Physics. — *"On geodesic precession."* By Prof. J. A. SCHOUTEN. (Communicated by Prof. H. A. LORENTZ).

(Communicated at the meeting of February 26, 1921).

In a preceeding communication I [1]) have demonstrated geometrically, that a system of axes, moved geodesically along a closed curve in a non-euclidean V_3, will show a deviation when returned to its starting point. For the special case that the linear element of the V_3 is the spacial part of the linear element of SCHWARZSCHILD and that the curve is a circle round the sun with a radius equal to the mean radius of the orbit of the earth, this deviation is 0.013″ after one revolution.

Now if firstly the fourdimensional problem of the motion of a material point in a static gravitational field, neglecting as usual quantities of order $\frac{\alpha^2}{R^2}$, could be reduced to a problem of classical mechanics (mechanics with the fundamental theorem: force = mass $\times$ geodesic acceleration) in a threedimensional non-euclidean space, and if secondly we could demonstrate that a geodesically moving system of axes may be regarded in first approximation as an inertial-system, than we might conclude for the earth to a deviation of the ordinary precession to the amount of 0.013″.

In the mean time FOKKER [2]), starting with the complete linear element of SCHWARZSCHILD, has demonstrated with a fourdimensional calculation, that, apart from other relativity-corrections on the ordinary precession, a geodesic precession exists, that is exactly $1^1/_2 \times 0.013''$.

Now we can show that this difference is caused by the fact, that the fourdimensional problem can be reduced then and only then to a threedimensional one, when the square of the velocity is of order $\frac{\alpha^2}{R^2}$, the square of the real occuring velocity in general being of order $\frac{\alpha}{R}$.

The world-line of a material point is given by the equation:

$$\delta \int ds = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

[1]) Proc Kon. Akad., XXI 1918, p. 533—539.
[2]) Proc. Kon. Akad. XXIII, 1921, p. 729.

Now if ds^2 has the form

$$ds^2 = \left(1 - \frac{\beta}{r}\right) dt^2 - dl^2 = \left(1 - \frac{\beta}{r}\right) dt^2 - \left(1 + \frac{\alpha}{r}\right) dr^2 - r^2 \sin^2\theta \, d\varphi^2 - r^2 d\theta^2, \quad (2)$$

than (1) can be replaced by

$$\left.\begin{aligned} 0 &= \delta \int_{t_2}^{t_1} \sqrt{\left\{\left(1 - \frac{\beta}{r}\right) - \left(\frac{dl}{dt}\right)^2\right\}}\, dt = \\ &= \delta \int_{t_2}^{t_1} \left\{1 - \frac{\beta}{2r} - {}^1/_2 \left(\frac{dl}{dt}\right)^2 - \frac{1}{8}\frac{\beta^2}{r^2} - \frac{1}{4}\frac{\beta}{r}\left(\frac{dl}{dt}\right)^2 - \frac{1}{8}\left(\frac{dl}{dt}\right)^4\right\} dt = \\ &= -\delta \int_{t_2}^{t_1} \left\{\frac{\beta}{2r} + \frac{1}{8}\frac{\beta^2}{r^2} + {}^1/_2\left(\frac{dl}{dt}\right)^2 + \frac{1}{4}\frac{\beta}{r}\left(\frac{dl}{dt}\right)^2 + \frac{1}{8}\left(\frac{dl}{dt}\right)^4\right\} dt. \end{aligned}\right\} \quad . \quad (3)$$

In this equation the second term and the two last terms only then can be neglected with respect to the other terms, when $\frac{\beta}{r}$ and consequently $\left(\frac{dl}{dt}\right)^2$ is of order $\frac{\alpha^2}{r^2}$. Then the equation changes into

$$\delta \int_{t_2}^{t_1} \left\{\frac{\beta}{2r} + {}^1/_2 \left(\frac{dl}{dt}\right)^2\right\} dt = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

But this is the equation of classical mechanics in a threedimensional space with the linear element dl and a potential function $U = \frac{\beta}{2r}$. (4) is equivalent to

$$\frac{\partial U}{\partial x^\mu} = g_{\varkappa\mu} \frac{d^2 x^\varkappa}{dt^2} + \begin{bmatrix} \lambda \nu \\ \mu \end{bmatrix} \frac{dx^\lambda}{dt} \frac{dx}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If $\frac{\beta}{2r}$ and consequently $\left(\frac{dl}{dt}\right)^2$ is of order $\frac{\alpha}{r}$, which in particular holds for the linear element of SCHWARZSCHILD, for which $\beta = \alpha$, the reduction to a threedimensional problem is not possible, at least not in this way [1]).

Now we will demonstrate, that in the threedimensional problem

[1]) Hence the equations derived Proc. Kon. Akad. XXI 1918, 1176—1183 on p. 1178—1180 hold only for velocities of order $\frac{\alpha}{r}$.

a geodesically moving system of axes is under certain conditions an inertialsystem. Therefore we firstly write out the equations (4) for the linear element dl.

