Mathematics. — "*Explanation of some Interference-Curves of Uni-axial and Bi-axial Crystals by Superposition of Elliptic Pencils.*" (Second paper). By J. W. N. LE HEUX. (Communicated by Prof. HENDRIK DE VRIES).

(Communicated at the meeting of October 29, 1921).

In a former paper [1]), it is shown, that the interference-curves of some crystals — without regard to the isogyres — may be considered as the "moiré"-images of two concentric elliptic pencils, each containing the curves of intersection of the two parts of the successive wave-surfaces with the upper side of the crystal plate.

These wave-surfaces were supposed to be homothetic and so, the velocities in directions, normal to the wave-fronts, were uniform.

From experimental results it is seen, that this supposition explains the phenominae, observed in a polarisation-microscope, only at some distance from the centre, not immediately around it (fig. 1 and 2). For the caracteristic black cross, which appears around the centre with uni-axial and bi-axial crystals, is in those figures indistinctly to be seen.

It is proposed in this paper to give an account of some further experiments, which enable us to obtain the black cross in the centre with a fair accurancy and so to find some further conditions as to the elliptic pencils.

Consider first the figure of the hyperbolas (fig. 1). If the inner curves were not little circles, but ellipses, ending in short, coïncident, straight lines, this image would become much better. For in this way, a nearly sufficient effect was obtained by superposing two excentric sets of ellipses, as is shown in my first paper.

By causing two of the four pendulums of the former described apparatus to begin a very short time after the two others, the ending-curves of both unissons are short, coïncident, straight lines and the results are indeed more satisfying, but not yet wholly right.

There is another condition to be fulfilled, as was remarked with the following experiment.

The name of unisson was given — according to LISSAJOUS — to the figure, described by two equal pendulums, with this extension, that the whole family of ellipses, described by altering the phase-difference from $\frac{\pi}{2}$ to 0, was also called unisson.

[1]) Proc. Royal Acad. XXIX, p. 1114—1117.

The beginning-curve was the circle, inscribed in the square upon the amplitudes, the ending-curve was a diagonal of this square.

A further extension is given by continuing the figure (the diagonal being run through), till another circle is described, with smaller radius, in consequence of the decrease of amplitudes, due to friction.

We will call this figure „continued unisson" — it shows a superposition of a larger and a smaller unisson and therefore interference-curves, which prove to be ellipses with their major axes in coïncidence with those of both unissons (fig. III).

This set of interference-ellipses is altered in a remarkable way, when the resistance of *one* of the composing movements increases. This may be obtained, with the apparatus used, by means of the screw-weight at the end of a ruler.

Successively, the ellipses are transformed into a flame-shaped image (fig. IV), then a black cross is formed around the centre (fig. V) and at last a set of hyperbolas appears (fig. VI), which is most plain, when the angle between the major axes of the beginning- and the ending-ellipse is between $\frac{\pi}{4}$ and $\frac{\pi}{2}$.

Increase of resistance of one of the composing movements induces a faster decrease of one of the amplitudes and so, the ellipses are no longer inscribed in squares but in rectangles. The major axis of each ellipse coïncides with the diagonal of the circumscribed rectangle. In this way, a rotation is caused around the centre, joined to the alteration in shape, already mentioned.

When the continued unisson is considered as a superposition of two common unissons, the rotations of the latter are in an opposite sense.

The principle of rotation arises immediately from the mathematical interpretation of the phenomenon.

In my first paper is already said, that in a superposition of two concentric pencils, the ellipses result from four vibrations.

Each pencil being given by

$$x = r \cos(\varphi + \alpha) + r \cos(\varphi + \gamma)$$
$$y = r \sin(\varphi + \beta) + r \sin(\varphi + \vartheta)$$

or by

$$x = 2r \cos\frac{\alpha - \gamma}{2} \cos\left(\varphi + \frac{\alpha + \gamma}{2}\right)$$
$$y = 2r \cos\frac{\beta - \vartheta}{2} \sin\left(\varphi + \frac{\beta + \vartheta}{2}\right)$$

and considering, that generally $\alpha - \gamma \neq \beta - \vartheta$, the ellipse with the

J. W. N. LE HEUX: "Explanation of some Interference-Curves of Uni-axial and Bi-axial Crystals by Superposition of Elliptic Pencils".

