the liquid moves from a small soap-bubble towards a large one, in the same way water that surrounds the ground grains moves from the fine towards the coarse grains when the state of equilibrium has not yet been reached. So far as I know, E. RAMANN was the first to point out this important influence of the grain-size on the moisture of the soil.[1] In the capillary spaces between the grains the height to which the water rises is also in inverse ratio to the cross-section of those interstitial spaces.

Now it seems to me that the same influence manifests itself in these phenomena of freezing and thawing of the ground.

For as soon as the frost penetrates into the ground, the ground grains become larger in consequence of their water-envelopes getting frozen, and then suck the water to them from the neighbouring, still unfrozen grains; this water freezes again, and thus the diameter of the solid grains gets greater and greater. In the same way the capillary spaces get narrower, so that ground water rises in them. The quantities of water that thus can be retained in the frozen parts of the ground, must be very considerable.

This appears in thawing weather from the muddy state of the ground at the surface, which thaws first. When also the lower layers are thawed, the water that has risen during the frost, can sink away, and return to the ground water.

Plants are not found uprooted through frost until it thaws. This may be explained in this way: when the ground thaws, differences of tension arise directed from below upward, through which the plants that have not yet firmly taken root, are ejected.

[1]) In the third edition of his "Bodenkunde", p. 332, (Berlin 1911).

Mathematics. — *"Two Representations of the Field of Circles on Point-Space."* By Prof. JAN DE VRIES.

(Communicated at the meeting of January 29, 1921).

1. In 1917 these Proceedings (Vol. 19, p. 1130) contained a paper of Dr. K. W. WALSTRA on the representation of the circles of a plane on the points of space. In this representation a pencil of circles is replaced by a point-range, a net of circles by a field of points, and two orthogonal circles are represented by two points that are harmonically separated by a paraboloid of revolution, the points of which are the images of the point-circles of the field of circles.

Lately this representation has been investigated more closely and applied further by Dr. J. SMIT in his thesis entitled: "A Representation of the Field of Circles on Point-Space" (Utrecht 1920). We arrive also at this representation in the following way. Let A be a point outside the plane Φ of the circles c; through c and A a sphere is passed. If we consider its centre as the image of c the representation defined in this way shows all the above mentioned peculiarities.

2. In order to arrive at another representation of the field of circles we transform in the first place the plane Φ by inversion with centre N into a sphere β; the circles c are in this way replaced by circles c' of β. Now we consider the pole C of the plane γ' of c' as the *image* of the circle c. The *point-circles* P of Φ are, evidently, represented by the points P' of β. A *straight line* l of Φ is transformed by the inversion into a circle λ through N, is therefore represented by a point L of the plane ν touching β at N. N is apparently the image of the straight line at infinity of Φ.

3. A *pencil of circles* (c) is transformed by inversion into a "pencil" (c'), i.e. a system of which there passes one circle through any point of β, so that the planes γ' of the circles c' form a pencil, pass therefore through a straight line r'. But then the poles C lie in a straight line r (the polar line of r' with respect to β). Also in this representation a *pencil of circles* is therefore transformed into a *point-range*.

If (c) has the base point B_1 and B_2 also the circles c' pass through two fixed points and the image r of (c) lies outside β.

If on the contrary (c) has two point-circles P_1 and P_2, r' is the intersection of the planes touching β at P'_1 and P'_2 and the image r cuts the sphere.

The image of a *pencil of concentrical circles* is evidently a straight line r through the point N.

A *parabolical pencil of circles* is represented by a tangent of β. Any two circles of such a pencil touch at a point P; the images of two touching circles are therefore joined by a tangent.

4. A *net of circles* $[c]$ is transformed by inversion into a "net" $[c']$; the planes γ' pass through a fixed point S, consequently the images C lie in a *plane* σ, the polar plane of S.

The image of a net with base-point P is the plane touching β at P'.

All the circles c cutting a circle c_0 *at right angles*, form a net $[c]$; to this belong the points P of c_0. As these points may be considered as circles touching c_0, they have their images in the points of contact of the tangents of β meeting in the image C_0. Consequently the net is represented by the polar plane of C_0. The images of two *orthogonal circles* are therefore *harmonically separated* by β. The sphere β plays here the same part as the paraboloid in the above mentioned representation.

All the circles intersecting c_0 *diametrically* also form a net, $[c^*]$. As $[c^*]$ has no circle in common with the net of the circles intersecting c_0 at right angles, the image σ^* is parallel to the plane σ of c_0. To $[c^*]$ belongs also the circle c_0: hence σ^* passes through C_0.