Since

$$\begin{aligned}
&\begin{bmatrix} rr \\ r \end{bmatrix} = -\tfrac{1}{2}\frac{\alpha}{r^2}, \quad \begin{bmatrix} \varphi\varphi \\ r \end{bmatrix} = -r \sin^2\theta, \quad \begin{bmatrix} \varphi r \\ \varphi \end{bmatrix} = +r\sin^2\theta, \\
&\begin{bmatrix} \varphi\varphi \\ \theta \end{bmatrix} = -r^2 \sin\theta\cos\theta, \quad \begin{bmatrix} \varphi\theta \\ \varphi \end{bmatrix} = +r^2\sin\theta\cos\theta, \\
&\begin{bmatrix} \theta\theta \\ r \end{bmatrix} = -r, \quad \begin{bmatrix} \theta r \\ \theta \end{bmatrix} = +r,
\end{aligned} \quad . \quad (6)$$

the other symbols of CHRISTOFFEL being zero, we have

$$\begin{aligned}
-\tfrac{1}{2}\frac{\beta}{r^2} &= \left(1+\frac{\alpha}{r}\right)\ddot{r} - \tfrac{1}{2}\frac{\alpha}{r^2}\dot{r}^2 - r\sin^2\theta\dot{\varphi}^2 - r\dot{\theta}^2 \\
0 &= r^2\sin^2\theta\,\ddot{\varphi} + 2r\sin^2\theta\,\dot{r}\,\dot{\varphi} + 2r^2\sin\theta\cos\theta\,\dot{\varphi}\,\dot{\theta} \\
0 &= r^2\ddot{\theta} - r^2\sin\theta\cos\theta\,\dot{\varphi}^2 + 2r\,\dot{r}\,\dot{\theta}
\end{aligned} \quad . \quad (7)$$

A motion, satisfying these equations is the circular motion:

$$r = R = \text{constant}, \quad \dot{\varphi}^2 = \omega^2_0 = \frac{\beta}{2R^3}, \quad \theta = \frac{\pi}{2} \quad . \quad . \quad . \quad . \quad (8)$$

When we consider only motions, deviating little from this circular one, we can put $\sin\theta = 1$, $\cos\theta = \frac{\pi}{2} - \theta$ and neglect in the first equation $\dot{r}^2$ and $\dot{\theta}^2$, in the second one $\cos\theta\,\dot{\theta}$ and in the third one $\dot{r}\,\dot{\theta}$. Then these equations pass into:

$$\begin{aligned}
-\tfrac{1}{2}\frac{\beta}{r^2} &= \left(1+\frac{\alpha}{r}\right)\ddot{r} - r\,\dot{\varphi}^2 \\
0 &= r\,\ddot{\varphi} + 2\dot{r}\,\dot{\varphi} \\
0 &= \ddot{\theta} - \cos\theta\,\dot{\varphi}^2
\end{aligned} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Now we introduce the variables x, y and z by the equations:

$$\begin{aligned}
r &= R + x\left(1 - \frac{\alpha}{2R}\right) \\
\varphi &= \omega_0 t + \frac{y}{R}, \quad \omega_0^2 = \frac{\beta}{2R^3} \\
\frac{\pi}{2} - \theta &= \cos\theta = \frac{z}{R}
\end{aligned} \quad . \quad . \quad . \quad . \quad (10)$$

x, y, z form a rectangular system of axes moving with a velocity

$R\omega_0 t$ along the orbit $r = R$, the axis of x having always the direction of the radius and the axis of y the direction of the motion. Then the equations pass into

$$\begin{aligned}
0 &= \left(1+\frac{\alpha}{2R}\right)\ddot{x} - 2\omega_0\dot{y} - 3\omega_0^2 x\left(1 - \frac{\alpha}{2R}\right) \\
0 &= \ddot{y} + 2\omega_0\left(1 - \frac{\alpha}{2R}\right)\dot{x} \\
0 &= -\ddot{z} - \omega_0^2 z
\end{aligned} \quad . \quad . \quad (11)$$

or

$$\begin{aligned}
\ddot{x} &= 2\left(1 - \frac{\alpha}{2R}\right)\omega_0\dot{y} + 3\left(1 - \frac{\alpha}{R}\right)\omega_0^2 x \\
\ddot{y} &= -2\left(1 - \frac{\alpha}{2R}\right)\omega_0\dot{x} \\
\ddot{z} &= -\omega_0^2 z.
\end{aligned} \quad . \quad . \quad . \quad (12)$$

