Physics. — *"Motion relativated by means of a hypothesis of* A. FÖPPL". By H. ZANSTRA. (Communicated by Prof. P. EHRENFEST).

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§ 1. *The fixation of "inertial systems" in classical mechanics without applying the principle of absolute motion.*

It is well known that in the equations of motion of classical mechanics for a system of n material points:

$$m_\nu \ddot{x}_\nu = X_\nu \qquad m_\nu \ddot{y}_\nu = Y_\nu \qquad m_\nu \ddot{z}_\nu = Z_\nu \quad . \quad . \quad . \quad (1)$$
$$(\nu = 1 \ldots . n.)$$

the position of those points is referred to a rectangular system of co-ordinates being at rest or moving uniformly in absolute space. But when we refer this position to another system of axes that does not move uniformly or rotates with respect to the above mentioned systems, the differential equation of motion assumes a more complicated form. E.g. if the system of axes has an absolute rotation, it is well known that we get on the left side of equation (1) terms of the type of centrifugal and Coriolis forces.

Those systems of co-ordinates in which the equation of motion assumes the simplest form (1), the so-called "inertial systems" [1]), are consequently defined by means of the idea of absolute space. This idea, introduced by NEWTON, was at first retained in the later elaboration of NEWTON's mechanics. NEWTON's contemporary BERKELEY however already gave a criticism of this principle of absolute motion (motion in "absolute space"). The purport of his demonstration is that motion of bodies must be referred to other bodies and not to an absolute space [2]).

In more recent times (± 1870) this question was taken up again, especially by C. NEUMANN, LANGE and MACH [3]). While NEUMANN still supported the idea of absolute motion, later criticism led to the general conviction that motion is relative. The latter being assumed, we have the following two problems:

1. How can inertial systems be fixed without the aid of an "absolute space".

2. Which are, according to this, the equations of motion of mechanics, if we require that absolute co-ordinates do no more occur in them, but exclusively relative co-ordinates (that fix the place of the material points with respect to each other) and their derivatives.

FÖPPL gave a solution of the first problem, which shall be treated in the next § [1]). In connection with this we shall give in § 4 a solution of the second problem. Here already can be remarked that this solution is quite different from the one given by the theory of EINSTEIN, in § 5 we return to this.

[1]) The idea of "inertial systems" was introduced by LANGE he calls them "Inertialsysteme".

[2]) G. BERKELEY. The Principles of Human Knowledge, section 111. e.v.

[3]) E. MACH. Die Mechanik in ihrer Entwickelung. As for LANGE and NEUMANN see § 5. Literature is mentioned in the Enzyklopedie der Math. Wiss. VI 1, p. 30. See moreover H. SEELIGER. Ueber die sogenannte absolute Bewegung. Sitzungsber. der Math. Phys. Klasse der Bayr. Ac. der Wiss. 1906. Bd. 36, p. 85.

2. *The hypothesis of* A. FÖPPL [2]), *by which special inertial systems are fixed.*

From here we will deal with motion in plane instead of space, for the sake of simplicity [3]).

With FÖPPL we assume that the total matter in space consists of a finite number n of material points. For the co-ordinates x', y' in an inertial system we then have the equations:

$$m_\nu \ddot{x}'_\nu = X_\nu \qquad m \ddot{y}'_\nu = Y_\nu \quad . \quad . \quad . \quad . \quad . \quad (2)$$
$$(\nu = 1 \ldots \ldots n)$$

We suppose further that for the quantities on the right side (components of force) the law of reciprocal action holds:

$$\Sigma X = 0 \quad \Sigma Y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3a)$$
$$\Sigma (x' Y - y' X) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3b)$$

The sign Σ includes all n points. This is the case e.g. if the points apply forces upon each other in the direction of the lines that join them.

From (2), (3a) and (3b) follows:

[1]) Some short critical remarks on this and some other solutions shall be given in § 5.

[2]) A. FÖPPL. Vorlesungen über technische Mechanik. VI. Erster Abschnitt. Die relative Bewegung.

[3]) For space we have a quite analogic reasoning. Then the use of vectors can be recommended.

$$\Sigma\, m\, \ddot{x}' = 0 \quad \Sigma\, m\, \ddot{y}' = 0 \quad \ldots \ldots \quad (4a)$$

$$\Sigma\, m\, (x'\, \ddot{y}' - y'\, \ddot{x}') = 0 \quad \ldots \ldots \quad (4b)$$

and after integration:

$$\Sigma\, m\, \dot{x}' = T_1 \quad \Sigma\, m\, \dot{y}' = T_2 \quad \ldots \ldots \quad (5a)$$

$$\Sigma\, m\, (x'\, \dot{y}' - y'\, \dot{x}') = R, \quad \ldots \ldots \quad (5b)$$

the 3 constants T_1, T_2 and R still being quite arbitrary. In analogic way we get in space 6 constants T_1, T_2, T_3, R_1, R_2, R_3. It is always possible to choose an inertial system in this way that 3 of these 6 quantities vanish, more of them cannot vanish without a special hypothesis.