5. An arbitrary conic δ^2 contains the image of a *system* $(c)_2$, with *index two*; for the tangent plane ϱ of a point R' has two points in common with δ and these are images of two circles c through the point R. The system $(c)_2$ belongs to the net $[c]$ which is represented by the plane of δ^2. [1])

If the plane ϱ touching β at R' also touches δ^2, R' is the central projection of a point R of the curve enveloped by $(c)_2$. Now let L be the image of the straight line l in Φ; the enveloping cone of β the vertex of which is L, has four tangent planes ϱ in common with the cone projecting δ out of L. From this it ensues that the system $(c)_2$ is enveloped by a *curve of the fourth order*.

The points of intersection of δ^2 with β are the image of four point-circles belonging to $(c)_2$; the points of intersection of δ^2 with the plane ν represent the two straight lines of $(c)_2$.

[1]) The orthoptical circles of a pencil of conics form a system $(c)_2$. For through a point of the straight line at infinity pass the degenerate circles consisting of l_∞ and the director lines of the two parabolas. The point-circles of the system are found in the double points of the three pairs of lines and in the centra (the orthogonal circle of the net to which $(c)_2$ belongs) having its centre in the point of intersection of the two director lines.

6. A twisted cubic δ^3 is the image of a system $(c)_3$ with *index three*. At such a system we can arrive in the following way. Let us consider three projective pencils of circles (c_1), (c_2), (c_3) in Φ; let c be the circle intersecting the homologous circles c_1, c_2 and c_3 at right angles. The image C of c is the point of intersection of the polar planes γ_1, γ_2, γ_3 of the images C_1, C_2, C_3. These planes revolve round the polar lines r_1, r_2, r_3 of the straight lines r^*_1, r^*_2, r^*_3 containing the images C_1, C_2, C_3; the locus of the point C is accordingly a twisted cubic, δ^3. Apparently we can inversely choose for r_1, r_2, r_3 three arbitrary chords of a given curve δ^3; their polar lines with respect to β define in Φ three projective pencils of circles, which in their turn define the system $(c)_3$ which has δ^3 its image.

7. A plane curve δ^3 is the image of a $(c)_3$ belonging to the net that is represented by the plane δ of δ^3. A tangent plane ϱ of β intersects δ^3 in three points of the straight line $\delta\varrho$; as a second tangent plane to β can be passed through this straight line, the system $(c)_3$ is characterized by the property that the three circles through a point P have another P^* in common. If $\delta\varrho$ is a tangent to β the three circles touch in a point P; this point of contact lies evidently on the orthogonal circle (diametrical circle) of the net. In a special case the orthogonal circle can be replaced by a straight line, containing in this case the centres of the circles of the system $(c)_3$.

8. Let O be the centre of the sphere β. If the images C_1 and C_0 of two circles c_1 and c_0 are such that $\angle OC_0C_1$ is a right angle, c_0 is intersected diametrically by c_1 (§ 4). If C_1 is fixed C_0 remains on the sphere Γ having OC_1 as diameter. This sphere is apparently the image of the twofold infinite system of circles c that are intersected diametrically by the fixed circle c_1. The intersection of two tangent planes of β has two points in common with Γ; hence through two points there generally pass two circles of the system. A pencil of circles contains also two circles of the system.

9. We arrive at *another representation* of the field of circles in the following way. In the plane Φ of the field there are assumed

three arbitrary points K, L, M; the powers of the circle c with respect to these points are considered to be the coordinates x, y, z of a point C with respect to an orthogonal system of axes.

The plane $x=0$ contains the images of the circles passing through K. As a *pencil of circles* (c) sends one circle through K, the image of (c) has one point in common with $x=0$ and is therefore also in this case a *straight line*. As further a pencil (c) has one circle in common with a net $[c]$, a *net* is represented by a *plane*.

A pencil (c) has two *pointcircles*; the locus of the images of the pointcircles P is again a *quadratic surface* Φ^2. We find its equation by making use of the well known relation between the sides of the complete quadrilateral $PKLM$.[1]) Substituting there $KL^2=h$, $LM^2=f$, $MK^2=g$, we find after some reduction,

$$fx^2+gy^2+hz^2+(h-f-g)(xy+hz)+(f-g-h)(yz+fx)+$$
$$+(g-h-f)(zx+gy)+fgh=0.$$

The plane $x=0$ contains only the image of the point-circle K; from this follows that Φ^2 touches the coordinate planes.

Any circle concentrical with the circle KLM, has equal powers relative to K, L and M, is therefore represented by a point of the straight line $x=y=z$; as a concentrical pencil contains only one point-circle at finite distance, Φ^2 must be an *elliptical paraboloid* the diameters of which make equal angles with the three coordinate axes.

If we choose K, L, M in the angular points of an equilateral triangle, so that $f=g=h$, Φ^2 becomes apparently a *paraboloid of revolution*.

1) See e.g. SALMON-FIEDLER, Anal. Geom. des Raumes I (1879) p. 74.