1}{3}\left[\frac{1-k^2}{k^2}\int_0^{\psi}\frac{d\psi}{\Delta\psi} + \frac{2k^2-1}{k^2}\int_0^{\psi}\Delta\psi.d\psi - sin\,\psi\,cos\,\psi\,\Delta\psi\right],$$

hence

$$\int_0^{1/2\pi} sin^2\psi\,\Delta\psi.d\psi = \frac{1}{3}\left[\frac{1-k^2}{k^2}F + \frac{2k^2-1}{k^2}E\right],$$

when the complete elliptical integral of the second kind, viz. $\int_0^{1/2\pi}\Delta\psi.d\psi$, is represented by E. Hence we find finally:

$$t = \sqrt{\frac{2k^2-1}{1-\alpha}}\sqrt{\frac{m}{2f}}\,F\ ;\qquad u_t^2 = \frac{u_0^2}{3k^2}\left[1+\frac{2k^2-1}{1-k^2}\frac{E}{F}\right], \quad . \quad (7)$$

because

$$\frac{\varphi}{\sqrt{w_1+w_2}} = \frac{\sqrt{\varphi}}{\sqrt[4]{\varphi^2(1-\alpha)^2+2\alpha}} = \sqrt{\frac{2k^2-1}{1-\alpha}}, \text{ and } \frac{w_1}{w_1+w_2} = 1-k^2.$$

We shall now compare the found formulae (7) with those found before, and again in the two limiting cases: high and low temperatures.

At *high* temperatures (φ small) k^2 approaches $^1/_2$ (if $\alpha < 1$), so that then (7) reduces to

$$t = 0 \quad ; \quad u_t^2 = {}^2/_3\,u_0^2\,(high\ temp.), \quad . \quad . \quad . \quad . \quad (7^a)$$

instead of $u_t^2 = {}^1/_2 u_0^2$ (weak collisions), as we found before. The

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[1]) See among others DURÈGE, Th. der ellipt. Funct., p. 65, formule (29), i. e. (with $\mu = 0, m = 0$)

$$sin\,\psi\,cos\,\psi\,\Delta\psi = \int_0^{\psi}\frac{d\psi}{\Delta\psi} - 2\,(1-k^2)\int_0^{\psi}\frac{sin^2\psi\,d\psi}{\Delta\psi} + 3\,k^2\int_0^{\psi}\frac{sin^4\psi\,d\psi}{\Delta\psi},$$

from which the integral with $sin^4\psi$ can be expressed in both the others, that with $sin^2\psi$ being expressed in F_ψ and E_ψ by the formula (see p. 69)

$$k^2\int_0^{\psi}\frac{sin^2\psi\,d\psi}{\Delta\psi} = F_\psi - E_\psi,$$

as can be easily verified by differentiation, after $\Delta\psi = \sqrt{1-k^2 sin^2\psi}$ has been put everywhere in the denominator. $\Bigg($The integral to be reduced by us then becomes

$$\int_0^{\psi}\frac{sin^2\psi - k^2 sin^4\psi}{\Delta\psi}d\psi\Bigg).$$

fall of the velocity during the action of the repulsive force — expressed as function of the time — is now less great than in fig. 1^a, so that the descending branch will be much more horizontal. The natural consequence of this is, that the time average gets much nearer to u_0^2. (c_v would now become $= {}^3/_2 \times {}^3/_2\,R = 4^1/_2$ instead of 6). However — the above calculation is certainly questionable at high temperatures, because then $log\left(1-\frac{x^2}{(l-s)^2}\right)$ may certainly not be expanded into a series, as at the culmination point of the impact x would become $= l-s\ (s' = s)$. The expansion into a series up to x^4 used by us, gives a too great value for ω_x, hence also a too great value for u_t^2. Instead of rather abruptly, the damping of this exceedingly great velocity would take place during a much too long interval — so great even that s' would lie far inside s, which is of course impossible.

