Mathematics. — "*Involutorial Correspondences* (2,2) *of the First Class*". By Prof. JAN DE VRIES.

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§ 1. An involutorial correspondence (2,2) of the *first class* is characterized by the property that an arbitrary straight line contains *one* pair of associated points P, P^*. If we associate to each other the straight lines joining a point P to the two homologous points P_1 and P_2, also the field of rays is arranged in an involutorial (2,2). At the same time there arises a null system, if we associate to P the straight lines PP_1 and PP_2; each straight line has in this case two null points, each point has two null rays.

If the point P describes the straight line r, its null rays envelop a curve (r) of the fourth class that has r as a double tangent. The six points V in which $(r)_4$ is cut by r, are evidently *branch points* of the (2,2). The *branch curve* (V) of the (2,2) is therefore a curve of the order *six*.

We shall now suppose that the locus of the coincidences $P \equiv P^*$ is a curve of the order n. If P describes the line r, the points P_1, P_2 associated to P, describe a curve ϱ, which has the n coincidences on r and the pair of associated points on r, P, P^*, in common with r.

Through this correspondence r is therefore transformed into a curve ϱ^{n+2} of the order $(n+2)$.

Let us now consider the curves ϱ_1^{n+2} and ϱ_2^{n+2} corresponding to the straight lines r_1 and r_2. Besides the two points associated to $S \equiv r_1 r_2$ they have the points P in common for which P_1 lies on r_1 and P_2 on r_2; the other common points are *singular*, i.e. each of them is associated to ∞^1 pairs P_1, P_2.

The curves $(r_1)_4$ and $(r_2)_4$ corresponding to the straight lines r_1 and r_2, have in the first place the two null rays of the point S in common. The line r_2 cuts ϱ_1^{n+2} in $(n+2)$ points P_2, which are associated to as many points P_1 on r_1 and accordingly define $(n+2)$ common tangents. The other $(12-n)$ common tangents are evidently *singular straight lines*; each of them bears ∞^1 pairs of points P, P^*.

Let us also consider the locus of the pairs of points P, P^* which are collinear with a point O. Let O_1 and O_2 be the points conjugated to O through (2,2); the curve in consideration ω is touched at O

by OO_1 and OO_2; it is therefore a *nodal* curve ω^4. Through O there pass six of its tangents; according to a theorem found by BERTINI the six points of contact, coincidences of the (2,2), lie on a conic [1]). The bearers of the *coincidences* of the (2,2) envelop consequently a curve of the *sixth class.*

§ 2. We arrive in the following way at a (2,2) for which $n = 2$. Let the conic a^2 and the pencil of conics (b^2) be given. To the point P we associate the points P_1 and P_2 in which the conic b^2 through P is cut by the polar line p of P relative to a^2. On a straight line r, (b^2) defines an involution; as a rule this has *one* pair of points in common with the involution on r of the pairs of points that are harmonically separated by a^2. This (2,2) belongs accordingly to the *first class.*

The points of a^2 are evidently the coincidences of this (2,2). The straight line r is transformed into a *nodal* ϱ^4, which has the pole R of r as double point. For when P moves along r, its polar line p revolves round R and bears the two points P_1, P_2 associated to P.

The base points B_k $(k = 1, 2, 3, 4)$ of (b^2) are *singular points.* On the polar line b_k of B_k (b^2) defines ∞^1 pairs of points P_1, P_2 which are associated to B_k. If P gets into the intersection of b_k with r, one of the points associated to P coincides with B_k; hence ϱ^4 passes through the four points B_k.

The conic b^2 through R cuts r in two points R_1, R_2, which are associated to R; hence ϱ^4 has a double point in R.

The six tangents of ϱ^4 meeting in R bear double points $P_1 \equiv P_2$; from this it follows again that the *branch curve* is a $(V)^6$. It has double points in the base points of (b^2); for the involution of the pairs of points on b_k associated to B_k contains two double points for which B_k is a branch point.

With a b^2 $(V)^6$ has four points in common besides the double points B_k; they are the branch points of the correspondence (2,2) on b^2. The curve $(V)^6$ touches a^2 in the six coincidences of the involution I^4 in which (b^2) cuts a^2.

§ 3. Any point A of a^2 is a coincidence of the (2,2), but it is also associated to the point A' which the tangent a at A has further in

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[1]) Relative to this conic ω^2 as an invariant curve, ω^4 is transformed into itself by a central quadratic involution (inversion) with centre O of which the other two fundamental points lie on the polar line of O relative to ω^2; this straight line contains the points of contact O_1, O_2 of O. (See J. DE VRIES, *La quartique nodale*, Archives Teyler, série II, tome IX, § 12).

common with the b^2 through A. Of the locus α of the points A' a b^2 contains four points besides the base points B; they are defined by the points of intersection of b^2 with a^2. On each of the two tangents a through B_k, A' coincides with B_k; hence α has double points in B_k.

Consequently the curve in question is an α^6. As it corresponds point for point to a^2 and is therefore rational, it must have six more double points. There are therefore six points A' each corresponding to two points A; the b^2 through such a point A' cuts a^2 in the two points A which it has in common with the polar line of A'.

The straight line b_k is transformed by (2,2) into a β^4 with *triple point* β_k. When P moves along b_k the polar line p continues to pass through β_k, so that always one of the points P_1, P_2 associated to P, coincides with β_k. If also the second point is to coincide with β_k, p must touch the b^2 through P at β_k. Now any straight line p through β_k touches one b^2; if we associate the points Q_1, Q_2 which this b^2 defines on b_k, to the pole P of p, there arises a correspondence (1,2) between P and Q. Hence Q coincides three times with P; but then the curve β_k into which b_k is transformed, has a threefold point in B_k and is therefore a *rational* β^4. [1])

§ 4. We shall now try to find the locus of the *double points* $P_1 \equiv P_2$. It has in the first place *threefold points* in B_k. On each b^2 there lie besides the base points four more points of the curve in question, namely the double points of the (2,2) in which the points of b are arranged. Consequently it is a δ^8. As it corresponds point for point to the branch curve $(V)^6$ it is just as the latter of the genus six; hence it must have *three* more *double points*. These we find in the double points of the three pairs of lines belonging to b^2.

