system, the origin of which is consequently situated in the centre of gravity of the system of points, while the total moment of momentum continuously vanishes (§ 2).

$$m_\nu \ddot{x}'_\nu = X_\nu \qquad m_\nu \ddot{y}'_\nu = Y_\nu \quad \ldots \ldots \quad (2)$$

$$\Sigma\, m x' = 0 \quad \Sigma\, m y' = 0 \quad \ldots \ldots \quad (10a)$$

$$\Sigma\, m\,(x'\dot{y}' - y'\dot{x}') = 0 \quad \ldots \ldots \quad (10b)$$

First we pass to a second system of axes, with its origin permanently in point 1 of the system of points and which is continuously parallel to the principal system, for this second system holds:

$$\ddot{x}_\nu = \frac{X_\nu}{m_\nu} - \frac{X_1}{m_1} \qquad \ddot{y}_\nu = \frac{Y_\nu}{m_\nu} - \frac{Y_1}{m_1} \quad \ldots \ldots \quad (11)$$

$$\Sigma\, m\, \Sigma\, m\,(x\dot{y} - y\dot{x}) - (\Sigma\, m x\, \Sigma\, m\dot{y} - \Sigma\, m y\, \Sigma\, m\dot{x}) = 0 \;. \quad (12)$$

(10*b*), becomes (12) because according to (10*a*) $x'_1\, \Sigma m = -\Sigma m x$, $y'_1 \Sigma m = -\Sigma\, m y$.

If we take a third system of axes with its origin also in point 1 and its axis of X permanently through point 2, and indicate the velocity of rotation of the third system of axes with respect to the second by one w, the equations (11) and (12) pass after transformation into the final equations (9*a*) and (*b*). Now w can be considered as an auxiliary quantity, that can be substituted from (9*b*) into (9*a*).

For the transformations of § 3 the first system of axes can also be chosen with its origin in the centre of gravity of the system of points, the calculations are quite analogous, however more complicated.

Physics. — "*Space-time symmetry*. I. *General Considerations*". By N. H. Kolkmeijer. Communication N°. 7*a* from the Laboratory of Physics and Physical Chemistry of the Veterinary College at Utrecht. (Communicated on behalf of Prof. W. H. Keesom, Director of the Laboratory, by Prof. H. Kamerlingh Onnes).

(Communicated at the meeting of December 18, 1920).

§ 1. *Introduction*. In the literature of later years some propositions are found for changes in the original atom-model of Rutherford-Bohr. These propositions are founded on different considerations. So Lewis [1]) and Langmuir [2]) were led by considerations on the chemical structure. Sommerfeld [3]) was brought to his combinations of ellipses by investigations of the cause of the fact that the defect of charge of the nucleus is not a whole number for the L-series. Born, Landé and Madelung [4]) again studied the absolute dimensions of the elementary cells of crystals, the transformations of energy, and especially the compressibility. By these considerations they were led (partly in cooperation) to the invention and nearer inspection of cubical models analogous to those of Lewis and Langmuir.

Evidently Born, Landé and Madelung especially felt the necessity of a change in the considerations on symmetry that were valid until now, because they considered moving systems. On the same ground I felt necessitated to introduce some new symmetry-elements, in which time also plays a role [5]). It now seemed desirable to consider the space-time-symmetry more systematically than could be done in Communication N°. 4. In this paper the way to attack this problem will be indicated.

§ 2. *Restriction to a definite kind of operations*. A symmetry operation will further on be denoted by Δ, a complex symmetry

1) G. N. Lewis, Journ. of the Amer. Chem. Soc. 38 (1916) p. 762.
2) I. Langmuir, Journ. of the Amer. Chem. Soc. 41 (1919) p. 868.
3) A. Sommerfeld, Physik. ZS. 19 (1918) p. 297.
4) M. Born and A. Landé, Verh. d. D. Phys. Ges. 20 (1918) p. 210.
M. Born, Verh. d. D. Phys. Ges. 20 (1918) p. 230.
A. Landé, Sitz.-Ber. d. Berl. Akad. 1919 p. 101.
Verh. d. D. Phys. Ges. 21 (1919) p. 2, 644, 653.
ZS. f. Phys. 2 (1920) p. 83.
E. Madelung and A. Landé, ZS. f. Phys. 2 (1920) p. 230.
5) N. H. Kolkmeijer, Comm. N°. 4, These Proceedings 23 (1920) p. 120.