The *hypothesis of* FÖPPL is:

There are inertial systems, for which all six constants T_1, T_2, T_3, R_1, R_2, R_3 vanish together.

Consequently for the plane:

$$T_1 = 0 \quad T_2 = 0 \quad R = 0 \quad \ldots \ldots \quad (6)$$

or also, after (5a) and (5b):

$$\Sigma\, m\, \dot{x}' = 0 \quad \Sigma\, m\, \dot{y}' = 0. \quad \ldots \ldots \quad (7a)$$

$$\Sigma\, m\, (x'\, \dot{y}' - y'\, \dot{x}') = 0 \quad \ldots \ldots \quad (7b)$$

To these special inertial systems belong also those for which moreover $\Sigma\, mx' = 0$ and $\Sigma\, my' = 0$, the origin thus coinciding permanently with the centre of gravity of the system of points, such a system is called by FÖPPL a *principal system of reference* (Hauptbezugssystem) [1].

Such a principal system $X_\beta\, Y_\beta$ can be constructed as follows: Take a system of axes $X_\alpha\, Y_\alpha$ of which the origin coincides permanently with the centre of gravity of the system of points, the axis of X passing permanently through one of the material points. Calculate $\Sigma\, mr^2 \theta$ in $X_\alpha\, Y_\alpha$. Take a second system of co-ordinates $X_\beta\, Y_\beta$ with its origin also permanently in the centre of gravity and give it in $X_\alpha\, Y_\alpha$ a velocity of rotation $\omega = -\dfrac{\Sigma\, mr^2\, \dot{\theta}}{\Sigma\, mr^2}$. In this way a principal system, consequently an inertial system, has been fixed without the aid of an "absolute space".

[1]) The hypothesis of FÖPPL in its original form is: for an inertial system with origin in the centre of gravity the total moment of momentum vanishes permanently.

3. *Motion relativated without the aid of the hypothesis of* FÖPPL.

First we will give a solution of the second problem of § 1, *without applying the hypothesis of* FÖPPL.

We assume the same as in § 2, so we start from the differential equation (2), for which hold the relations (4a) and (4b), we assume moreover that the forces, of which X_ν and Y_ν are the components only depend on the relative position of the points (e.g. Newton force).

Take now a new system of axes XY with its origin permanently in an arbitrary material point, which we will call point 1, and with its axis of X permanently through a second arbitrary-point 2 of the system of points. The new co-ordinates x and y are now relative co-ordinates as meant in § 1. After transformation of the former co-ordinates to the new system of axes [1]), the equations of motion (2) pass, considering (4a) and (4b), into:

$$\left.\begin{aligned} \ddot{x}_\nu - \dot{w}\, y_\nu - w^2\, x_\nu - 2\, w\, \dot{y}_\nu &= \frac{X_\nu}{m_\nu} - \frac{X_1}{m_1} \\ \ddot{y}_\nu + \dot{w}\, x_\nu - w^2\, y_\nu + 2\, w\, \dot{x}_\nu &= \frac{Y_\nu}{m_\nu} - \frac{Y_1}{m_1} \end{aligned}\right\} \quad \ldots \quad (8a)$$

in which w is given by:

$$a\, \dot{w} + 2\, bw + c = 0 \quad \ldots \ldots \quad (8b)$$

a, b and c are functions of the x, y, $\dot{x}$, $\dot{y}$ and $\ddot{x}$, $\ddot{y}$ of the n points

$$\left.\begin{aligned} a &= \Sigma\, m\, \Sigma\, m\, (x^2 + y^2) - (\Sigma\, mx)^2 - (\Sigma\, my)^2. \\ b &= \Sigma\, m\, \Sigma\, m\, (x\dot{x} + y\dot{y}) - \Sigma\, mx\, \Sigma\, m\dot{x} - \Sigma\, my\, \Sigma\, m\dot{y}. \\ c &= \Sigma\, m\, \Sigma\, m\, (x\ddot{y} - y\ddot{x}) - (\Sigma\, mx\, \Sigma\, m\ddot{y} - \Sigma\, my\, \Sigma\, m\ddot{x}). \end{aligned}\right\} \quad (8c)$$

The sign Σ includes all n points.

There are $2n$—3 co-ordinates x_ν, y_ν and according to this $2n$—3 equations (8a) of the second order, the auxiliary quantity w occurs in one equation (8b) of the first order in w. So (8a) and (8b) form together a system of order $2\,(2n-3) + 1 = 4n-5$. After elimination of w there remain $2n$—3 co-ordinates, so that e.g. the system can be reduced to $2n$—4 *equations of the second and one of the third order*: In these equations only the relative co-ordinates, their derivatives and the quantities X_ν Y_ν occur. As we supposed the components $X_\nu Y_\nu$ dependent on the relative position of the material points only, the problem is solved.

