Mathematics. — *"On the Path of a Ray of Light in the Field of Gravitation of a Single Material Centre."* By Prof. W. VAN DER WOUDE. (Communicated by Prof. J. C. KLUYVER).

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1. Starting from the line element of the field of gravitation of a single material centre proposed at the same time by SCHWARZSCHILD [1]) and DROSTE [2]), I wish to demonstrate in this paper:

1. the path of any ray of light is (with allowable neglect) a hyperbola of which the sun is one of the foci; all these light paths have equal major axes;

2. by these geometrical data 8 hyperbolas are defined passing through 2 given points; from physical considerations however it appears easily that only one of the 8 hyperbolas connecting in this way the earth to an arbitrary star, is a path along which the light starting from the star, reaches the earth.

To this I add a remark on the determination of the magnitude of the deviation. [3])

2. The line element of SCHWARZSCHILD—DROSTE has the following expression:

$$ds^2 = \left(1 - \frac{\mu}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{\mu}{r}} - r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right), \quad . \quad . \quad (1)$$

or, after the substitution

$$r = \varrho\left(1 + \frac{\mu}{4\varrho}\right)^2,$$

$$ds^2 = \left\{1 - \frac{\mu}{\varrho\left(1 + \frac{\mu}{4\varrho}\right)^2}\right\} dt^2 - \left(1 + \frac{\mu}{4\varrho}\right)^4 \left\{d\varrho^2 + \varrho^2\left(\sin^2\theta \, d\varphi^2 + d\theta^2\right)\right\} \quad (1')$$

[1]) K. SCHWARZSCHILD: Sitzungsberichte der Kön. Preuss. Akad. der Wissenschaften, 1916.

[2]) J. DROSTE: Het zwaarteveld van een of meer lichamen volgens de theorie van EINSTEIN. Thesis for the doctorate, Leiden, 1916.

[3]) For a prior discussion of the path of light in this field cf: W. DE SITTER: On EINSTEIN's Theory of Gravitation (Monthly Notices R. A. S., vol. LXXVI, p. 717).

A. S. EDDINGTON: Report on the Relativity Theory of Gravitation (p. 53—56).

H. VAN DER LINDEN: La Trajectoire d'un rayon lumineux dans le champ de gravitation d'EINSTEIN-SCHWARZSCHILD (Académie royale de Belgique, Bulletin, t. VI, 1920, p. 90—97).

Here μ is the mass of the centre of gravitation C.

Concerning the "space-coordinates" r (or ϱ), θ, φ, it is assumed that θ and φ have the same values as would be ascribed to them by an observer who made his observations and calculations in the conviction that space is Euclidian; for the rest neither r nor ϱ need be exactly equal to the distance R measured from C on Euclidian suppositions, but r and R or ϱ and R are supposed to be univalent functions of each other, so that $\frac{r}{R}$ and $\frac{\varrho}{R}$ differ so little from 1 that, at least for not too small values of R, $\left(\frac{r}{R}\right)^2$ and $\left(\frac{\varrho}{R}\right)^2$ may be neglected relatively to unity.

The units of length and time are assumed such that the light covers the unit of length in the unit of time, so that we may for instance consider 1 km. and $\frac{1}{3.10^5}$ sec. as those units; finally the mass μ is expressed in gravitation-units. If e.g. we consider the centre of the sun to be the centre of gravitation, we have $\mu = 1{,}47$, and already immediately outside the surface of the sun $\frac{\mu}{\varrho}$ will be very small; in the field outside the sun we may therefore, as a first approximation, neglect the second and higher powers of $\frac{\mu}{\varrho}$ and replace (1′) by

$$ds^2 = \left(1 - \frac{\mu}{\varrho}\right) dt^2 - \left(1 + \frac{\mu}{\varrho}\right) \{ d\varrho^2 + \varrho^2 (sin^2 \theta \, d\varphi^2 + d\theta^2) \} \quad . \quad (2)$$

3. In the space (2) the propagation of light occurs along a minimum line, i. e. along a line for which

$$ds = 0,$$

or for which

$$\left(1 - \frac{\mu}{\varrho}\right) dt^2 = \left(1 + \frac{\mu}{\varrho}\right) \{d\varrho^2 + \varrho^2 (sin^2 \theta \, d\varphi^2 + d\theta^2)\};$$

as the nature of the field makes it at once clear that the path will lie in a "plane" through C, we may here put $\theta = \frac{1}{2}\pi$, so that for the path of the light we have:

$$\left(1 - \frac{\mu}{\varrho}\right) dt^2 = \left(1 + \frac{\mu}{\varrho}\right) (d\varrho^2 + \varrho^2 \, d\varphi^2) \quad . \quad . \quad . \quad . \quad (3)$$

It is *assumed* that even now the path between 2 points A and B in three dimensional space may be found by making the condition that the first variation of the integral:

$$I = \int_A^B dt = \int_A^B \sqrt{\frac{1+\frac{\mu}{\varrho}}{1-\frac{\mu}{\varrho}}(d\varrho^2 + \varrho^2\, d\varphi^2)}$$

must be equal to zero. Neglecting the same quantities as before we may therefore say that we must determine φ as such a function of ϱ that

$$\delta \int_A^B \sqrt{\left(1 + \frac{2\mu}{\varrho}\right)\left[1 + \varrho^2 \left(\frac{d\varphi}{d\varrho}\right)^2\right]}\, d\varrho = 0 \quad . \quad . \quad . \quad (4)$$

In order to solve this problem we point to another, which from a purely analytical point of view is equivalent to it. If in space, now thought to be Euclidian, a planet with the unit of mass moves according to NEWTON's law in the field of gravitation of the sun, according to the principle of least action its course is given by

$$\delta \int \sqrt{2\left(\frac{\mu}{\varrho} + h\right)\left[1 + \varrho^2 \left(\frac{d\varphi}{d\varrho}\right)^2\right]}\, d\varrho = 0,$$

where h is the constant of the living force.

