Physiology. — "*On the Influence of the Season on Laboratory Animals*". By Prof. H. ZWAARDEMAKER.

(Communicated at the meeting of January 29, 1921).

This technical subject appears to be of general application. In previous publications the present writer and his co-workers have superadded to J. LOEB's balancing of ions of the circulating fluids, expressed in the equation $\frac{Na + K}{Mg + Ca}$ = constant, the balancing formula $\frac{K + (U\,O)_2 + Th}{Ca + Sr + Ba}$ = constant. In the latter formula the radio-physiological antagonism between K and $(U\,O)_2 + Th$. need not be taken into account [1]).

Moreover, in earlier discourses the replacement of potassium by the other radio-active elements Rb, $(U\,O)_2$, U, Th, Io, Ra, Em, has been repeatedly discussed [2]).

Now the present writer wishes to point out that the dosages in which these elements are to be administered must be much smaller in summer than in winter. Of course, this difference is not brought about by the radio-active elements as such, but by the fact that in summer the organs are more sensitized by certain substances [3]).

These substances can be washed out, so that in the transition periods the functionating of a summer-organ during some hours' perfusion with an artificial but nonetheless efficient circulating fluid, suffices to transform a summer-organ into a winter-organ.

As regards sensitizing power, that of the washed-out substances is analogous to that of adrenalin.

The organs operated upon were the hearts of frogs and of eels. A detailed publication will appear elsewhere.

1) C. R. des Séances de la Soc. de Biologie 7 Juin 1919.
2) Journal of Physiology Vol. 53 p. 273 1920.
3) Proceedings of this Acad. 25 Sept. 1920.

Physics. — "*On the principles of the theory of quanta*. By PAUL S. EPSTEIN. (Communicated by Prof. P. EHRENFEST).

(Communicated at the meeting of January, 29, 1921.)

1. *Introduction*. The quantum-theory in the form, which in 1911 PLANCK [1]) has given it, depends on the application of statistical mechanics in the so-called "phase-space" of the canonical position- and impulse-coordinates $q_1\, q_2 \ldots q_f;\ p_1\, p_2 \ldots p_f$, and consists in dividing this space into elementary regions of probability. The method obtains a considerable simplification for the soluble mechanical systems, since for them each impulse-coordinate $p_i = p_i\,(q_i)$. Instead of the $2f$-dimensional phase-space (f being the number of degrees of freedom of the system) it is then sufficient to consider the f "phase-planes" (p_i, q_i), which, as the author showed a few years ago [2]), gives great advantages in the treatment of these systems. In each of these planes the successive conditions of the system are represented by a curve. For the class of the "conditioned-periodic motions", the only ones for which so far quantum-conditions have been established, the curves in question are as a rule closed. The only exception is formed by the "cyclic coordinates" which bear the character of a plane angle; a cyclic coordinate varies from 0 to 2π and the corresponding impulse is constant; hence the representive curve becomes a segment of a straight line parallel to the axis of abscissae. [3])

PLANCK's hypothesis, as extended by SOMMERFELD and the author, consists in the assumption of the existence among the states of the system of certain preferential or "stationary" motions, which are represented by discrete curves in the diagram, the area of the phase-plane between two successive stationary curves being equal to the universal constant h

$$\iint dp\, dq = h. \qquad (1)$$

If the area of the narrowest of these curves (or for cyclic coor-

1) M. PLANCK. Verhandelingen van het Solvay-congres.
2) P. S. EPSTEIN. Ann. d. Phys. **50**, p. 489; **51**, p. 168, 1916.
3) This case was discussed for the first time by P. EHRENFEST. Verh. d. D. phys. Ges. **15**, p. 451. 1913.

dinates the one nearest the axis of abscissae) is equal to h_0, that of the $(n+1)\,st$ stationary orbit will be

$$\int p dq = h_0 + nh \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

h_0 has therefore to be determined, in order that all the stationary curves be fixed.

