Mathematics. — "*On analytic functions defined by certain* LAMBERT *series.*" By J. C. KLUYVER.

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The definition of the analytic function was based by WEIERSTRASS on his theory of power-series. From a given analytic expression we deduce an element of the analytic function, that is a power-series converging within a determinate circle, and by the continuation of this element an analytic function is defined existing within the region that is covered by the set of the circles of convergence. One and the same analytic expression in distinct regions may define several functions. So, for instance, TANNERY's series

$$\sum_{n=0}^{n=\infty} \frac{z^{2^n}}{1-z^{2^{n+1}}}$$

for $|z|<1$ will represent the analytic function $\varphi_1(z)=\sum_{k=1}^{k=\infty} z^k = \frac{z}{1-z}$, whereas for $|z|>1$ the expression defines the analytic function $\varphi_2(z)=-\sum_{k=1}^{k=\infty} z^{-k} = -\frac{1}{z-1}$. Both functions, each of them defined in a separate region, can be continued over the whole plane, but manifestly they remain everywhere essentially distinct.

In fact, from the general theory it follows that the concept of an analytic function is not co-extensive with the concept of functionality as expressed by an analytic expression and it is precisely this fundamental idea that, as BOREL repeatedly pointed out, sometimes leads to conclusions which are not always in every respect satisfactory [1]).

BOREL supposes that a given analytic expression $F(z)$ defines a function $\varphi_1(z)$ inside a certain closed curve C and moreover a second function $\varphi_2(z)$ in the region outside C, the singularities of these functions being everywhere-dense on the curve, so that C for both functions constitutes a socalled natural limit. He then shows that the series of polynomials representing $\varphi_1(z)$ under certain conditions remains convergent, absolutely and uniformly, when the variable z along certain radii crosses the boundary C. Otherwise said, it occurs that the value of an analytic expression, coinciding at first with that of the function $\varphi_1(z)$, can be made to change continuously into the value of the function $\varphi_2(z)$ and this possibility more or less seems to be incompatible with the theory, according to which the functions $\varphi_1(z)$ and $\varphi_2(z)$ are wholly unconnected.

[1]) Leçons sur les Fonctions monogènes uniformes d'une variable complexe. Chap. III.

In the present paper I propose to treat two simple examples in which the transformation of $\varphi_1(z)$ into a series of polynomials is not necessary, and that, as I believe, yet give an insight into the tendency of BOREL's remarks.

Let the given analytic expression be the series of LAMBERT

$$F(z)=\sum_{n=1}^{n=\infty} \frac{1}{n^s}\cdot\frac{z^n}{1-z^n},$$

where the exponent s may supposed to be real.

Clearly, whatever be the value of s, we can expand $F(z)$ into an integral series, and as for $|z|<1$ we have

$$\left|\frac{1}{n^s}\cdot\frac{z^n}{1-z^n}\right|<\frac{1}{n^s}|z|^n\cdot\frac{1}{1-|z|},$$

$F(z)$ defines an analytic function $\varphi_1(z)$ inside the circle C of radius unity. However, if $s>1$, we may write

$$F(z)=-\sum_{n=1}^{n=\infty}\left(\frac{1}{n^s}+\frac{1}{n^s}\cdot\frac{\frac{1}{z^n}}{1-\frac{1}{z^n}}\right)=-\zeta(s)-\sum_{n=1}^{n=\infty}\frac{1}{n^s}\cdot\frac{\frac{1}{z^n}}{1-\frac{1}{z^n}},$$

and from $F(z)$ we derive also an integral series in $\frac{1}{z}$, that is a second analytic function $\varphi_2(z)$ existing in the region outside C.

The functions $\varphi_1(z)$ and $\varphi_2(z)$ represented in distinct regions by the same analytic expression satisfy the relation

$$\varphi_1(z)+\varphi_2\left(\frac{1}{z}\right)=-\zeta(s), \qquad (|z|<1)$$

but the main question is, whether either of them is, or is not an analytic continuation of the other. The decision can be based on a transformation of $F(z)$. Corresponding to the rational numbers $\frac{p}{q}$ of the interval (0,1) we can arrange the so-called rational points $a_n = e^{2\pi i\frac{p}{q}}$ on the circle C as a sequence (a_n) and denoting by q the denominator of the rational fraction that corresponds to a_n, it will be seen that we have

$$F(z)=-z\,\zeta(s+1)\sum_{n=}^{n=\infty}\frac{1}{q^{s+1}}\cdot\frac{1}{z-a_n}.$$