[1]) See § 5. g.

4. *Motion relativated with the aid of the hypothesis of* FÖPPL.

In this § we shall give a much simpler solution of the second problem of § 1 by making the same suppositions as in the preceding §, *moreover making use of the hypothesis of* FÖPPL. Instead of (4*a*) and (4*b*) we take the equations (7*a*) and (7*b*), but further we proceed in exactly the same way.

After transformation to the new, relative co-ordinates [1]), the equations of motion (2) pass, considering (7*a*) and (7*b*), into:

$$\left.\begin{aligned}\ddot{x}_\nu - \dot{w} y_\nu - w^2 x_\nu - 2 w \dot{y}_\nu &= \frac{X_\nu}{m_\nu} - \frac{X_1}{m_1}\\ \ddot{y}_\nu + \dot{w} x_\nu - w^2 y_\nu + 2 w \dot{x}_\nu &= \frac{Y_\nu}{m_\nu} - \frac{Y_1}{m_1}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (9a)$$

in which

$$w = -\frac{\Sigma m \, \Sigma m (x\dot{y} - y\dot{x}) - (\Sigma mx \, \Sigma m\dot{y} - \Sigma my \, \Sigma m\dot{x})}{\Sigma m \, \Sigma m (x^2 + y^2) - (\Sigma mx)^2 - (\Sigma my)^2} \quad . \quad (9b)$$

The sign Σ includes all n points.

The equations (9*a*) are the desired equations of motion in relative co-ordinates, if for w we substitute the value (9*b*). After this substitution we get $2n$—3 *independent differential equations of the second order*.

Without introducing FÖPPL's hypothesis we came in § 3 to $2n$—4 equations of the second and one of the third order. Owing to the hypothesis of FÖPPL we have found in this § $2n$—3 equations of the second order. For this reason the equations (9*a*) and (9*b*), we found here, are preferable to the equations (8*a*), (8*b*) and (8*c*).

5. *Remarks.*

a. LANGE's method of trial bodies [2]) gives an experimental way of finding inertial systems. He does not discuss however their connection with the total of matter in space.

b. NEUMANN [3]) and afterwards BOLTZMANN [4]) try to do this by referring the place of the material points to the principal axes of inertia of the total system. They do not give differential equations. A further consideration of this question will bring the conviction,

[1]) See § 5, g.

[2]) L. LANGE. Geschichtliche Entwickelung des Bewegungsbegriffs.

[3]) C. NEUMANN. Ueber die Prinzipien der Galilei-Newtonschen Mechanik.

[4]) BOLTZMANN. Vorlesungen über die Prinzipien der Mechanik. Leipzig 1904. II, p. 333.

that the differential equations holding in such a system must be of a different form from NEWTON's. On the other hand the system of FÖPPL does not require any alteration of NEWTON's differential equations, but only a definite value of some integration constants (§ 2).

c. The problems studied by KOLKMEIJER [1]) are in many respects more extensive. So he eliminates also absolute time. In this connection it is sufficient to state, that he does not find equations of the type of § 4, because he does not make use of FÖPPL's hypothesis.

d. EINSTEIN gives in his theory of relativity a way of relativating, founded on quite different principles from those held in this essay. This is connected with the fact that he wants to relativate not only mechanical but also electromagnetical and optical phenomena.

e. In connection with an experiment, made by FÖPPL [2]) and a remark by FREUNDLICH [3]), the following thought-experiment may be discussed: At the north-pole of the earth the pendulum-experiment of FOUCAULT is made. Under the pendulum a heavy flywheel with vertical axis of rotation has been mounted. Problem: Does the pendulum's motion alter, when we revolve the flywheel? FÖPPL's hypothesis and our equations give us the following answers: With respect to the principal system remains: 1°. the rotation of the pendulum's plane permanently = 0; 2°. the sum of moments of momentum of flywheel and earth = a constant, also this sum for the rest of the bodies of the universe, because the total sum has to remain = 0. Consequently the rotation of the pendulum's plane with respect to the rest of the bodies (the fixed stars) does not alter (it does alter with respect to the earth, the rotation of the latter having undergone some change).

f. From the point of view of NEWTON's mechanics can be said: 1°. If only two celestial bodies were in universe, it were possible that they moved round each other at a constant distance. 2°. A liquid mass, supposed to be the only body in space, can be flattened by centrifugal forces, though no relative motion of its particles is observed. With the view taken here, which is based on FÖPPL's hypothesis, this is impossible.

g. For the transformations of § 4 we can start from a principal

[1]) N. H. KOLKMEYER. Eliminatie van de begrippen assenstelsel, lengte en tijd uit de vergelijkingen voor de planetenbewegingen. Dissertation Amsterdam 1915.