operation [1]) by $\Delta\Delta$. SCHOENFLIES [2]) and his predecessors only consider such Δ's that an application of them to a point A produces a point B, the coordinates of which are found from those of A by a linear orthogonal substitution. By these operations the distance between two points therefore does not change.

It seems natural to introduce for space-time symmetry-operations too the restriction that an application of them to two four-dimensional *xyz-ict*-points (or rather $x^1x^2x^3x^4$-points) A and B does not change the four-dimensional distance $A\,B$.

In the first place we thus limit our considerations to those Δ's the algebraic representation of which is a linear orthogonal four dimensional substitution and to corresponding symmetry-elements.

Secondly, (as was also done by SCHOENFLIES and his predecessors in an analogous sense) we exclude those Δ's, the repeated application of which to a point A gives an infinite number of points at the same time within a finite space or within a finite time-interval at the same place.

Thirdly it will prove desirable to introduce still one restriction, which has not its analogue in the three-dimensional problem. From the algebraic substitution mentioned we see, that x^1, x^2, x^3 of the new point B depend on x^4 of the original point A. Thus, application of the Δ's in question to A gives a point B that is displaced in the course of time to an infinite distance even when A remains on the spot. This fact is an objection against the consideration of such a Δ; an objection however that may be avoided by considering only the final result of subsequent applications of more than one Δ of the kind mentioned to a point A, of a $\Delta\Delta$ therefore.

So we limit ourselves to the consideration of such Δ's, the application of which to a point A gives a point B with a world-line parallel with the x^4-axis, when the world-line of A has that direction.

In the next §§ we shall see to which kind of Δ's we are led by this restriction.

§ 3. *Geometrical meaning and analytical indication of one of the kinds of operations considered.* In a R_4 with coordinates x^1, x^2, x^3 and $x^4 = ict$, R_3 be an arbitrary linear space of three dimensions. We shall call R_3 a s y m m e t r y - s p a c e [3]) (symbol $\mathfrak{r}$) when the corresponding operation (symbol $\mathfrak{R}$, name space-time-reflection) changes

[1]) In the same sense as f.i. a rotatory-reflection is a $\Delta\Delta$.

[2]) A. SCHOENFLIES. Krystallsysteme und Krystallstructur, Leipzig 1891.

[3]) This name has already been used by P. H. SCHOUTE, Verh. Kon. Ak. Amst. Eerste Sectie II 7 (1894) p. 16.

a point $A(x_1^1, x_1^2, x_1^3, x_1^4)$ into a point $B(x_2^1, x_2^2, x_2^3, x_2^4)$ that is geometrically to be found in the following way.

Draw through A a perpendicular to $\mathfrak{r}$ and measure the length of that perpendicular on the other side of its point of intersection with $\mathfrak{r}$. The end-point of this stretch is the point B.

When $\mathfrak{R}$ is given by the length l_1 of the perpendicular from the origin of R_4 on $\mathfrak{r}$ and its direction cosines $\varphi_1^1, \varphi_1^2, \varphi_1^3$ and φ_1^4, where

$$\varphi_1^{1^2} + \varphi_1^{2^2} + \varphi_1^{3^2} + \varphi_1^{4^2} = 1, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

then we find by substitution of one of the four indices 1, 2, 3 and 4 for n

$$x_2^n = x_1^n + 2\,\varphi_1^n\,(l_1 - \varphi_1^1\,x_1^1 - \varphi_1^2\,x_1^2 - \varphi_1^3\,x_1^3 - \varphi_1^4\,x_1^4) \quad . \quad . \quad (2)$$

When in a three-dimensional $x^1x^2x^3$-system we consider a plane V through the origin, the direction cosines of its normal being in the ratio φ_1^1, φ_1^2 and φ_1^3, then the points A' and B', corresponding in this system to A and B in the four-dimensional system, are lying on the same perpendicular to V, $x_2^n - x_1^n$ being proportional to φ_1^n for the values $n = 1, 2, 3$.