This series of fractions represents $\varphi_1(z)$, if $|z| < 1$, $s > 0$, on the other hand it is equal to $\varphi_2(z)$ as soon as $|z| > 1$ and at the same time $s > 1$. We now can apply a theorem due to Goursat [1]) and conclude that the points a_n without exception are singular points of the functions $\varphi_1(z)$ and $\varphi_2(z)$. Hence, as these points form a set dense on C, the continuation of either of the functions across the circle is excluded. [2])

By application of EULER's summation-formula we can calculate the values taken by the functions $\varphi_1(z)$ and $\varphi_2(z)$, when z along the radius approaches one of the singular points. In this way I find in the first place, when z has a positive value $x < 1$, the following asymptotic expression for $\varphi_1(z)$

$$\varphi_1(x) = \frac{1}{\log\frac{1}{x}} \cdot \zeta(s+1) + \left(\log\frac{1}{x}\right)^{s-1} \Gamma(1-s)\,\zeta(1-s) - \tfrac{1}{2}\zeta(s) +$$

$$+ \frac{B_1}{2!}\log\frac{1}{x} \cdot \zeta(s-1) - \frac{B_2}{4!}\left(\log\frac{1}{x}\right)^3 \zeta(s-3) + \frac{B_3}{6!}\left(\log\frac{1}{x}\right)^5 \zeta(s-5) - \ldots.$$

holding for all non-integer values of s.

The result is less simple, when z tends along the radius to the point $e^{2\pi i\frac{p}{q}} = e^{i\beta}$. Putting $z = \varrho e^{i\beta}$, I get for $\varrho < 1$ and supposing again s to be a non-integer [3])

[1]) Bulletin des Sciences Math., t. XI, p. 109. Sur les fonctions à espaces lacunaires.

[2]) This results also from one of the propositions concerning the series $\sum\limits_{n=1}^{n=\infty} b_n \frac{z^n}{1-z^n}$, enunciated in a previous communication (Verslagen en Mededeelingen. XXVIII. p. 269) according to which the continuation of the function across the circle is impossible, as soon as $b_n > 0$ and $\lim\limits_{n=\infty} b_n = 0$.

[3]) For integer values of s the result is obtained by making s tend to the integer limit. So for instance, if s tends to zero, we will find

$$\varphi_1(x) = \frac{C - \log\log\frac{1}{x}}{\log\frac{1}{x}} + \tfrac{1}{4} - \frac{B_1^2}{2\,.\,2!}\log\frac{1}{x} - \frac{B_2^2}{4\,.\,4!}\left(\log\frac{1}{x}\right)^3 - \ldots\ldots,$$

and

$$\lim_{\rho\to 1}\left\{\varphi_1(\varrho e^{i\beta}) - \frac{C - 2\log q - \log\log\frac{1}{\varrho}}{q\log\frac{1}{\varrho}}\right\} = \tfrac{1}{4} - \frac{i}{2q}\sum_{h=1}^{h=q-1} h\cot\frac{h\beta}{2}.$$

The former of these formulae was obtained by SCHLÖMILCH, the latter I deduced in a previous paper: On LAMBERT's series.

$$\varphi_1(\varrho e^{i\beta}) = \frac{1}{q^{s+1}\log\frac{1}{\varrho}} \cdot \zeta(s+1) + q^{s-1}\left(\log\frac{1}{\varrho}\right)^{s-1} \Gamma(1-s)\,\zeta(1-s) - \tfrac{1}{2}\zeta(s) +$$

$$+ \frac{i}{2q^s}\sum_{h=1}^{h=q-1}\zeta\left(s,\frac{h}{q}\right)\cot\frac{h\beta}{2} +$$

$$+\log\frac{1}{\varrho}\cdot\left\{+\frac{B_1}{2!\,q^{s-1}}\zeta(s-1) - \frac{1}{1!\,2^2q^{s-1}}\sum_{h=1}^{h=q-1}\zeta\left(s-1,\frac{h}{q}\right)(D\cot v)_{v=\frac{h\beta}{2}}\right\} +$$

$$+\left(\log\frac{1}{\varrho}\right)^2\cdot\left\{ - \frac{i}{2!\,2^3q^{s-2}}\sum_{h=1}^{h=q-1}\zeta\left(s-2,\frac{h}{q}\right)(D^2\cot v)_{v=\frac{h\beta}{2}}\right\} +$$