We further pass to a system of axes x', y', z', which revolves with respect to x, y, z around the ayis of z with an angular velocity ω in the sense of y to x, the axis of z' coinciding with the axis of z:

$$\begin{aligned}
x &= x'\cos\omega t + y'\sin\omega t \\
y &= -x'\sin\omega t + y'\cos\omega t \\
z &= z'.
\end{aligned} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Then the equations pass into:

$$\begin{aligned}
\ddot{x}' &= -2\left\{\omega - \omega_0\left(1 - \frac{\alpha}{2R}\right)\right\}\dot{y}' + 3\omega_0^2\left(1 - \frac{\alpha}{R}\right)(x'\cos^2\omega t + y'\sin\omega t\cos\omega t) + \\
&\qquad + \left\{\omega^2 - 2\omega\,\omega_0\left(1 - \frac{\alpha}{2R}\right)\right\}x' \\
\ddot{y}' &= +2\left\{\omega - \omega_0\left(1 - \frac{\alpha}{2R}\right)\right\}\dot{z}' + 3\omega_0^2\left(1 - \frac{\alpha}{2R}\right)(x'\cos\omega t\sin\omega t + y'\sin^2\omega t) - \\
&\qquad + \left\{\omega^2 - 2\omega\,\omega_0\left(1 - \frac{\alpha}{2R}\right)\right\}y' \\
\ddot{z}' &= -\omega_0^2 z'.
\end{aligned} \quad (14)$$

Being given a spherical body with centre in the origine of the system x, y, z and so small, that the squares of its dimensions may be neglected. Then, supposing the body with this neglection to be rigid, in the expressions of the moments $\Sigma\, m\,(y'\ddot{z}' - z'\ddot{y}')$ cycl. the terms with x', y', z' will all contain an inertial product or a difference of two equal inertial moments and consequently this terms will vanish. The terms with $\dot{x}'$, $\dot{y}'$ and $\dot{z}'$ then and only then vanish for every kind of motion with respect to x', y' and z' if ω be choosen in such way that the terms with $\dot{x}'$, $\dot{y}'$ and $\dot{z}'$ vanish in the equations (14), i.e. if:

$$\omega = \omega_0 \left(1 - \frac{a}{2R}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

But in this case x', y', z' is extactly a geodesically moving system of axes as I demonstrated in the publication referred to on the first page. After one revolution this system has turned over $\frac{\pi a}{R}$.

This can of course also be calculated in the fourdimensional way. Starting with the linear element (2) we find a precession passing for $\beta = 0$ (velocity approaching to zero) in the above calculated value $\frac{\pi a}{R}$ and for $\beta = a$ in $1^1/_2 \times$ this value.

It is worth observing the ordinary precession gets possibly also another value in relativistic mechanics than in classical mechanics, a possibility pointed out by DE SITTER. By means of the equations given by FOKKER it will be possible to calculate the deviation caused by this, at least so far as it is not influenced by forces caused by the mutual attractions of the parts of the planet.

Physics. — *Mutual Influence of Neighbouring Fraunhofer Lines*"[1]). By Prof. W. H. JULIUS.

(Communicated at the meeting of January 29, 1921.)

If the hypothesis holds good that the darkness of Fraunhofer lines is *not* a pure absorption effect — as it is commonly supposed to be — but chiefly due to anomalous dispersion (showing itself both in molecular diffusion and irregular ray-curving), we may expect on theoretical grounds[2]) that neighbouring Fraunhofer lines will, as a rule, seem to repel each other. If, now, such a mutual influence is actually found to exist, a mighty support will thus be given to the said interpretation of the solar spectrum, as long as it remains impossible to explain that phenomenon on the basis of the current view that one is dealing with mere absorption lines.

In a communication on "The general relativity theory and the solar spectrum"[3]) we have made use of the already reliable and striking results obtained in a preliminary research on the manifestations of mutual influence of Fraunhofer lines as appearing in the limb-centre displacements measured by ADAMS[4]) about the year 1910. At my request Dr. P. H. VAN CITTERT and Dr. M. MINNAERT have, however, once more examined the same data with the utmost care, using still more rigorously defined criteria, in order that every trace of bias might be avoided in selecting the lines. Besides, the investigation has been extended over the observation son limb-centre displacements published by EVERSHED, NARAYANA AYYER and ROYDS[5]) in 1914—1916. It will appear that this extension of the field has led to a considerable corroboration of the former inferences, so as to put the existence of mutual influence practically beyond doubt.

Care has been taken, of course, that during the act of selecting lines that would probably be influenced, one was ignorant of the observed displacements. Basing ourselves on the conception how,

[1]) This paper is an abstract of an ampler article that has since appeared in the Astrophysical Journal **54**, 92 (1921). (Note, added January 1922).

[2]) Cf. Astroph. Journ. **43**, 49—53 (1916).

[3]) W. H. JULIUS and P. H. VAN CITTERT. These Proc. **23**, 522 (1920).

[4]) W. S. ADAMS, Astroph. Journal **31**, 30 (1910); Mt. Wilson Contrib. No. 43.

[5]) EVERSHED and ROYDS, Kodaik. Bull. **39** (1914); NARAYANA AYYER, Kodaik. Bull. **44** (1914); ROYDS, Kodaik. Bull. **53** (1916).