When the distances from A' and B' to V are denoted by $-y_m$ with $m = 1$ and 2 resp., then we have

$$y_m = \frac{\varphi_1^1\,x_m^1 + \varphi_1^2\,x_m^2 + \varphi_1^3\,x_m^3}{\sqrt{1-\varphi_1^{4^2}}} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and therefore

$$\left.\begin{aligned} y_2 &= y_1 + 2\sqrt{1-\varphi_1^{4^2}}\,(l_1 - \sqrt{1-\varphi_1^{4^2}}\,y_1 - \varphi_1^4\,x_1^4) \\ x_2^4 &= x_1^4 + 2\,\varphi_1^4\,(l_1 - \sqrt{1-\varphi_1^{4^2}}\,y_1 - \varphi_1^4\,x_1^4) \end{aligned}\right\} \quad . \quad . \quad (4)$$

Thus the distance from B' to V and the new value of *ict* are evidently found from the values of these quantities for the point-incident A by drawing in a two-dimensional yx^4-system a line in such a way, that the perpendicular to it from the origin has a length l_1 and forms with the x^4 axis an angle with a cosine $= \varphi_1^4$ and by reflecting the point A'' with the coordinates x_1^4 and y_1 in that line. The coordinates of the point B'' thus found give the new value of the time and the new distance to V in the three-dimensional figure.

§ 4. *Each space-time-symmetry-operation of the considered kind may be regarded as a complex symmetry-operation of space-time-reflections.*

The Δ's treated by SCHOENFLIES and predecessors, reflection in a plane, inversion about a centre, translation, rotation through $\frac{2\pi}{2}, \frac{2\pi}{3}, \frac{2\pi}{4}$ and $\frac{2\pi}{6}$, 2-, 3-, 4- and 6-al screws, 2-, 3-, 4- and 6-al rotatory-reflections and gliding reflection may all be regarded as complex reflections in one or more planes [1]). This is a consequence of the linear-orthogonal character of the substitution by which only congruent and symmetrical figures are possible.

For the same reason we must also assume that each imaginable space-time-Δ of the considered kind may be considered as a complex reflection in one or more symmetry-spaces. This shows at once the way in which each space-time-symmetry-element can be found.

§ 5. *General formula for the coordinates of the point B, found from the point A by application of an arbitrary complex space-time-symmetry-operation of the above considered kind.*

$$x_m{}^n = x_0{}^n + 2 \sum_{k=1}^{k=m} \{ (l_k - \varphi_k{}^1\, x_0{}^1 - \varphi_k{}^2\, x_0{}^2 - \varphi_k{}^3\, x_0{}^3 - \varphi_k{}^4\, x_0{}^4) \, [\varphi_k{}^n +$$

$$+ \sum_{l=k+1}^{l=m} \varphi_l{}^n \Big((l,k) + \Sigma\, (l,p)\,(p,k) + \Sigma\, (l,p)\,(p,q)\,(q,k) + \ldots \Big)] \} \quad . \; . \; (5)$$

where (p, q) has been written for

$$-2\,(\varphi_p{}^1\,\varphi_q{}^1 + \varphi_p{}^2\,\varphi_q{}^2 + \varphi_p{}^3\,\varphi_q{}^3 + \varphi_p{}^4\,\varphi_q{}^4),$$

while we must take $l > p > q > k$ etc.