For this purpose PLANCK [1]) lays down the principle, that the narrowest orbit must coincide with the *natural boundary of the phase-plane*; i.e. if on any grounds, connected with the nature of the system, the integral (1), which is essentially positive, cannot fall below a definite value, the latter has to be taken as h_0. In most cases a lower limit of that kind does not exist and the integral may be taken equal to zero, whence

$$\int p dq = nh \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2')$$

In his treatment of the relativistic Kepler-motion SOMMERFELD [2]) found the case to be different; he there gave a lower limit $p_0 = \varkappa e^2/c$ [3]) for the *constant azimuthal impulse*; this would give $h_0 = 2\pi p_0$. It would therefore, as pointed out by PLANCK, be necessary to take (2) as the fundamental relation, whereas experiment (the Balmer-series) can only be reconciled with supposition (2′). SOMMERFELD [4]) tried to remove this contradiction by pointing out, that when the motion of the nucleus is taken into account the numerical value of the limiting impulse is smaller than $\varkappa e^2/c$. In what follows we hope to prove that *the limitation of the phase-plane by the value $p = p_0$ is only an apparent one, even if the motion of the nucleus is left out of account,* and that p can very well fall below this value: at the same time the character of the motion is then essentially changed.

The admissibility of stationary orbits of azimuthal impulse $p = 0$ which on SOMMERFELD's theory seemed to be excluded is thereby proved in principle. As long as we are dealing with attractive forces (nucleus and electron) these orbits are hardly of practical importance, as they must lead to a collision of electron and nucleus. But the case changes, when the forces are repulsive (nucleus and α-particle); the orbits are then hyperbolic. If the quantization of such orbits is admitted, interesting physical conclusions follow which appear to

[1]) M. PLANCK. Ann. d. Phys. **50**, p. 385. 1916.

[2]) A. SOMMERFELD. Ann. d. Phys. **51**, p. 57. 1916.

[3]) Here e is the charge of an electron, $\varkappa e$ that of the atomic nucleus and c the velocity of light.

[4]) A. SOMMERFELD. Münchener Ber., p. 137, 1916.

give an explanation of certain recent experimental results of RUTHERFORD's [1]). The question raised by the author [2]) before as to the quantization of non-periodic motions is therefore put once more and discussed in a different manner (§ 4, 7).

§ 2. *The apparent boundary of the phase-plane $p = p_0$.*

The relativistic Kepler-motion is given by the following equation between the polar coordinates r, φ (cf. l.c. p. 819).

$$\frac{1}{r} = \frac{B}{p^2 - p_0^2}\left[1 - \varepsilon \cos \frac{\sqrt{p^2 - p_0^2}}{p^2}(\varphi - \varphi_0)\right] \quad . \quad . \quad . \quad . \quad (3)$$

with the abbreviations

$$\left.\begin{aligned} B &= \pm p_0 c\left(m + \frac{\alpha}{c^2}\right) \quad ; \quad \varepsilon = \frac{\sqrt{B^2 + A(p^2 - p_0^2)}}{B} \\ A &= \alpha\left(\frac{\alpha}{c^2} + 2m\right) \quad ; \end{aligned}\right| \quad . \quad . \quad . \quad (4)$$

α represents the energy of the system, c the velocity of light, m the mass of the moving particle. The positive sign of B refers to the case of attraction, the negative sign to repulsion; φ_0 is the azimuth of the radius vector with respect to the aphelion.

For $p > p_0$ with negative energies $(A < 0)$ and attracting forces $(B > 0)$ the orbit is an ellips with perihelion-motion. The procession of the perihelion increases in speed, the smaller the difference $p^2 - p_0^2$, and in the limiting case $p = p_0$ the orbit converges on the nucleus in a manner similar to an Archimedian spiral [3]):

$$\frac{1}{r} = \frac{B}{p^2}(\varphi - \varphi_0)^2 - \frac{A}{B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

But nothing prevents us from now taking $p < p_0$; the expression (3) then assumes the form:

$$\frac{1}{r} = \frac{B}{p_0^2 - p^2}\left\{\varepsilon \cosh \frac{\sqrt{p_0^2 - p^2}}{p^2}(\varphi - \varphi_0) - 1\right\} \quad . \quad . \quad . \quad . \quad (6)$$

The right-hand side of this expression for a very large positive or negative value of φ becomes exponentially infinite independently of the value of the excentricity ε. The two extremities of the orbit thus approach logarithmic spirals. It further follows from (4) that

[1]) E. RUTHERFORD. Phil. Mag. **37**, p. 537, 1919.