$$+\left(\log\frac{1}{\varrho}\right)^3\cdot\left\{-\frac{B_2}{4!\,q^{s-3}}\zeta(s-3) + \frac{1}{3!\,2^4q^{s-3}}\sum_{h=1}^{h=q-1}\zeta\left(s-3,\frac{h}{q}\right)(D^3\cot v)_{v=\frac{h\beta}{2}}\right\} +$$

$$+\left(\log\frac{1}{\varrho}\right)^4\cdot\left\{ + \frac{i}{4!\,2^5q^{s-4}}\sum_{h=1}^{h=q-1}\zeta\left(s-4,\frac{h}{q}\right)(D^4\cot v)_{v=\frac{h\beta}{2}}\right\} +$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

In this formula $\zeta(p,\alpha)$ stands for the function that, if $p > 1$ and $0 < \alpha < 1$, is represented by the series $\sum\limits_{n=0}^{n=\infty}\frac{1}{(\alpha+n)^p}$.

It may be noticed that in both equations the absolute value of the error committed by stopping at any particular stage in the series is always less than a finite multiple of that of the last written term.

In particular we may deduce, supposing $s > 1$,

$$\lim_{x\to 1}\left\{\varphi_1(x) - \frac{1}{\log\frac{1}{x}}\zeta(s+1)\right\} = -\tfrac{1}{2}\zeta(s),$$

$$\lim_{\rho\to 1}\left\{\varphi_1(\varrho e^{i\beta}) - \frac{1}{q^{s+1}\log\frac{1}{\varrho}}\zeta(s+1)\right\} = -\tfrac{1}{2}\zeta(s) + \frac{i}{2q^s}\sum_{h=1}^{h=q-1}\zeta\left(s,\frac{h}{q}\right)\cot\frac{h\beta}{2}.$$

$$\left(\beta = 2\pi\frac{p}{q}\right)$$

Hence, as z approaches along the radius a rational point $e^{2\pi i\frac{p}{q}}$ it is only the real part of the value of the function that increases indefinitely and at all points $e^{2\pi\frac{ip}{q}}$ which correspond to the same denominator q the real parts are ultimately equal.

The function $\varphi_2(z)$ behaves in quite similar manner because of the relation

$$\varphi_1\left(\frac{1}{z}\right) + \varphi_2(z) = -\zeta(s)\ ,\ (|z| > 1)$$

by means of which $\varphi_2(z)$, as soon as z along the radius tends to $e^{2\pi i\frac{p}{q}}$ from the outside of the circle, is expressed in $\varphi_1\left(\frac{1}{z}\right)$.

The rational points on C thus having been recognized as singularities of $\varphi_1(z)$ and $\varphi_2(z)$, we now must turn our attention to other points on the curve, and as such I will consider the points $e^{2\mu i\xi}$, where ξ is a root of an irreducible algebraic equation of degree $\mu>1$ with integer coefficients. Evidently these points $e^{2\mu i\xi}$ which I will call the algebraic points of order μ on C, determine a new enumerable set, everywhere-dense on the circle.

Let $z=\varrho\, e^{2\mu i\xi}$, then it is readily seen that for all values of ϱ

$$\left|\frac{1}{z^n}-1\right|>1 \quad , \text{ if } \cos 2\pi\, n\,\xi<0,$$

$$\left|\frac{1}{z^n}-1\right|>|\sin 2\pi\, n\,\xi| \quad , \text{ if } \cos 2\pi\, n\,\xi>0.$$

Now in the latter case $n\xi$ is an irrational number increasing with the index n, hence there exists an integer k, such that $|n\xi-k|<\frac{1}{2}$. But, as $\cos 2\mu(n\xi-k)=\cos 2\mu n\xi>0$, we must have $|n\xi-k|<\frac{1}{4}$ and $\sin 2\mu\,|n\xi-k|$ being the sine of an acute angle is greater than the angle itself multiplied by $\frac{2}{\pi}$.

Therefore, if $\cos 2\pi n\xi>0$, we may write

$$|\sin 2\pi\, n\,\xi|=\sin 2\pi\,|n\,\xi-k|$$

and

$$\left|\frac{1}{z^n}-1\right|>4n\left|\xi-\frac{k}{n}\right|.$$

Now according to Liouville's known theorem about algebraic numbers, we have

$$\left|\xi-\frac{k}{n}\right|>\frac{1}{Mn^{\mu}},$$

where M is a finite number independent of n and only depending on the coefficients of the equation of which ξ is a root.