§ 6. *We can derive all space-time-symmetry-operations by simply combining all space-symmetry-operations that have been mentioned without the limiting to 2-, 3-, 4- or 6-al axes with time-symmetry-operations.* The linear orthogonal substitution, expressed by formula (5) will have a scheme of coefficients:

$$\left.\begin{matrix} {}_1a_1 & {}_1a_2 & {}_1a_3 & {}_1a_4 & {}_1a_5 \\ {}_2a_1 & {}_2a_2 & {}_2a_3 & {}_2a_4 & {}_2a_5 \\ {}_3a_1 & {}_3a_2 & {}_3a_3 & {}_3a_4 & {}_3a_5 \\ {}_4a_1 & {}_4a_2 & {}_4a_3 & {}_4a_4 & {}_4a_5 \end{matrix}\right\}$$

while these coefficients are connected by the following relations:

$$\left.\begin{aligned} {}_1a_1{}^2 + {}_2a_1{}^2 + {}_3a_1{}^2 + {}_4a_1{}^2 &= 1 \\ {}_1a_2{}^2 + {}_2a_2{}^2 + {}_3a_2{}^2 + {}_4a_2{}^2 &= 1 \\ {}_1a_3{}^2 + {}_2a_3{}^2 + {}_3a_3{}^2 + {}_4a_3{}^2 &= 1 \\ {}_1a_4{}^2 + {}_2a_4{}^2 + {}_3a_4{}^2 + {}_4a_4{}^2 &= 1 \end{aligned}\right\} (6) \text{ and } \left.\begin{aligned} {}_1a_1\,{}_1a_2 + {}_2a_1\,{}_2a_2 + {}_3a_1\,{}_3a_2 + {}_4a_1\,{}_4a_2 &= 0 \\ {}_1a_1\,{}_1a_3 + {}_2a_1\,{}_2a_3 + {}_3a_1\,{}_3a_3 + {}_4a_1\,{}_4a_3 &= 0 \\ {}_1a_1\,{}_1a_4 + {}_2a_1\,{}_2a_4 + {}_3a_1\,{}_3a_4 + {}_4a_1\,{}_4a_4 &= 0 \\ {}_1a_2\,{}_1a_3 + {}_2a_2\,{}_2a_3 + {}_3a_2\,{}_3a_3 + {}_4a_2\,{}_4a_3 &= 0 \\ {}_1a_2\,{}_1a_4 + {}_2a_2\,{}_2a_4 + {}_3a_2\,{}_3a_4 + {}_4a_2\,{}_4a_4 &= 0 \\ {}_1a_3\,{}_1a_4 + {}_2a_3\,{}_2a_4 + {}_3a_3\,{}_3a_4 + {}_4a_3\,{}_4a_4 &= 0 \end{aligned}\right\} (7)$$

1) (Note added during translation). See f.i. C. VIOLA N. Jahrb. f. Miner., Geol. und Pal. Beil. Bd. 10 p. 495 1896. G. WULFF Zs f. Kryst. u. Miner. 27 p. 556 1896.

Because of the third restriction introduced in § 2 we must suppose ${}_1a_4$, ${}_2a_4$ and ${}_3a_4$ to equal zero. Substituting this in (6) and (7) we find:

$${}_4a_4{}^2 = 1 \quad {}_4a_1 = 0 \quad {}_4a_2 = 0 \quad {}_4a_3 = 0 \quad . \; . \; . \; . \; . \; (8)$$

while (6) and (7) are then reduced to:

$$\left.\begin{aligned} {}_1a_1{}^2 + {}_2a_1{}^2 + {}_3a_1{}^2 &= 1 \\ {}_1a_2{}^2 + {}_2a_2{}^2 + {}_3a_2{}^2 &= 1 \\ {}_1a_3{}^2 + {}_2a_3{}^2 + {}_3a_3{}^2 &= 1 \end{aligned}\right\} \; . \; (9) \text{ and } \left.\begin{aligned} {}_1a_1\,{}_1a_2 + {}_2a_1\,{}_2a_2 + {}_3a_1\,{}_3a_2 &= 0 \\ {}_1a_1\,{}_1a_3 + {}_2a_1\,{}_2a_3 + {}_3a_1\,{}_3a_3 &= 0 \\ {}_1a_2\,{}_1a_3 + {}_2a_2\,{}_2a_3 + {}_3a_2\,{}_3a_3 &= 0 \end{aligned}\right\} \; . \; (10)$$