[2]) P. S. EPSTEIN. Ann. d. Phys. **50**, p. 815, 1916. This paper will be quoted here as l.c.

[3]) Comp. A. SOMMERFELD Ann. d. Physik. **51**, p. 50 1916.

for $A \lesseqgtr 0$, $\varepsilon \gtreqless 1$. Thus with negative energy r always remains finite, the particle moves out from the centre and again returns to it.

When the energy disappears or becomes positive the orbit divides into two branches which run from the centre to infinity or vice versa. In the limiting case $p = 0$ r is only finite for $\varphi = \varphi_0$, i.e. the motion is rectilinear.

Thus it is seen, that in reality there is not a limit $p = p_0$ at all: with small positive values of $p - p_0$ the orbit encircles the centre many times, while r diminishes, but remains at a finite distance from it which passes through a minimum and then increases again. For $p = p_0$ the curve runs into the centre as an Archimedian spiral. The approach to the centre is even more rapid when $p < p_0$, the spiral becoming logarithmic. It must not be supposed that the particle in its motion on the spiral will permanently remain near the centre: for although the spiral encircles the point an infinite number of times, its total length is finite and the time to describe it from a finite distance, as a simple calculation shows, is also finite and practically very small. Therefore the collision will occur very soon.

§ 3. *Quantization of the spiral orbits.* In the last section we have shown, that in the relativistic Kepler-motion, even with negative energy, besides the ellips-like orbits other forms are possible which are of finite length and are only once described.

The question now arises, whether these motions can be submitted to quantum-conditions and in what manner this would have to be done. Our answer to the first question is implicitly contained in the above discussion: the disappearance of the limiting value h_0 in assumption (2) we have explained by the fact that orbits have to be taken into account for which p is less than the azimuthal quantum p_0. It follows that these orbits join on continuously to the others and must be equivalent to them from the point of view of the quantum theory. Since for $p > p_0$ the stationary motions are given by the relation $p = nh/2\pi$ $(n = 1, 2, \ldots)$, it follows that for $p < p_0$ the only possible stationary condition is $p = 0$. This conclusion is strengthened by the circumstance that when the movement of the nucleus is taken into account (as proposed by SOMMERFELD) similar spiral-shaped orbits have to be considered in order to explain the possibility of $p = 0$: this can be easily shown to be the case.

We have therefore only to discuss the quantisation of the radial impulse: its dependence on the radius vector r and on the constants of the problem is given by the equation (l. c. p. 823).

$$p_r = \sqrt{A + 2B\frac{1}{r} + (p_0^2 - p^2)\frac{1}{r^2}} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which is represented graphically in Fig. 1 for $p < p_0$. The curves nearest the axis of ordinates correspond to large negative values of the energy constant α. With increasing energy the curves bend out more and more and for $\alpha = 0$ they divide into two branches which approach asymptotically to the axis of abscissae. For α positive the asymptotes are straight lines parallel to this axis.

For small values of r (7) reduces to

$$p_r = \sqrt{p_0^2 - p^2}\,\frac{1}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7')$$

i.e. at a distance from the axis of abscissae the curves are hyperbolic. The area of such a curve is logarithmically infinite and the difference between the area of two curves is also always infinite, unless we apply artificial means such as the formation of the principal values of the integral. Since according to the quantum theory the areas of two successive stationary curves must differ by the finite quantity h, it follows that in this case the stationary energy stages must be infinitely dense, i.e. all values of the energy are "selected" in the sense of the theory. Whereas the selected values of p form a series of discrete numbers, those of α form a continuum. There are thus an infinite number of motions which starting from the zero reach as far as we like. All these orbits lead to a collision with the nucleus and for this reason they are not very important physically.