In this way we conclude that

$$\left|\frac{1}{z^n}-1\right|>\frac{4}{Mn^{\mu-1}}$$

and consequently that we have for all values of $\varrho=|z|$

$$\left|\frac{1}{n^s}\cdot\frac{z^n}{1-z^n}\right|<\frac{1}{n^s} \quad , \text{ if } \cos 2\pi\, n\xi<0,$$

$$\left|\frac{1}{n^s}\cdot\frac{z^n}{1-z^n}\right|<\frac{M}{4}\cdot\frac{1}{n^{s-\mu+1}} \quad , \text{ if } \cos 2\pi\, n\xi>0.$$

Therefore the series of Lambert F (z) converges absolutely on the radius of the point $e^{2\pi i\xi}$, as soon as $s>\mu$, the convergence being then independent of ϱ and uniform on any segment of the radius. Supposing z to move continuously along that radius, the value of the analytic expression F (z) which for $s<1$ is equal to that of the function $\varphi_1(z)$ changes also continuously into the value of the function $\varphi_2(z)$ as soon as ϱ becomes greater than unity. Besides, if s is taken sufficiently above the number μ, for instance, if we take $s>2\mu-1$, the series obtained by differentiating term-by-term the series F (z) with regard to ϱ in exactly the same way will give the value of $\frac{d\varphi_1(z)}{dz}$ or that of $\frac{d\varphi_2(z)}{dz}$ according to the value of ϱ. In this order of thought we may ascribe to the functions $\varphi_1(z)$ and $\varphi_2(z)$ a common definite value at the point $e^{2\pi i\xi}$, though of course that point is not an ordinary point. Making $\varphi_1(e^{2\pi i\xi})$ and $\varphi_2(e^{2\pi i\xi})$ both equal to the finite limit $\lim\limits_{\rho\to 1}$ F $(\varrho e^{2\pi i\xi})$, we obtain

$$\varphi_1(e^{2\pi i\xi})=\varphi_2(e^{2\pi i\xi})=-\tfrac{1}{2}\zeta(s)+\frac{i}{2}\sum_{n=1}^{n=\infty}\frac{\cot\pi n\xi}{n^s}$$

and the series $\sum\limits_{n=1}^{n=\infty}\frac{\cot\pi n\xi}{n^s}$ will certainly be convergent, if only $s>\mu$.

Hence, we have established a certain connexion between the functions $\varphi_1(z)$ and $\varphi_2(z)$ which according to Weierstrass's theory we must regard as essential distinct and in no wise connected. In fact, we have shown that in this very special case in which the classical continuation by means of power-series is impossible, a new kind of continuation, as complete as could be desired, is furnished by the series of Lambert along the radii of an enumerable infinite set.

The question arises, whether cases exist in which the continuation by means of a series of Lambert is effected along the radii of a set having the power c of the continuum. The answer is in the affirmative, we only want to choose a Lambert series the coefficients of which are decreasing more rapidly. For instance I will consider the series

$$G(z)=\sum_{n=1}^{n=\infty}\frac{1}{n!}\cdot\frac{z^n}{1-z^n}.$$

Again in this new series the coefficients are positive and zero is their common limit, hence according to the proposition mentioned in the footnote on p. 1228, the rational points on the circle C are singularities of the analytic functions $\psi_1(z)$ and $\psi_2(z)$ defined by $G(z)$ inside and outside C.

Again some insight in the behaviour of these functions in the neighbourhood of the singularities is obtained by the application of EULER's summation-formula. Giving in the first place z the positive value $x < 1$, I find

$$\psi_1(x) = \frac{1}{\log \frac{1}{x}} \{\mathrm{Li}(e) - \mathrm{C}\} - \tfrac{1}{2}(e-1) + \frac{\mathrm{B}_1}{2!} \cdot e \log \frac{1}{x} - \frac{\mathrm{B}_2}{4!} \cdot 5e \left(\log \frac{1}{x}\right)^3 +$$

$$+ \frac{\mathrm{B}_3}{6!} \cdot 52\, e \left(\log \frac{1}{x}\right)^5 \cdot \cdot \cdot \cdot \cdot \cdot$$

and the absolute value of the error committed by stopping at any particular stage in the series always will be less than that of the last written term.