At low temperatures (φ great) on the other hand there can be no objection to applying the expansion into a series up to x^4, because then the velocity is so small that it will be reduced to 0 already within a very short interval. Now the modulus k approaches to 1, hence E to $\int_0^{1/2\pi} cos\,\psi\,d\psi = (sin\,\psi)_0^{1/2\pi}$, i.e. also to 1; but F will approach to

$$\int_0^{1/2\pi}\frac{d\psi}{cos\,\psi} = log\,tg\,(45 + \tfrac{1}{2}\psi)_0^{1/2\pi} = log\,\infty - log\,1, \text{ i.e. to } log\,\infty.$$

As, however, at the same time $1-k^2$ approaches 0, we must examine what value $(1-k^2)\,F$ assumes in (7), when k^2 is near 1.

According to a well-known theorem[1]) F approaches to $log\frac{4}{\sqrt{1-k^2}}$ in this case.

Hence we get for t and u_t^2, when k approaches 1, from (7):

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[1]) Cf. among others LAMB, Treatise on Hydrodynamics, p. 170; CAYLEY, Ellipt. funct., Art. 72; MAXWELL, Elect. and Magn. II, p. 311—316; DURÈGE, p. 190 et seq., particularly p. 213; KIRCHHOFF, Vorl. p. 270; etc. Better than DURÈGE's derivation, which is based on LANDEN's transformation, is KIRCHHOFF's beautiful derivation. The latter is founded on the splitting up of the integral into two parts, viz. $\int_0^{1/2\pi} = \int_0^{1/2\pi-\theta} + \int_{1/2\pi-\theta}^{1/2\pi}$, in which θ is a small quantity, which is, however, supposed to be great with respect to $\sqrt{1-k^2}$. But in both derivations only the *limiting* value of F is reached.

It is in my opinion a better method to start from JACOBI's relation

$$t=\frac{1}{\sqrt{1-\alpha}}\log\frac{4}{\sqrt{1-k^2}}\sqrt{\frac{m}{2f}}\quad;\quad u_t^{\,2}=\frac{u_0^{\,2}}{3\,(1-k^2)\,\log{}^4/\sqrt{1-k^2}},$$

because E approaches 1, and 1 may be omitted by the side of $\frac{2k^2-1}{1-k^2}\,\frac{E}{F}=\frac{1}{(1-k^2)F}$. Now from (6) follows for large values of $\varphi\ (\alpha<1)$:

$$k^2=\frac{1}{2}\left[1+\left(1-\frac{\alpha}{\varphi^2\,(1-\alpha)^2}\right)\right]=1-\frac{{}^1/_2\,\alpha}{\varphi^2\,(1-\alpha)^2},$$

so that we obtain with $\log\frac{4}{\sqrt{1-k^2}}=\frac{1}{2}\log\frac{16}{1-k^2}$:

$$\left.\begin{aligned}t&=\frac{1}{2\sqrt{(1-\alpha)}}\log\left(\frac{32\,(1-\alpha)^2}{\alpha}\,\varphi^2\right)\sqrt{\frac{m}{2f}}\\ u_t^{\,2}&=\frac{\frac{4}{3}\,\frac{(1-\alpha)^2}{\alpha}\,\varphi^2}{\log\left(\frac{32\,(1-\alpha)^2}{\alpha}\varphi^2\right)}\,u_0^{\,2}\end{aligned}\right\}.\ (\textit{low temp.})\ .\ \ .\quad(7^b)$$

$F=-\frac{\log q'}{\pi}F'$, in which F' refers to the integral with the complementary modulus $k'=\sqrt{1-k^2}$, and q' is one of the auxiliary quantities q and q', introduced by JACOBI. From the relation $\sqrt{k'}=\frac{2\sqrt[4]{q'}(1+q'^2+q'^6+\textit{etc.})}{1+2q'+2q'^4+2q'^9+\textit{etc.}}$ follows first of all $\frac{1}{16}k'^2=q'\left(\frac{1+q'^2+q'^6+\dots}{1+2q'+2q'^4+\dots}\right)^4$, from which $q'=\frac{1}{16}k'^2\left(1+\frac{1}{2}k'^2+\frac{21}{64}k'^4+\dots\right)$. And from this follows:

$$-\frac{1}{2}\log q'=\log\frac{4}{k'}-\left(\frac{1}{4}k'^2+\frac{13}{128}k'^4+\dots\right).$$