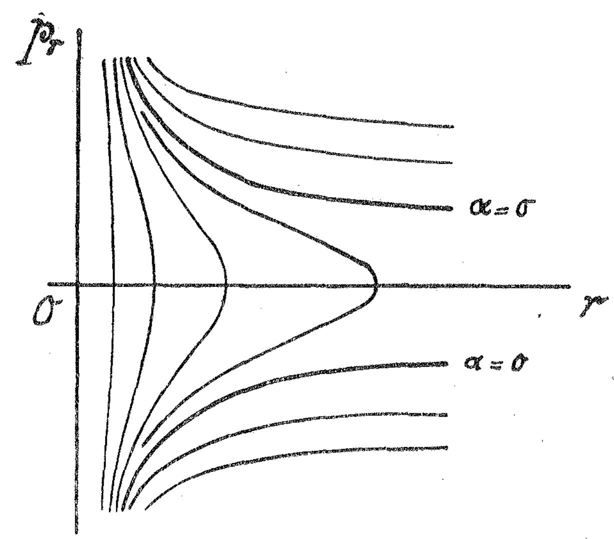

Fig. 1.

But for our purpose it is important that these orbits are possible in principle irrespectively of how long an electron can move along it.

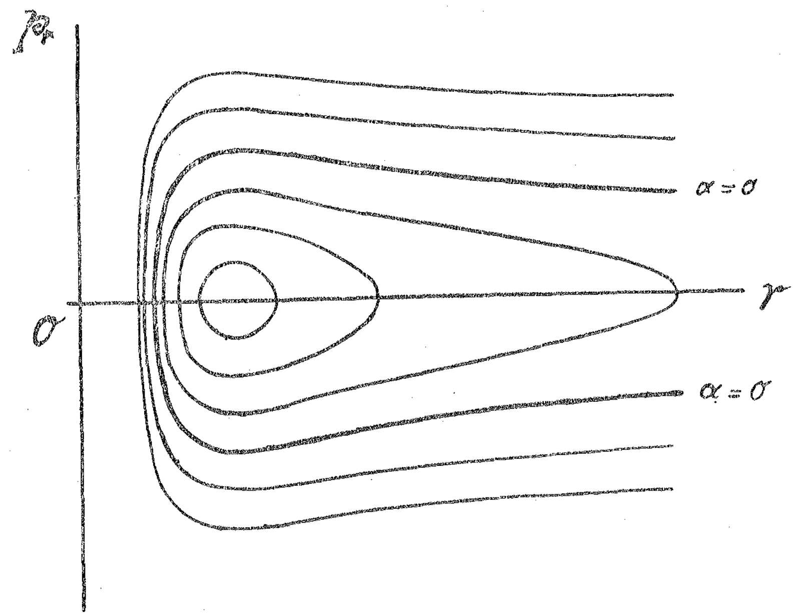

Fig. 2.

§ 4. *Quantization of the hyperbolic curves.* The problem becomes of greater importance physically, if repulsive forces are considered, so that the orbits are hyperbolic. The question, how these orbits have to be quantizized was discussed by me several years ago (l.c.). The method adopted then, which was explicitly stated to be provisional, I do not wish to adhere to in all its particulars. But the fundamental idea of submitting such orbits to quantum-conditions still appears to me a sound one. Quite a long time ago I have in the Munich colloquium developed certain views on this subject which appear to me still to deserve attention. For simplicity we shall here disregard the relativity correction ($c = \infty$): the radial impulse according to (7) and (4) then assumes the form:

$$p_r = \sqrt{2 m \alpha + 2 \varkappa m e^2 \frac{1}{r} - p^2 \frac{1}{r^2}} \quad . \quad . \quad . \quad . \quad (8)$$