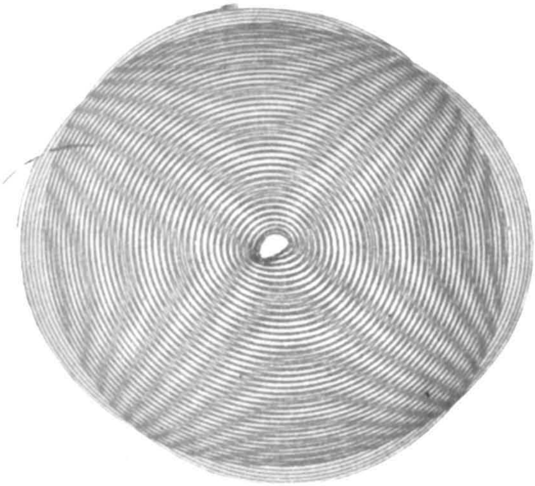

Fig. 1.

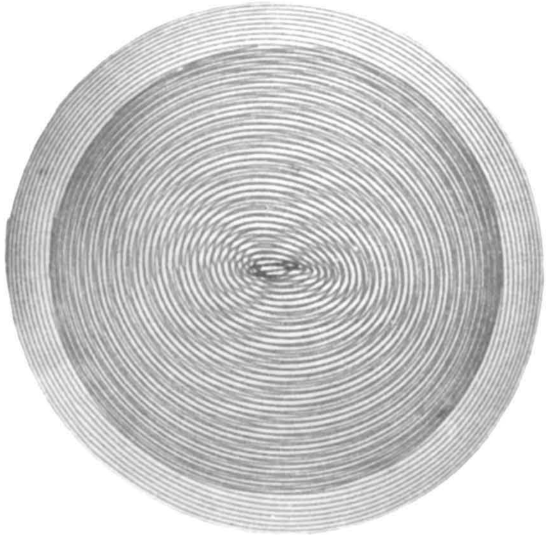

Fig. 2.

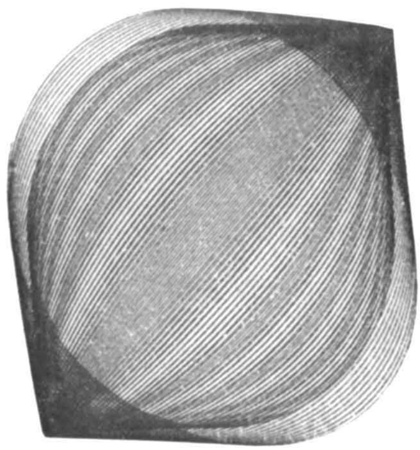

Fig. 3.

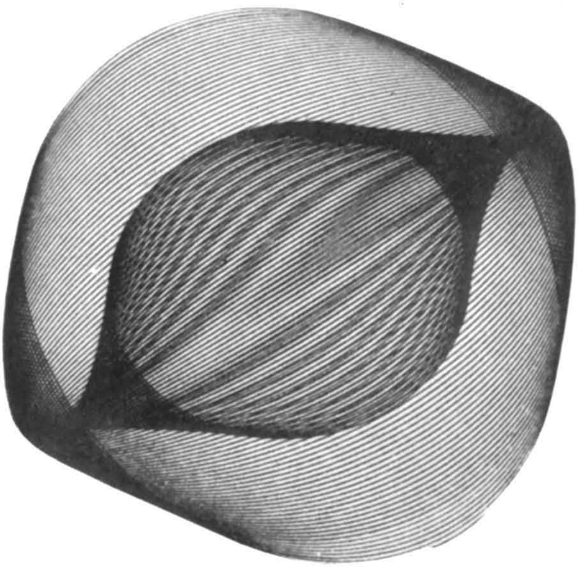

Fig. 4.