Putting then $z = \varrho e^{2\pi i \frac{p}{q}} = \varrho e^{i\beta}$ and making ϱ tend to unity, we will find

$$\lim_{\varrho \to 1} \left\{ \psi_1(\varrho e^{i\beta}) - \frac{1}{\log \frac{1}{\varrho}} \sum_{k=1}^{k=\infty} \frac{1}{(kq)!\, kq} \right\} = -\tfrac{1}{2}(e-1) - \sum_{h=1}^{h=q-1} \left(\frac{h}{q} - \tfrac{1}{2}\right) e^{\cos h\beta + i \sin h\beta}$$

The function $\psi_2(z)$ behaves in the neighbourhood of a singular point in a similar manner because of the relation

$$\psi_2(z) + \psi_1\left(\frac{1}{z}\right) = -(e-1). \qquad (|z| > 1)$$

Now, let ξ be a transcendental number of the interval (0,1) the expansion of which in a continued fraction gives

$$\frac{1}{a_1 +} \frac{1}{a_2 +} \frac{1}{a_3 +} \cdot \cdot \cdot \cdot \frac{1}{a_k +} \cdot \cdot \cdot \cdot,$$

where all integers a_k are less than a given finite number l.

Evidently these numbers ξ, and therefore also the points $e^{2\pi i \xi}$ form a set of power c, the set of points $e^{2\pi i \xi}$ however being not dense on the circle. By the known properties of continued fractions we have, k being an arbitrary integer, $\frac{\mathrm{T}_n}{\mathrm{N}_n}$ the n-th convergent

$$\left|\xi - \frac{k}{n}\right| > \left|\xi - \frac{\mathrm{T}_n}{\mathrm{N}_n}\right| > \left|\frac{\mathrm{T}_{n+2}}{\mathrm{N}_{n+2}} - \frac{\mathrm{T}_n}{\mathrm{N}_n}\right| =$$

$$= \frac{a_{n+2}}{\mathrm{N}_n (a_{n+2} \mathrm{N}_{n+1} + \mathrm{N}_n)} > \frac{1}{2\mathrm{N}_n \mathrm{N}_{n+1}} > \frac{1}{2(a_{n+1}+1)\mathrm{N}_n^2} > \frac{1}{2(l+1)\mathrm{N}_n^2},$$

and as N_n is manifestly always less than $(l+1)^n$, we may write

$$\left|\xi - \frac{k}{n}\right| > \frac{1}{2(l+1)^{2n+1}}.$$

Determining then the integer k by the condition $|n\xi - k| < \frac{1}{2}$ and putting $z = \varrho e^{2\pi i \xi}$, we get by the same reasoning as before

$$\left|\frac{1}{z^n} - 1\right| > 1 \qquad , \text{ if } \cos 2\pi n\xi < 0,$$

$$\left|\frac{1}{z^n} - 1\right| > 4n \left|\xi - \frac{k}{n}\right| > \frac{2n}{(l+1)^{2n+1}} \qquad , \text{ if } \cos 2\pi n\xi > 0,$$

and consequently

$$\left|\frac{1}{n!} \cdot \frac{z^n}{1-z^n}\right| < \frac{1}{n!} \qquad , \text{ if } \cos 2\pi n\xi < 0,$$

$$\left|\frac{1}{n!} \cdot \frac{z^n}{1-z^n}\right| < \frac{(l+1)^{2n+1}}{2n \cdot n!} \qquad , \text{ if } \cos 2\pi n\xi > 0.$$

Hence the series $G(z)$ will converge absolutely on the radius of the point $e^{2\pi i \xi}$ and the convergence will be uniform on any segment of that radius.

Thus then, we have shown that in this case the functions $\psi_1(z)$ and $\psi_2(z)$ are connected at all points of an aggregate of power c and that along the radii of these points the series of LAMBERT $G(z)$ procures a faultless continuation, whereas the analytic continuation necessarily fails [1]).

The elementary examples I discussed show as well as the examples of BOREL that sometimes we are led to regard as a single function a group of distinct analytic functions existing in separate regions. And from the fact that in these cases a non-analytic continuation can be effectuated, the question arises whether a certain extension should not be given to the concept of functionality. BOREL made a step in this direction by developing the theory of a class of non-analytic, monogenic functions existing in a so-called domain of CAUCHY [2]).

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1) As we have $\frac{1}{n!} < \frac{1}{n^s}$ for all values of s, if only n is sufficiently large, we are certain that the series $G(z)$ also furnishes the continuation along the radii of algebraic points of order whatever.

2) Leçons sur les fonctions monogènes uniformes d'une variable complexe. Chap. V.