For $\alpha < 0$ the motion is elliptical, for $\alpha = 0$ parabolic, for $\alpha > 0$ hyperbolic. The aspect of the curves in the phase-plane (p, r) is seen in Fig. 2. The part of the plane, where $\alpha < 0$ is bounded by the heavily drawn curve $\alpha = 0$, both ends of which approach the axis of abscissae asymptotically. Inside this region the curves are elliptic and it is easy to fulfil the condition that the area between

two successive curves be equal to h. In the region outside the curve $\alpha = 0$ each one of the curves possesses two asymptotes parallel to the axis of abscissae. The strip between two curves whose energy-constants differ by a finite amount has an infinite area. Just as in the case of the spiral orbits we may conclude that the energy stages of the stationary motions must be infinitely dense. *Every positive value of the energy-constant is therefore a "selected" value in the sense of the quantum theory.* That hyperbolic orbits with all values of the energy are present, was already enunciated by Bohr on the ground of experimental results (by Wagner and others). From our point of view this does not prove that the hyperbolic motion is beyond the controll of the quantum theory; on the contrary this fact is a natural inference of a consistent application of this theory.

This view naturally implies that the azimuthal impulse must also be subjected to quantic conditions. What these are cannot immediately be deduced from the case of the elliptic motion. Two possibilities seem to present themselves: we must extend the integral $\int p d\varphi$ either over the range of change of the coördinate φ, i. e. over the angle enclosed between the asymptotes, or, as in the case of the elliptic motion, from 0 to 2π. The former assumption would according to 2′ give

$$p = \frac{nh}{2\varphi}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9a)$$

the latter

$$p = \frac{nh}{2\pi}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9b)$$

In § 7 we shall meet with an argument in favour of the second assumption, but a decision between the two can ultimately only be brought by experiment.

§ 5. *Collision between an α-particle and an atomic nucleus.* We shall now investigate the case of repulsive forces in detail and thereby take into account the motion of the nucleus, neglecting the relativistic correction which is of no importance for our purpose. In the usual manner by means of the principle of the centre of mass we eliminate the co-ordinates of the one body and so reduce the problem to that of a system of two degrees of freedom. We shall choose as the variables the relative polar co-ordinates of the two particles, i. e. their distance and the angle φ under which the α-particle appears for an observer moving with the atomic nucleus, and shall call p the impulse corresponding canonically to φ, m and M the masses, e, E the charges of the α-particle and nucleus respectively, and finally v the initial velocity of the α-particle, the atom being originally supposed at rest. The equation to the orbit then assumes the simple form

$$\frac{1}{r} = \frac{\mu e E}{p^2}\left[\varepsilon \cos(\varphi - \varphi_0) - 1\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where

$$\frac{1}{\mu} = \frac{1}{m} + \frac{1}{M} \quad ; \quad \varepsilon = \sqrt{1 + \left(\frac{pv}{eE}\right)^2} \quad . \quad . \quad . \quad (11)$$

Hence the angle $\overline{\varphi}$ between the axis and the asymptote of the relative hyperbolic orbit of the α-particle is given by

$$\cos \overline{\varphi} = \frac{1}{\varepsilon}$$

or

$$tg\,\overline{\varphi} = \frac{pv}{eE}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

We can now change to the absolute motion by considering, that the centre of gravity of the two bodies must move uniformly; originally this point moves in the direction of the α particle with the velocity $mv/(M+m)$ and this motion has therefore to be superposed on the relative motion. A simple calculation gives the following result[1]): after a sufficient time both bodies have assumed a uniform rectilinear motion. The direction of the final motion of the α-particle encloses an angle Φ with the initial direction (through this angle the α-particle is deflected by the collision)

$$tg\,\Phi = \frac{2\,tg\,\overline{\varphi}}{(m - M) + (M + m)\,tg^2\,\overline{\varphi}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

the velocity V obtaining the value

$$V = \frac{v}{M+m}\sqrt{m^2 + M^2 - 2\,m\,M \cos 2\,\overline{\varphi}}\,. \quad . \quad . \quad . \quad (14)$$

The angle between the direction in which the atom is propelled and the original direction of the α-particle is exactly equal to the angle $\overline{\varphi}$ of equation (12). The velocity of the atom is

$$u = 2v\frac{\mu}{M}\cos\overline{\varphi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