Through expansion into series $F'=\frac{1}{2}\pi\left(1+\frac{1}{4}k'^2+\frac{9}{64}k'^4+\dots\right)$ is easily derived for $F'=\int_0^{{}^1/_2\pi}\frac{d\psi}{\sqrt{1-k'^2\sin^2\psi}}$, so that from $F=\frac{F'}{{}^1/_2\pi}\left(-\frac{1}{2}\log q'\right)$ finally ensues $F=\left(1+\frac{1}{4}k'^2+\frac{9}{64}k'^4\right)\left[\log\frac{4}{k'}-\left(\frac{1}{4}k'^2+\frac{13}{128}k'^4+.\right)\right]$, or approximated $F=\left(1+\frac{1}{4}k'^2\right)\left(\log\frac{4}{k'}-\frac{1}{4}k'^2\right)$, the limiting value of which is evidently $\log\frac{4}{k'}$.

We may still point out that the auxiliary quantity q always remains very small. q' is $=0$ for $k'=0$, but q' is only 0,043 for $k'={}^1/_2\sqrt{2}$ (the same for k and q). It is to this fact that the exceedingly strong convergence of the JACOBI series for elliptical functions is owing.

Again t approaches logarithmically to ∞ and $u_t^{\,2}$ to $\frac{A\,\varphi^2}{\log B\,\varphi^2}u_0^{\,2}$, just as in ($c_2$), derived in § 3 with two separate forces.

In order to render a comparison possible, we must now again introduce the maximum work Δ performed by the attractive forces. From (4^a), i.e.

$$\omega_x=\frac{2f}{m}(l-s)^2\left[(1-\alpha)\frac{x^2}{(l-s)^2}-\frac{\alpha}{2}\,\frac{x^4}{(l-s)^4}\right],$$

follows that this will be maximum, when $\frac{x^2}{(l-s)^2}=\frac{1-\alpha}{\alpha}$, so that

$$(\omega_x)_{max}=\frac{2f}{m}(l-s)^2\,\frac{(1-\alpha)^2}{2\alpha}.$$

Multiplied by ${}^1/_2\,Nm$, we get accordingly (Cf. Note 1)

$$Nf\,(l-s)^2\,\frac{(1-\alpha)^2}{2\alpha}=\frac{2}{3}\Delta.$$

As further $\varphi^2=\frac{(l-s)^2}{u_0^{\,2}}\,\frac{2f}{m}$, we get:

$${}^1/_2\,Nm\,.\,\frac{32\,(1-\alpha)^2}{\alpha}\varphi^2\,u_0^{\,2}=Nf\,(l-s)^2\,.\,\frac{32\,(1-\alpha)^2}{\alpha}=64\times\frac{2}{3}\Delta\,.$$

Thus we find for ${}^1/_2\,Nm\,u_t^{\,2}=RT$, because ${}^1/_2\,Nm\,u_0^{\,2}={}^2/_3\,L_0=$ $={}^2/_3\,(E-\Delta)$:

$$RT=\frac{{}^1/_{24}\times64\times{}^2/_3\,\Delta}{\log\frac{64\times{}^2/_3\,\Delta}{{}^2/_3\,L_0}}=\frac{{}^{16}/_9\,\Delta}{\log\frac{64\,\Delta}{E-\Delta}},\ .\ \ .\ \ .\ \ .\quad(8)$$

as against $RT=\frac{{}^2/_3\,\theta\Delta}{\log\frac{4\Delta}{E-\Delta}}$ on the former assumption of two separate forces (see Note 3). The coefficients are different, but the logarithmic relation has remained *entirely the same.*

As however our former assumption leads to better coefficients than the assumption (2) of Note 4, elaborated by us in this paper, and as it does so both for high and for low temperatures, we can in future, by the side of the latter procedure, also base ourselves on the supposition — which is *simpler for the calculations* — of *two separate* forces, in which the repulsive force begins to act at $x=l-\sigma$, after the attractive force has reached its highest point.