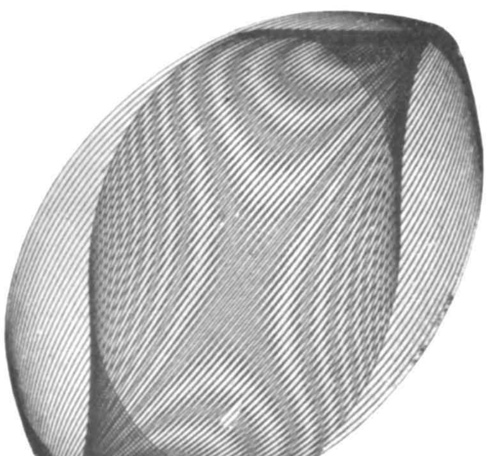

Fig. 5.

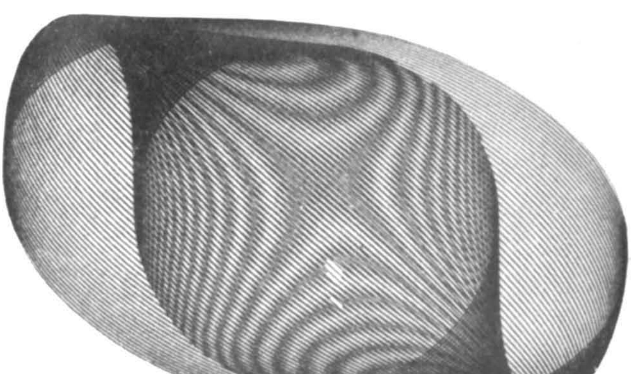

Fig. 6.

variable difference of phase $\frac{\alpha-\beta+\gamma-\vartheta}{2}$ is inscribed in a variable rectangle with sides $4r\cos\frac{\alpha-\gamma}{2}$ and $4r\cos\frac{\beta-\vartheta}{2}$ and therefore will rotate around the centre.

This rotation of the singular ellipses may not be confounded with the rotation of the whole unisson around the centre.

In this case, interference curves are also seen by superposing a unisson and the same figure after rotation — the image shows however slightly curved, parallel lines, cutting orthogonally the bisectrix of the angle between the axes, when the latter is about 15°.

At last, it is evident, that a sufficient result will be obtained in combining all the conditions, above mentioned, that is: The image of the hyperbolas proceeds from the superposition of two concentric pencils of ellipses (each ellipse resulting from four vibrations). The beginning-curves of these pencils are an ellipse and a circle, having common tangents in the extremities of the minor axis of the ellipse; the ending-curves are short, coincident, straight lines. The curves between show both a regular alteration in shape and a rotation.

The latter has an opposite sense for both pencils, but good results were also obtained in the case of rotation of one of the pencils only.

When these conditions are not observed, great differences become immediately visible.

So, the figure of the hyperbolas degenerates into a Maltese cross, when there is but a small angle between the ending-ellipses. The four arms of the cross point to the extremities of the major axes. Such a cross is also formed, when these ellipses (straight lines) are coïncident, but differ to much in magnitude.

The rotation of the ellipses being too fast, the "asymptotes" of the hyperbolas are curved in the same sense; when the two pencils have but a slight difference besides, the figure shows a pencil of curved radii.

If both ending-ellipses are coincident, but too large, the figure of the hyperbolas is exactly formed, but the upper and the lower part are translated in opposite sense.

The same remarks are applicable to the black cross around the centre in the image of the lemniscates.

The here described method of moiré figures to explain interference curves needs not to suppose (as is done with the commonly used method of isophase-surfaces of BERTIN), that both broken normals to the wave-fronts follow the same way in the crystal.