According to the view set forth in § 4 certain special motions of

[1]) Comp. C. Darwin. Phil. Mag. **27**, p. 499, 1914.

the system are to be allowed, namely those for which the azimuthal impulse p has a value satisfying the conditions ($9a, b$). In these the letter n represents a positive whole number, but $n = 0$ which would be excluded according to Sommerfeld must also be admitted on the point of view explained in § 2. In the latter case the assumptions ($9a$) and ($9b$) both give $p = 0$; hence

$$\overline{\varphi}_0 = 0 \quad ; \quad u_0 = 2\frac{\mu}{M}v\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

in other words: *the nuclei or "recoilrays"* as Rutherford has called them, *have for $n = 0$ the direction of the primary α-rays.*

§ 6. *Recoil-rays of hydrogen.* Whereas the values (16) which obtain for $n = 0$, hold generally for all kinds of atoms, the results are less general for $n = 1$, 2 etc.; we shall only discuss the special case of the collision with a hydrogen atom. On the assumption ($9a$) we have according to (12)

$$\overline{\varphi}\,tg\,\overline{\varphi} = \frac{hv}{2\,e\,E}\,n\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

on assumption ($9b$)

$$tg\,\overline{\varphi} = \frac{hv}{2\pi\,e\,E}\,n\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

In these expressions we may substitute $h = 6.55 \times 10^{-27}$ erg. sec. $E = 4.77.10^{-10}$ e. s. units $e = 2E$; for v we shall take the velocity of the α-rays of RaC, for which Rutherford gives the value $1.92.10^9$ cm/sec. we then obtain

$$\overline{\varphi}\,tg\,\overline{\varphi} = 13.8\,n, \quad \text{or} \quad tg\,\overline{\varphi} = 4.40\,n \quad . \quad . \quad . \quad . \quad (19)$$

The first of these equations gives

$$n = 1, \quad \overline{\varphi}_1 = 84°, \quad u_1 = 0{,}10\,u_0, \quad R_1 = 0{,}001\,R_0,$$
$$n = 2, \quad \overline{\varphi}_2 = 86°{,}50', \quad u_2 = 0{,}055\,u_0, \quad R_2 = 0{,}0002\,R_0.$$

The velocities $u_1\,u_2$ are computed from (15), the corresponding ranges R_1, R_2 from the empirical equation $R : R_0 = u^3 : u^3$.

Similarly the second hypothesis gives

$$n = 1, \quad \overline{\varphi}_1 = 77°, \quad u_1 = 0{,}22, \quad R_1 = 0{,}011\,R_0,$$
$$n = 2, \quad \overline{\varphi}_2 = 83°{,}30', \quad u_2 = 0{,}11, \quad R_2 = 0{,}001\,R_0.$$

On the view that the particles can only move on the special orbits allowed by the quantum theory, we obtain the following result: a portion of the recoil rays are emitted in the direction of the primary rays ($n = 0$); besides there are only particles which start at considerable angles to that direction, the smallest angle in the one case (9b) being 77° in the second (9a) even 84°. The corresponding ranges are exceedingly much smaller than for the H-atoms emitted in the direction of the primary α-particles.

These results agree with the result of RUTHERFORD's experiments [1]), who found all the H-atoms to be propelled in the direction of the primary rays. The range of this secondary radiation was 28 cms., which gives $R_1 = 0.028$ cm. or $R_1 = 0.31$ cm. according as we use (9a) or (9b). These values are too small for experimental verification, and were bound to escape detection.

§ 7. *Transition to the stationary orbits.* Up to the present time the quantum theory has only been applied to systems whose members permanently move round each other at a finite distance, i.e. systems which in the LAPLACE-sense are stable. My attempt of 1916 (l.c.) to apply the theory to the single passage of a particle through the sphere of action of a nucleus has not met with much sympathy among physicists. It therefore seems necessary to submit the difference between the two cases to a careful conceptual analysis.