I hope that the foregoing Notes go to clear up some of the difficulties that might have presented themselves in the reading of my first paper.[1]) Now we will proceed with the task which we had set ourselves, and examine the Virial of attraction and of repulsion with a view to the drawing up of the equation of state in case of small volumes, especially at very low temperatures.

(To be concluded).

La Tour près Vevry, Autumn 1920.

I gratefully express my thanks again to the VAN 'T HOFF-fund, which has greatly facilitated the execution of this work.

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[1]) Though not *all* the objections, which some time ago Prof. LORENTZ was so kind as to communicate to me in a letter, may have been removed by this paper, yet I hope that some have been solved. Nobody can be more fully conscious of the great difficulties that are to be surmounted here, than the author himself.

In the autumn of 1919 I had the privilege of having a discussion with Prof. EHRENFEST — to whom we owe the so important theory of the *Adiabatic Invariants* (1916), which theory was later so felicitously continued by BURGERS (1917) and KRUTKOW (1919—1920) — on the contents of my first paper. He advanced, among others, the objection that not *all* the molecules on approach to other molecules would come in collision, after which they would again move away from them, but that some of them would remain for a time in the neighbourhood of them. This is perfectly true, but in case of *gases* we should then have to do with *association*, a case that was purposely left out of consideration by me as an unnecessary complication. But as we have to deal here not with gases, but chiefly with *solid* bodies, where the molecules only move to and fro between the neighbouring ones, this complication is, indeed, quite excluded. Besides already for a long time all the more recent theories of structure have rejected the idea of association in the solid, crystallized state, so that we are justified in leaving it quite out of consideration. But in my opinion there are greater difficulties, of an *entirely different* nature, to which I hope to return later.

Physiology. — *"The function of the Otolithes".* By Prof. R. MAGNUS and A. DE KLEYN.

(Communicated at the meeting of Sept. 25, 1920).

In the course of the last half century an infinite amount of literature has appeared upon the functions of the vestibular organ. From the first, anatomical research rendered it probable that a sharp distinction had to be made between the sensory epithelium in the cristae of the semi-circular canals covered by the so-called cupula and which can move freely in the endolymph, and between the sensory epithelium of the maculae of sacculus and utriculus which, covered by the otolithes with their greater specific weight, appears specially suited to react upon the greater or lesser pressure of the said otolithes.

But, whereas our knowledge of the function of the semi-circular canals is fairly extensive and, moreover, the anatomical data agree fairly well with the clinical and experimental data, this is by no means the case with respect to the knowledge and theories regarding the function of the otolithes.

This is due to various causes. If, as has been supposed in particular by MACH and BREUER, the function of the otolithes is determined by the greater or lesser pressure upon the sensory epithelium beneath, we may expect that the otolithes will have some influence upon those reflexes which come into play by changes in the position of the head, and that their influence will continue as long as this position remains unchanged and the pressure of the otolithes is constant. Or, in other words, that the otolithic reflexes are more particularly tonic reflexes. Up till a few years ago, tonic reflexes of the labyrinth were known only in the form of compensatory positions of the eyeball, and therefore the function of the otolithes had to be studied exclusively from these reflexes. As, however, our knowledge of these positions was still far from complete, and sufficient investigations had not been made, it is obvious that the literature on this subject will contain opinions of a more or less speculative nature. Moreover, the influence of the clinique was inhibitory. While, clinically, the different vestibular reactions upon (rotatory) movements were investigated more and more carefully and began to assume an ever-increasing importance in the diagnostics of the diseases of the labyrinth,