Equations (8) say, that the transformed time depends on the time only, and that the transformed space-coordinates are dependent of the original ones only. Equations (9) and (10) show, that this last transformation is linear orthogonal. By this we have proved the proposition stated at the beginning of this §. We need therefore only apply equation (5) for values of $\varphi_m{}^4 = 0$ viz. for a pure space-transformation and of $\varphi_m{}^4 = 1$ viz. for a pure time-transformation.

§ 7. *Meaning of the cases* $\varphi_m{}^4 = 0$ *and* $\varphi_m{}^4 = 1$. A $\mathfrak{R}$ with $\varphi_m{}^4 = 0$ is nothing else than a reflection. (Symbol $\mathfrak{S}$, symbol of the symmetry-plane $\mathfrak{s}$).

It might seem interesting to derive all imaginable space-Δ's by investigating which combinations of $\mathfrak{S}$'s when considered as complex Δ, are compatible with the restrictions 1 and 2 of § 2 [1]). A point of consideration could be whether the order of application of the reflections in the $\Delta\Delta$ should be chosen arbitrarily or not. In the first case [2]), we find, that each $\Delta\Delta$ may be regarded as a combination of those already used by SCHOENFLIES and predecessors, but we might say just as well, that the Δ's used by SCHOENFLIES are but combinations of $\mathfrak{S}$'s and that there exist combinations, which were not treated by him. Proceeding in the indicated way, we find some Δ's that are aequivalent with point- and space-*groups* of SCHOENFLIES. After this we might investigate which space-groups can be formed from those Δ's.

As however the result of such an investigation has already been obtained by SCHOENFLIES we shall do better to combine each of his

1) (Note added during translation). C. VIOLA and G. WULFF partly executed such a plan (l.c.).

2) This seems natural by analogy with Δ's that were known before and is also demanded by the principle that around each particle the configuration of the other particles is the same. An exception to this last demand is formed by the definition of the sense of rotation and translation resp. dilation for a screw resp. time-rotation (see further on).

Δ's without restriction to 2-, 3-, 4- or 6-al axes with the possible time-Δ's [1]). Which are these?

One single $\mathfrak{R}$ with $\varphi^4 = 1$ will be called a "retroduction" [2]), the corresponding symmetry-element a "symmetry-moment". In a moving system of particles there exists a $\mathfrak{M}$ (symbol for a retroduction), when each point P, where at the moment t a particle A is present, is also occupied by a particle B at the moment $2\mathfrak{m}-t$, where $\mathfrak{m}$ is the symbol for a symmetry-moment and so the value of its t too. When then at the moment $t+\Delta t$ A is at Q, there must also be present a particle at Q at the moment $2\mathfrak{m}-t-\Delta t$. Because of the second restriction of § 2 we conclude that this last particle must be particle B. The velocities of A and B at P are therefore equal and opposite. At the moment $\mathfrak{m}$ there would thus be at the same place two particles with opposite velocities. This would be in conflict with the impermeability of matter (which we shall assume to hold for the electrons too), unless the two particles are identical [3]). Let us therefore suppose this to be the case. Then each particle must have come to rest at the moment $\mathfrak{m}$ and hence describe its path in the opposite direction.

When we have however a ΔΔ of a $\mathfrak{M}$ and a $\mathfrak{S}$ we must change the above "at the same place" into "at the image of the place in $\mathfrak{s}$". Then the difficulty of two particles with different velocities being at the same moment (at the moment $\mathfrak{m}$) at the same place, would be avoided, unless at the moment $\mathfrak{m}$ the particles were lying in the $\mathfrak{s}$. In this case the velocity at that moment would not necessarily be 0, when only the two particles were supposed to be identical. Having passed the $\mathfrak{s}$ the particle then describes the symmetrical path and when moreover the $\mathfrak{s}$ was intersected perpendicularly by the path there would not be any discontinuity in the motion. In Comm. n°. 4 l.c. the symmetry-element of such a Δ (symbol $\mathfrak{MS}$) has been called "reversal-symmetry-plane". Further on we shall call it reversal-plane and the operation reversal-reflexion.