The hypothesis of the theory as established by BOHR consists of two parts: 1. There are certain preferential or stationary orbits in which the system moves without radiation. 2. If the initial state is not a stationary one, the system passes into a stationary state with the emission of energy in the form of radiation. It is quite possible, that the real process is only formally represented by this division, but it has been confirmed in several cases and it forms for the present the only basis on which we can erect our further structures.

As regards the existence of stationary orbits, there does not seem to be any reason, why the quantum conditions should be solely applicable to finite orbits. Our views on this point have been expounded in §§ 3 and 4; but we shall try to strengthen them from a fresh point of view. The difference between motions which are finite and those which reach to infinity is expressed analytically by the fact, that for the former each cartesian co-ordinate may be represented as a FOURIER-series according to angular variables, whereas this is impossible for the latter. BOHR has established a relation between the terms of this FOURIER-series and the transitions which on the quantum-theory are possible from one stationary orbit to another.

In the case of the relativistic Kepler-motion the Cartesian coordinates are $x = r \cos \varphi$, $y = r \sin \varphi$. For shortness putting

1) E. RUTHERFORD, l.c.

$$\frac{\sqrt{p^2 - p_0^2}}{p}(\varphi - \varphi_0) = \psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

it follows from (3) that

$$x = \frac{p^2 - p_0^2}{B} \frac{\cos \varphi}{1 - \varepsilon \cos \psi} \quad ; \quad y = \frac{p^2 - p_0^2}{B} \frac{\sin \varphi}{1 - \varepsilon \cos \psi} \quad . \quad . \quad (21)$$

For a motion of elliptic type ($\varepsilon < 1$) x and y are periodic in φ and ψ with a period 2π; φ and ψ are therefore angular coordinates of the problem and a FOURIER-expansion is possible [1]). Passing to the case $\varepsilon > 1$ the angle ψ becomes limited and varies between the limits $\pm \, arc \cos \left(\frac{1}{\varepsilon}\right) = \pm \bar{\psi}$. Only between these limits x and y have the meaning of the functions given in (21); hence they may now be represented by a FOURIER-integral

$$x = \frac{2}{\pi} \frac{p^2 - p_0^2}{B} \cos \varphi \int_0^\infty \cos s \psi \, ds \int_{-\bar{\varphi}}^{+\bar{\varphi}} \frac{\cos \lambda}{1 - \varepsilon \cos \lambda} d\lambda .$$

The case is different for φ: this angle also varies between two extreme values $\frac{\pm p \bar{\psi}}{\sqrt{p^2 - p_0^2}} = \pm \bar{\varphi}$; but in contrast with ψ it possesses a physical periodicity: on changing φ by the amount 2π the same point of space is reached, so that x and y remain periodic with respect to φ. We may continue the dependence on φ in the ranges $\bar{\varphi} < \varphi < \pi$ and $-\pi < \varphi < -\bar{\varphi}$, where no motion takes place, just as we like. It would be simplest to assume the continuance of the law expressed by $\cos \varphi$ and $\sin \varphi$ over the whole range from 0 to 2π, in which case we should get

$$x = \frac{p^2 - p_0^2}{\pi B} \int_0^\infty [\cos (s\psi + \varphi) + \cos (s\psi - \varphi)] \, ds \int_{-\bar{\varphi}}^{+\bar{\varphi}} \frac{\cos s \lambda}{1 - \varepsilon \cos \lambda} d\lambda .$$

It seems to me that in this result lies a confirmation of the reasoning of § 4. The coefficient of ψ is the number s which may assume any value, whereas the coefficient of φ is the whole number 1. Extending BOHR's principles to this case we might conclude that the radial quantum which is subordinated to ψ underlies no limitations, whereas the azimuthal quantic number can only change by

1) These coordinates are not linear functions of the time. If we wish them to satify the latter condition, they have to be defined differently. But the conclusions to be drawn remain valid with this change in the definition.

1 each time. In this way it is made probable, that the azimuthal impulse possesses discrete special values and the analogy with the case of the elliptic motion imparts special probability to the hypothesis (9*b*).

