Mathematics. — *"Properties of Congruences of Rays"*. By H. J. VAN VEEN. (Communicated by Prof. J. CARDINAAL).

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§ **1**. The rays of the space Σ_3 can be represented by a quadratic hypersurface O^2_4 in a five dimensional point-space Σ_5. By the aid of this representation I have derived the properties of a few complexes of rays (Nieuw Archief voor Wiskunde R 2, dl. XI, p. 232; id. dl. XII, p. 19; Handel. 17e Nederl. Natuur- en Geneesk. Congres 1919, p. 171). By the method followed there also the characteristic numbers for an arbitrary complex of rays can be determined. In what follows I shall make use of the representation mentioned to derive some of the principal properties of *congruences of rays*.

§ **2**. In the first place I consider an *arbitrary congruence* of the *field-degree* p and the *sheaf-degree* q. This congruence is represented in Σ_5 by a surface V_2 of O^2_4, which has p points in common with any α-plane (representation of a field of rays) and q points with any β-plane (representation of a sheaf of rays).

§ **3**. Let P be an arbitrary point of V_2. A hyperplane through P cuts V_2 in a curve that has one tangent in P; the tangents in P to V_2 form, therefore, a plane pencil. 2 of the straight lines through P on O^2_4 (images of plane pencils of Σ_3) lie in a linear space R_3 through P; for a ray of Σ_3 lies in 2 plane pencils of a bilinear congruence to which it belongs. The straight lines of O^2_4 through P form accordingly a threedimensional quadratic cone. The plane pencil and the cone lie in the hyperplane touching O^2_4 at P and have therefore two straight lines in common. Consequently:

Any ray of a congruence is intersected by 2 *consecutive rays*, or

On any ray of the congruence there lie 2 *focal points and through any ray of the congruence there pass* 2 *focal planes.*

The identity of the surface of the focal points with that of the focal planes (*the focal surface*) can easily be demonstrated now in Σ_3 (See e.g. STURM, Liniengeometrie II).

§ **4**. The hyperplane R_4 touching O^2_4 at a point P, cuts V_2 in a curve V_1. This I project out of P on a linear space R_3 in R_4.

Of the α- (β-) planes through P two cut a straight line of R_3; they intersect R_3 therefore in the a- and b-lines of a quadratic surface O^2. On O^2 lies the projection k of the curve V_1; k has the a- and b-lines as p- and q-fold secants. The projections of k out of a point O of O^2 on a plane τ of R^3 gives a curve k', which has a p-fold and a q-fold point in the passages A_1 and B_1 of the a- and b-lines through O. The straight line A_1B_1 has only the points A_1 and B_1 in common with k', hence k' is of the order $p+q$. If r is the number of double points of k, hence also of k', the class of k' is:

$$(p+q)(p+q-1)-p(p-1)-q(q-1)-2r=2(pq-r)$$

so that out of A there can be drawn

$$2(pq-r)-2q=2q(p-1)-2r$$

and out of B

$$2(pq-r)-2p=2p(q-1)-2r$$

tangents to k', touching this curve elsewhere.

These numbers are at the same time the numbers of b- and a-lines touching k and also the numbers of the β- and the α-planes through P that have two coinciding points in common with V_2.

Now r is the number of bisecants of V_2 through P or the *axis-degree* (*rank*) of the congruence in consideration, hence:

The focal surface of a congruence of the field-degree p, the sheaf-degree q and the axis-degree r is of the order $2p(q-1)-2r$ and of the class $2q(p-1)-2r$.

§ **5**. I shall now consider the *complete congruence of intersection* of two complexes.

If these are of the order m and n, they have as images the multiplicities which O^2_4 has in common with two hypersurfaces V_4^m and V_4^n. These two hypersurfaces cut each other in a multiplicity V_3^{mn}. The bisecants of V_3^{mn} passing through a point P and cutting a plane π lie in the linear space $R_3 \equiv (P, \pi)$; they are also the bisecants through P of a curve in R_3 which is the complete intersection of a surface of the m^{th} order and a surface of the n^{th} order. The number of these bisecants is $\frac{1}{2}mn(m-1)(n-1)$, hence the bisecants of V_3^{mn} through P form a cone of 3 dimensions and of the order $\frac{1}{2}mn(m-1)(n-1)$. By choosing P on O^2_4 it appears that through this point there pass $mn(m-1)(n-1)$ bisecants of V_3^{mn} that lie on O^2_4, accordingly:

The congruence of intersection of a complex of the m^{th} order and a complex of the n^{th} order has the axis-degree $mn(m-n)(n-1)$.

§ **6**. In order to find the order and the class of the focal surface of the congruence of intersection of two complexes, I pass two α-planes α_1 and α_2 through a point P of O^2_4. The β-plane through a ray a_1 of the plane pencil P, α_1 cuts α_2 in a straight line a_2. A plane γ through a_1 which cuts α_2 in a straight line a^1_2 and has two coinciding points in common with V_3^{mn}, touches the curve which the space $R_3 \equiv (\alpha_1 \alpha_2)$ cuts out of V_3^{mn}; as this curve is the curve of intersection of a surface of the m^{th} order and a surface of the n^{th} order, it has the rank $mn(m+n-2)$; accordingly this is the number of the planes through a_1 intersecting α_2 in a straight line a_2^1 and at the same time touching V_3^{mn}. Between the rays a_2 and a_2^1 belonging to the same ray a_1, there exists a $[mn(m+n-2), mn(m+n-2)]$ correspondence, hence there pass through P $2mn(m+n-2)$ β-planes (and as many α-planes) touching V_3^{mn}, or:

The focal surface of the congruence of intersection of a complex of the m^{th} *and a complex of the* n^{th} *order, has the order and the class* $2mn(m+n-2)$.

Of course this result can also be derived from § 4 and § 5.

Anatomy. — "OMBREDANNE's *Theory of the "lames vasculaires" and the anatomy of the canalis cruralis*". By G. C. HERINGA. (Communicated by Prof. J. BOEKE).

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When adopting the reasonable and general view that the muscular fasciae are to be considered as compressions of loose connective tissue, originated under mechanic influences exerted by the surrounding individual muscles, a conception has been propounded, which brings before our mind the "muscle-compartments" in a clear and comprehensible way, and which is also of great practical value. Still, it would seem that the scientific value of this view may justly be contested. It would seem also that in order to obtain a clear notion of the matter these formations of connective tissue themselves should receive more of our attention than the spaces invested by the fasciae and filled up with muscles.

I would call upon the reader to consider a fascia as a thin layer of undifferentiated connective tissue, bounded on either side by a lamina of fibrillary connective tissue. Further we conceive a blood-vessel running in the middle layer and we imagine, in accordance with the publica opinio, that the two plane faces have, as it were, been attached through a polishing process to this interstitium of connective tissue by the mechanic action of the surrounding muscles, or by other tissues. This hypothesis approaches real facts, for a similar position we observe of the vasa plantaria med. und lat. in the septa intermuscularia pedis; a similar location we observe of the vena jugularis ext. in the fascia superficialis colli, of the aa. meningeae and the sinus durae matris in the hard cerebral membrane, of the vena saphena magna in the fascia lata, of the vasa epigastrica in the fascia transversalis [1]), and finally in a similar way numerous nerves — suffice it to mention only the n. cutaneus femoris lateralis, the branches of the n. femoralis, of the nervi superficialis colli — are running in the fasciae, prior to their ultimate intrusion into the skin.

The instances here enumerated, could easily be increased. They lend support to the roughly phrased conception that, generally

[1]) TESTUT et JACOB, Traité d'Anat. topogr., p. 45.