Other ΔΔ's of time- and space-Δ's may be investigated in the indicated way.

[1]) After this we have still to form groups with the Δ's used by Schoenflies and with the newly introduced ones.

[2]) This name (from retro = back and duco = I lead) and the name dilation (from differo = I postpone) introduced later on have been chosen in consultation with Prof. Damsté of Utrecht.

[3]) We exclude therefore cases as imagined by Landé l.c., in which after a collision two electrons suddenly get each others velocities in direction and magnitude. Landé himself designs these cases as improbable.

§ 8. *The ΔΔ's of two and more* $\mathfrak{M}$'s. Regarding time-Δ's as special cases of $\mathfrak{R}$'s, we see from the end of § 4, that we can find all kinds of time-Δ's by only studying ΔΔ's of $\mathfrak{M}$'s.

The applicability of the complex symmetry-operation of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ (symbol $\mathfrak{P}$, name "dilation", symbol and name of the symmetry-element $\mathfrak{p}$ and "period") to a system of particles, means that when at a moment t a particle A is at the point P there are also particles B resp. C at P at the moments $2\mathfrak{m}_1-2\mathfrak{m}_2+t$ and $2\mathfrak{m}_2-2\mathfrak{m}_1+t$ respectively. In this case P is every time occupied by a particle after a lapse of time, $2(\mathfrak{m}_1-\mathfrak{m}_2)$. When the number of particles at our disposition is not infinite, the same particle A must necessarily at the end arrive at P again. Moreover, each particle, when arriving at P must have the same velocity and the same direction of motion, which will become evident, when we consider the state at moments $2(\mathfrak{m}_1-\mathfrak{m}_2)+t+\Delta t$. All particles are thus distributed in unequal numbers over differently shaped closed paths in which they circulate with phase-differences, that are the same for the different paths and also for the different particles in one and the same path. The times of revolution in two paths are proportional to the numbers of the circulating particles.

A ΔΔ of a $\mathfrak{P}$ and a rotation through $\frac{2\pi}{n}$ about a n-al symmetry-axis was already used in Comm. n°. 4 l.c. We shall call its symmetry-element n-al time-axis, the symmetry-operation time-rotation [1]) (symbol $\mathfrak{PA}$).

The complex operation of $\mathfrak{M}_1$, $\mathfrak{M}_2$ and $\mathfrak{M}_3$ (symbol $\mathfrak{Q}$, name reversal-dilation) is a symmetry-operation of a system of particles, when it fulfills this condition: When at the moment t the point P is occupied by a particle A, we shall find there particles B, C etc. at the moments:

$-2\mathfrak{m}_1+2\mathfrak{m}_2+2\mathfrak{m}_3-t$, $2\mathfrak{m}_1-2\mathfrak{m}_2+2\mathfrak{m}_3-t$ and $2\mathfrak{m}_1+2\mathfrak{m}_2-2\mathfrak{m}_3-t$.

In the first place we have therefore three symmetry moments. At those moments all particles must therefore return in their paths. As this must happen at more than one moment each particle oscillates in a different path of arbitrary form, while the moments of returning are the same for all paths. It is evident that in each path one particle only can circulate now. Secondly there evidently exists a period. To find it the following considerations will be of use: When to a moment t we apply the order $\mathfrak{M}_2\mathfrak{M}_3\mathfrak{M}_1$ and to the

[1]) The distinction we must make here between the two possible combinations of sense of rotation and sense of dilation is analogous to that which Schoenflies and his predecessors made between left- and right-handed screws.