Although the existence of stationary orbits is thus rendered probable, it does not follow that a particle which to begin with is not moving in a stationary orbit will have time and opportunity to pass into one. The giving off of energy requires time which is always available in the case of stable motion (in LAPLACE's sense). But for hyperbolic motion the case is different: the energy is not limited by any conditions, but the rotational impulse tends towards definite values which can only be reached by the process of radiation of electromagnetic moment of momentum. For this radiation the time available is only the one motion past the nucleus, and it is thus quite possible that the impulse lost by radiation is not sufficient and that the particle returns to infinity without having reached a stationary condition. On the basis of MAXWELL's theory this would even be the usual case. Calculation gives for the radiated impulse (for $p >> p_0$)

$$2\frac{e^2v^2}{c^3}\,\frac{\varkappa e^2}{pv}\left\{\left(\frac{\varkappa e^2}{pv}+\frac{1}{3}\,\frac{pv}{\varkappa e^2}\right) bg\, tg\, \frac{pv}{\varkappa e^2}-1\right\},$$

that is an amount of the order 10^{-31} erg sec., whereas the steps of the constant p are about 10^{-27} erg sec., or about 1000 times larger. Under these circumstances no fraction of the particles worth mentioning could attain stationary orbits.

On the other hand we have the experimental fact, mentioned in the previous section, that the *H*-atoms are preferably emitted in the direction of the incident α-particles and it seems difficult to interpret this otherwise than on the quantum-theory. One of the possible explanations of RUTHERFORD's results seems therefore to be that the radiation is really stronger than would follow from MAXWELL's theory, sufficiently so to carry a considerable portion of the systems into the stationary condition. When we consider that even in the radiation of the hydrogen spectrum, where the distances from the nucleus are greater than 2×10^{-8} cms, a considerable deviation exists from MAXWELL's theory, the supposition in RUTHERFORD's case of a very much larger deviation does not appear to us too hazardous. For the distance from the nucleus is here of the order 3.5×10^{-13} and thus the acceleration about 1.5×10^{6} times larger than in the emission of the hydrogen lines. Moreover EINSTEIN [1]) has postulated a complete breach with MAXWELL's theory for elementary processes

[1]) A. EINSTEIN. Kleiner-Festschrift, Zürich 1918.

of this kind and we further know from the existence of a limit of the Röntgen-radiation on the side of the small wavelengths, that they cannot take place in accordance with the theory. According to RUTHERFORD's experiments the relative number of the emitted recoil-rays is strongly dependent on the rapidity or range R of the primary α-particles, as shown by the following table:

$R = 7{,}0$	5,3	4,5	3,7	3,0	etc.
$N = 100$	77	51	25	0.	

This might be interpreted as indicating that the radiation of rotational impulse decreases rapidly with the speed v, so that with falling v there are less and less particles which are able to reach the stationary orbit. If this view is correct, we would have in RUTHERFORD's table a new way along which to penetrate into the riddle of the quantum theory.

Side by side with particles which have completed the transition into the stationary orbit, others are to be expected, even with the highest velocities, which owing to a higher initial impulse have not succeeded in doing so. The directed radiation must therefore be surrounded with a scattered radiation. According to a kind personal communication of sir ERNEST RUTHERFORD's something of that kind is found experimentally: a new experimental method has shown that the recoil-rays are in reality less homogeneous than appeared originally and that side by side with the rapid H-particles observed at first, there are others of smaller speed [1]).

As suggested above it is probable that the large deviations from MAXWELL's theory, as required for a sufficiently strong radiation, are limited to the range of very high accelerations. This makes it doubtful, whether a similar approach to the stationary orbit $n = 1$ is to be expected as to the orbit $n = 0$; for the range near it correspondends to a much greater distance from the nuclens.

On a different occasion we hope to discuss the question, how the stationary orbits are distributed for nuclei other than of hydrogen. We shall only mention here, that for heavy atoms the equations (17) and (18) owing to the high value of the nucleus charge E make the steps of the discrete angular distribution so small, that the result cannot differ appreciably from a distribution in accordance with classical statistics.

[1]) Cf. E. RUTHERFORD, Phil. Mag. **41** (6), p. 307, 1921.