From this it appears that the solution of (4)

$$\varphi = f(\varrho)$$

represents at the same time the orbit of a planet round the sun, where $h = \frac{1}{2}$, and that also the reverse is true.

Now each of these orbits of a planet is a conic section. However it would not be *quite* exact to say that the light path in three dimensional space is a conic section (unless we define a conic section in a non-Euclidian plane by a curve that has the same polar equation as a conic section; in differential geometry, however, the names ellipse and hyperbola are already given to different curves); by means of

$$\varphi = \varphi_1 \quad , \quad \varrho = \varrho_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

it is represented in a Euclidian plane by a conic section, where ϱ_1 and φ_1 are polar coordinates.

It is of some importance to remark that the formulas (5) represent the plane with the line element

$$ds^2 = \left(1 + \frac{\mu}{\varrho}\right)(d\varrho^2 + \varrho^2\, d\varphi^2)$$

conformly on the Euclidian plane as the coefficients of the two fundamental forms are proportional.

If therefore it appears that in the image plane the tangent at B

to the light path AB makes with the radius vector CB a larger or a smaller angle than the straight line AB, it does not only follow from this that indeed according to EINSTEIN's theory out of B the point A will be seen in a different direction from that which was to be expected from the former theories, but also that the numerical value of this deviation may be read directly from the image.

Notwithstanding the objection just mentioned, confusion being excluded, we shall not hesitate henceforth to call the curve $\varphi = f(\varrho)$ in the EINSTEIN-plane a conic section; we only wanted to point out that for the final conclusion an appeal to the characteristic of conform representation is necessary.

4. Let us first consider once more the paths of material points moving with the unit of mass according to NEWTON's law in the field of gravitation of the sun C, thought to be Euclidian, while for all those paths the constant h has the same value; it is already certain that all those paths form a system of conic sections with a common focus C. It is further known that the semi major axis of such a conic section is determined by:

$$a = -\frac{\mu}{h}.$$ [1]

Hence all the conic sections have equal major axes; the sign of the axis indicates whether we have to do with an ellipse, a hyperbola or a parabola.

Applied to the problem in question this means:

The course of any ray of light is a hyperbola of which the sun is one of the foci; the length of the semi major axis is always equal to 2μ ($= \pm$ 3 k.m.).

5. Now we shall determine the path of the light between 2 given points A and B.

With a view to this we describe out of C a circle γ with a radius 2μ and out of A and B two circles touching γ; the points of intersection of γ with AC and the produced part of AC are called resp. A'' and A'; B'' and B' are defined in the same way. Each point of intersection of one of the circles with centre A and one of the circles with centre B forms together with C the foci of a hyperbola through A and B, the major axis of which is 4μ. To begin with we find therefore 8 hyperbolas; let us now examine which of them gives a possible light path between A and B.

If S_4 is a point of intersection of the circles AA'' and BB', we have

[1] Cf. e.g. P. APPELL: Traité de Mécanique rationelle, I p. 393.

$$AC - AS_4 = 4\mu = BS_4 - BC,$$

i.e. A and B lie on different branches of the hyperbola having C and S_4 for foci. This hyperbola is therefore a light path, but not one from A to B; neither is this the case with the two hyperbolas that have a focus in one of the points of intersection of the circles AA' and BB''.

The two circles AA'' and BB'' can have imaginary points of intersection; but it is also possible that these points of intersection are real. If on the latter supposition S_3 is such a point, we have

$$AC - AS_3 = 4\mu = BC - BS_3,$$

i.e. A and B lie on that branch of the hyperbola having C and S_3 for foci, that is not curved towards the sun, hence just on that branch that is not a light path.

If however S_1 and S_2 are the points of intersection of the circles AA' and BB', C and S_1 as well as C and S_2 are the foci of a hyperbola of which the same branch, the branch curved towards C, passes through A and B.

Of the 8 hyperbolas through A and B which have C for a focus and of which the semi major axis is equal to 2μ, there are only two that give a possible light path; indeed, both are light paths, if only the branch on which A and B lie, does not intersect the sun.

Here the question arises:

Let B be a point of the path of the earth and C the centre of the sun; can we place A in such a way that B is reached by 2 rays of light from A?

We call S_2 that point of intersection of the circles AA' and BB' that lies on the same side of AB as C; we put $\angle A'SB = \sigma$. It is clear that now CS_2 is less than CP, if P is the point of intersection of the tangents at A' and B' to γ; hence

$$CS_2 < 4\mu \cot g \tfrac{1}{2} \sigma.$$

The point S_2 lies therefore inside the sun (as $2\mu = \pm 3$) and the hyperbola with C and S_2 for foci will not be a light path connecting A and B, unless perhaps if σ is very small.

We shall therefore assume, in order to give CS_2 as great a value as possible, that A lies on the production of BC at infinite distance. We find

$$CS_2 < B'S_2 = \sqrt{8\mu \times 2BB'} = \pm 6 \times 10^4 \ km,$$

so that even now S_2 lies inside the sun. Accordingly: *It is impossible to see a point in 2 different directions out of a point of the path of the earth.*