The bearers of the double point $P_1 \equiv P_2$ envelop a curve of the sixth class (§ 1) of the same genus as the branch curve, hence with four double tangents; these we find in the straight lines b_k.

For the points where b_k is touched by two of the conics b^2, correspond as double points to the branch point B_k.

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[1]) On b_k there lie 2 points that are associated in the (2,2) to each other and at the same time to B_k, and which therefore together with that point form a polar triangle of a^2. The b^2 containing them is consequently circumscribed to ∞^1 polar triangles so that on it the (2,2) has been transformed into a *cubic involution* In this involution *each* base point B is associated to the points of intersection of b^2 with the polar line of B.

If we define the pencil (b^2) by two conics, each circumscribed to a polar triangle of a^2, each b^2 bears a cubic involution and the whole correspondence (2,2) is transformed into a system of ∞^1 involutorial triplets.

§ 5. Each straight line $B_k B_l$ is evidently *singular*, for it bears ∞^1 pairs of points that are harmonically separated by a^2.

A straight line would also be singular if the involution in which it is cut by (b^2), coincided with the involution of the pairs of points that are harmonically separated by a^2. And this will be the case when this straight line is touched in its two points of intersection with a^2 by conics b^2.

Now the straight lines t that touch b^2 at its points of intersection with a^2, envelop a curve of the class *six*. For the points of contact of the tangents out of any point to the conics b^2 lie on a cubic and this meets a^2 in six points, each of which defines a straight line t. This envelope is *rational*; it has therefore *ten double tangents*; to them belong evidently the six straight lines $B_k B_l$.

Hence there are, besides these, *four more singular straight lines*, s_k.

The straight line s_k is transformed through (2,2) into the system of s_k and a nodal cubic that has its double point in the pole of s_k. The straight line $B_k B_l$ is transformed into the system of $B_k B_l$, $B_m B_n$, b_k and b_l.

§ 6. The points P_1 and P_2 associated to P in the (2,2), correspond to each other in another (2,2), which may be called the derivative of the former. This (2,2)* is likewise of the *first class*; for on a straight line p there lies only the pair in which p cuts the conic b^2 passing through the pole P of p.

Also this (2,2)* has singular points in B_k; for if P describes the polar line b_k, P_1 remains in B_1 and P_2 describes the above mentioned rational curve β_k^4.

The curves ϱ_1^4 and ϱ_2^4 corresponding in the (2,2) to the straight lines r_1 and r_2, have (§ 1) 10 points P in common for which P_1 lies on r_1, P_2 on r_2. Hence P_2 describes a curve ϱ^{10} when P_1 describes the straight line r_1. This ϱ^{10} has *quadruple points* in B_k, for r_1 cuts the curve β_k^4 in four points P_1.

Each branch point of the (2,2) is at the same time a branch point of the (2,2)*; accordingly they have also *the same branch curve* $(V)^6$. The *coincidences* of the (2,2)* are the *double points* of the (2,2); the *curve of coincidence* is therefore the above mentioned δ^8, which passes three times through B_k, twice through the double points of the pairs of lines. We find the points of intersection of r with ϱ^{10} in the eight points which r has in common with δ^8 and in the pair of points P_1, P_2 on r.

The four singular straight lines (§ 1) of the (2,2)* are found in the straight lines b_k.

§ 7. In the following way we arrive at a (2,2) for which $n = 3$. Let a^3 be a cubic, p^2 the polar conic, p the polar straight line of P. To P we associate the two points of intersection P_1 and P_2 of p^2 with p. The correspondence (2,2) arising in this way, is involutorial, because P and P_1 may be considered as threefold elements in a cubic involution where the points of intersection of PP_1 with a^3 form a group [1]), or as the double points of the cyclic projectivity defined by this group. The *class* of this (2,2) is therefore *one*.

If P gets on a^3, P_1 and P_2 coincide with P; P is in this case a branch point coinciding with the corresponding double point. If on the other hand P gets into a *point of inflexion* B, p is a part of p^2, so that B is a *singular point* and the stationary tangent is a *singular straight line*.

If P gets on the *Hessian* H^3 of a^3, p passes through the double point of p^2, also lying on the Hessian, and P_1 coincides with P_2, so that P is a branch point. The *branch curve* $(V)^6$ consists therefore of a^3 and H^3 and these curves are at the same time the locus of the *double points*.

When P describes the straight line r, p^2 describes a pencil and p envelops a conic. In each base point of (p^2) there lie therefore two points associated to P. As a p^2 contains moreover the two points of intersection with the corresponding p, the straight line r is transformed into a *quadrinodal curve* ϱ^6. This contains the nine points of inflexion of a^3, as these correspond to the points in which r cuts the stationary tangents. Consequently ϱ^6 touches a^3 in the three points of intersection of a^3 with r.

The derivative of this (2,2) is of the *fourth class*. For a straight line p has four poles and contains therefore the four pairs P_1, P_2 in which it is cut by the corresponding four polar conics p^2.

[1]) KOHN, *Zur Theorie der harmonischen Mittelpunkte.* (Sitz. ber. der Akad. der Wiss. Wien, Bd. LXXXVIII, S. 424).