[2]) A. FÖPPL. Sitzungsber. der math.-phys. Klasse der Bayr. Ac. der Wiss. 1904. Bd. 34, p. 3.

[3]) ERWIN FREUNDLICH. Die Grundlagen der EINSTEINschen Gravitationstheorie. Berlin 1916, p. 27.

system, the origin of which is consequently situated in the centre of gravity of the system of points, while the total moment of momentum continuously vanishes (§ 2).

$$m_\nu \ddot{x}'_\nu = X_\nu \qquad m_\nu \ddot{y}'_\nu = Y_\nu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\Sigma\, m\dot{x}' = 0 \quad \Sigma\, m\dot{y}' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10a)$$

$$\Sigma\, m\,(x'\dot{y}' - y'\dot{x}') = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (10b)$$

First we pass to a second system of axes, with its origin permanently in point 1 of the system of points and which is continuously parallel to the principal system, for this second system holds:

$$\ddot{x}_\nu = \frac{X_\nu}{m_\nu} - \frac{X_1}{m_1} \qquad \ddot{y}_\nu = \frac{Y_\nu}{m_\nu} - \frac{Y_1}{m_1} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\Sigma\, m\, \Sigma\, m\,(x\dot{y} - y\dot{x}) - (\Sigma\, mx\, \Sigma\, m\dot{y} - \Sigma\, my\, \Sigma\, m\dot{x}) = 0 \quad . \quad (12)$$

(10*b*), becomes (12) because according to (10*a*) $x'_1\, \Sigma m = -\Sigma mx$, $y'_1 \Sigma m = -\Sigma\, my$.

If we take a third system of axes with its origin also in point 1 and its axis of X permanently through point 2, and indicate the velocity of rotation of the third system of axes with respect to the second by one w, the equations (11) and (12) pass after transformation into the final equations (9*a*) and (*b*). Now w can be considered as an auxiliary quantity, that can be substituted from (9*b*) into (9*a*).

For the transformations of § 3 the first system of axes can also be chosen with its origin in the centre of gravity of the system of points, the calculations are quite analogous, however more complicated.

Physics. — *"Space-time symmetry. I. General Considerations"*. By N. H. KOLKMEIJER. Communication N°. 7*a* from the Laboratory of Physics and Physical Chemistry of the Veterinary College at Utrecht. (Communicated on behalf of Prof. W. H. KEESOM, Director of the Laboratory, by Prof. H. KAMERLINGH ONNES).

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§ 1. *Introduction.* In the literature of later years some propositions are found for changes in the original atom-model of RUTHERFORD-BOHR. These propositions are founded on different considerations. So LEWIS [1]) and LANGMUIR [2]) were led by considerations on the chemical structure. SOMMERFELD [3]) was brought to his combinations of ellipses by investigations of the cause of the fact that the defect of charge of the nucleus is not a whole number for the L-series. BORN, LANDÉ and MADELUNG [4]) again studied the absolute dimensions of the elementary cells of crystals, the transformations of energy, and especially the compressibility. By these considerations they were led (partly in cooperation) to the invention and nearer inspection of cubical models analogous to those of LEWIS and LANGMUIR.

Evidently BORN, LANDÉ and MADELUNG especially felt the necessity of a change in the considerations on symmetry that were valid until now, because they considered moving systems. On the same ground I felt necessitated to introduce some new symmetry-elements, in which time also plays a role [5]). It now seemed desirable to consider the space-time-symmetry more systematically than could be done in Communication N°. 4. In this paper the way to attack this problem will be indicated.

§ 2. *Restriction to a definite kind of operations.* A symmetry operation will further on be denoted by Δ, a complex symmetry

1) G. N. LEWIS, Journ. of the Amer. Chem. Soc. 38 (1916) p. 762.
2) I. LANGMUIR, Journ. of the Amer. Chem. Soc. 41 (1919) p. 868.
3) A. SOMMERFELD, Physik. ZS. 19 (1918) p. 297.
4) M. BORN and A. LANDÉ, Verh. d. D. Phys. Ges. 20 (1918) p. 210.
M. BORN, Verh. d. D. Phys. Ges. 20 (1918) p. 230.
A. LANDÉ, Sitz.-Ber. d. Berl. Akad. 1919 p. 101.
Verh. d. D. Phys. Ges. 21 (1919) p. 2, 644, 653.
ZS. f. Phys. 2 (1920) p. 83.
E. MADELUNG and A. LANDÉ, ZS. f. Phys. 2 (1920) p. 230.
5) N. H. KOLKMEIJER, Comm. N°. 4, These Proceedings 23 (1920) p. 120.