result of this operation the order $\mathfrak{M}_2\mathfrak{M}_3\mathfrak{M}_1$, the influences of $\mathfrak{M}_1$ neutralize each other, so that in fact we have only applied the double dilation $\mathfrak{M}_2\mathfrak{M}_3\mathfrak{M}_2\mathfrak{M}_3$ [1]). Besides the intervals of time $4(\mathfrak{m}_2-\mathfrak{m}_3)$ between the passages of particles by P, we find the intervals $4(\mathfrak{m}_3-\mathfrak{m}_1)$ and $4(\mathfrak{m}_1-\mathfrak{m}_2)$ too. Now we come into conflict with the second restriction of § 2, unless the quantities $\mathfrak{m}_2-\mathfrak{m}_3$ and $\mathfrak{m}_3-\mathfrak{m}_1$ have a greatest common measure. This is the time of oscillation. We can easily prove that then all demands of § 2 are satisfied.

By the investigation of $\Delta\Delta$'s of $\mathfrak{Q}$'s and space-Δ's we shall find i.a. that the paths may be closed in the same way as has been found for $\mathfrak{P}$, but that then half of the paths (chosen in a definite way) is described in the opposite direction.

§ 9. *There are no other time Δ's than $\mathfrak{M}$, $\mathfrak{P}$ and $\mathfrak{Q}$.* For all $\Delta\Delta$'s of even numbers of $\mathfrak{M}$'s the same considerations hold as the following for four $\mathfrak{M}$'s. When at a moment t the particle A is at P, and when we have to do with a $\Delta\Delta$ of four $\mathfrak{M}$'s we must find at P also particles B, C etc. at the moments $\pm 2\mathfrak{m}_1 \pm 2\mathfrak{m}_2 \pm 2\mathfrak{m}_3 \pm 2\mathfrak{m}_4 + t$, where the sign $+$ has to be chosen for half of the $\pm$-signs, the sign $-$ for the other half. This gives therefore more than one dilation, which together yield however (comp. the considerations on $\mathfrak{Q}$) only a dilation equal to their greatest common measure, which case is already comprised in $\mathfrak{P}$.

For all $\Delta\Delta$'s of uneven numbers of $\mathfrak{M}$'s we can follow the reasoning on the case of $\mathfrak{Q}$. Thus this neither gives something new.

Combinations of time-Δ's yield nothing that has not yet been treated.

§ 10. *Symbols for the new symmetry-operations and symmetry elements.* For shortness sake we shall give names and symbols to the *s.-t.*-Δ's and symmetry-elements. As a preliminary system we propose the following:

With a small change now and then we retain the names and symbols of Schoenflies. When now a Δ of Schoenflies is combined with a retroduction the name of the first Δ might be changed by joining to it the prefix reversal. The same may be done with the names of the symmetry-elements. Before the symbols of Δ's and symmetry-elements we add $\mathfrak{M}$ and $\mathfrak{m}$ resp. When the change relates to a dilation the prefix is "time", for the symbols this becomes

[1]) In the here indicated way the treatment of $\Delta\Delta$'s of $\mathfrak{M}$'s (and therefore of $\mathfrak{M}$'s and $\mathfrak{S}$'s) is much simplified. By applying one of the Δ's thus found to the symmetry-elements of another one we can see whether this brings us into conflict with the restrictions of § 2. A $\Delta\Delta$ found in this way evidently forms a *group* of Δ's.

$\mathfrak{P}$ and $\mathfrak{p}$. When an operation is combined with a reversal-dilation we add the prefix reversal-time and for the symbols $\mathfrak{Q}$ and $\mathfrak{q}$. Sometimes the name obtained in this way is still somewhat shortened. In the following table we find these provisionally fixed names together with the symbols.

Without time Δ			*With $\mathfrak{M}$*		
Identity	1		Retroduction (symm.-moment)	$\mathfrak{M}$	$\mathfrak{m}$
Inversion (centre)	$\mathfrak{J}$	$\mathfrak{i}$	Reversal-inversion	$\mathfrak{M}\mathfrak{J}$	$\mathfrak{mi}$
Reflection (symmetry-plane)	$\mathfrak{S}$	$\mathfrak{s}$	Reversal-reflection (reversal-plane)	$\mathfrak{M}\mathfrak{S}$	$\mathfrak{ms}$
Rotation (n-al axis)	$\mathfrak{A}$	$\mathfrak{a}$	Reversal-rotation	$\mathfrak{M}\mathfrak{A}$	$\mathfrak{ma}$
Rotatory-reflection (n-al reflect.-axis)	$\overline{\mathfrak{A}}$		Reversal-rotatory reflection	$\mathfrak{M}\overline{\mathfrak{A}}$	
Translation (place-period)	$\mathfrak{T}$	$\mathfrak{t}$	Reversal-translation	$\mathfrak{M}\mathfrak{T}$	$\mathfrak{mt}$
Gliding-reflection (gliding plane)	$\overline{\mathfrak{T}}$		Reversal-gliding-reflection	$\mathfrak{M}\overline{\mathfrak{T}}$	
Screw (n-al screw axis)	$\mathfrak{X}$		Reversal screw	$\mathfrak{M}\mathfrak{X}$	
With $\mathfrak{P}$			*With $\mathfrak{Q}$*		
Dilation (period)	$\mathfrak{P}$	$\mathfrak{p}$	Reversal-dilation	$\mathfrak{Q}$	
Time-inversion	$\mathfrak{P}\mathfrak{J}$	$\mathfrak{pi}$	Reversal-time-inversion	$\mathfrak{Q}\mathfrak{J}$	$\mathfrak{qt}$
Time-reflection (time-plane)	$\mathfrak{P}\mathfrak{S}$	$\mathfrak{ps}$	Reversal-time-reflection	$\mathfrak{Q}\mathfrak{S}$	$\mathfrak{qs}$
Time-rotation	$\mathfrak{P}\mathfrak{A}$	$\mathfrak{pa}$	Reversal-time-rotation	$\mathfrak{Q}\mathfrak{A}$	$\mathfrak{qa}$
Time-rotatory-reflection	$\mathfrak{P}\overline{\mathfrak{A}}$		Reversal-time-rotatory-reflection	$\mathfrak{Q}\overline{\mathfrak{A}}$	
Time-translation	$\mathfrak{P}\mathfrak{T}$	$\mathfrak{pt}$	Reversal-time-translation	$\mathfrak{Q}\mathfrak{T}$	$\mathfrak{qt}$
Time-gliding-reflection	$\mathfrak{P}\overline{\mathfrak{T}}$		Reversal-time-gliding-reflection	$\mathfrak{Q}\overline{\mathfrak{T}}$	
Time-screw	$\mathfrak{P}\mathfrak{X}$		Reversal-time screw	$\mathfrak{Q}\mathfrak{X}$	

§ 11. *The way in which s.-t.-symmetry-operations may be combined into groups.* When the point groups of Schoenflies are completed by those, which contain other than 2-, 3-, 4- and 6-al rotations etc. we can form from each of the thus found groups, *s.-t.*-groups by combining each of the non-aequivalent operations of a group with either no time-operation or with a $\mathfrak{M}$ or with a $\mathfrak{P}$, or with a $\mathfrak{Q}$. Each of the thus found groups must then still be examined to find out whether the time-operations added are perhaps in conflict with each other. Several of the groups obtained will also be found to be the same.

The same might be done with the translation-groups [1]), which are formed by Schoenflies as a means to change point-groups into space-groups. After this, all obtained *s.-t.*-point-groups are multiplied by each of the *s.-t.*-translation-groups found. Examples of such groups will be given in a following paper (N°. 7*b*).

[1]) In the case of translation-groups we have no longer a ground for the assumption that a $\mathfrak{P}$ and a $\mathfrak{Q}$ cause the paths to be closed. The only thing we should have won by omitting this hypothesis however would be the allowance of a continuous translatory motion of the whole system of particles. It would not be desirable to include this motion in our considerations.