

**Physics.** — “*On the Equation of State for Arbitrary Temperatures and Volumes. III. On the Law of Force between the Molecules of Mon-atomic Substances*”. By Dr. J. J. VAN LAAR. (Communicated by Prof. H. A. LORENTZ).

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### § 8. General Considerations.

In the two preceding papers <sup>1)</sup> it has been demonstrated that a closer consideration of the problem of the movement of a molecule to and fro between the two adjacent molecules (for the sake of simplicity reduced to a problem of one-dimension) necessarily leads at *low* temperatures to an expression of the form

$$\varepsilon = \Delta + \frac{2\Delta}{e \frac{2/3\Delta}{RT} - 1},$$

in which  $\Delta$  represents the *zero-point energy*, i.e. the energy of the active forces, which remains when the temperature (determined by the time-average of  $u^2$ ) has become  $= 0$  [loc. cit. p. 1198 ( $\Delta$  is there represented by  $E_0$ ) and p. 905].

If it could be proved that in this  $\Delta = 3/2, Nhv$ , the analogy with PLANCK'S formula would become identity. But to reach this, we should have to know the accurate law of attraction, i.e. a law which takes into account the *motion in closed orbits* of the negative electrons round the positive nucleus of the atoms. The prevalent laws of attraction have not taken this into account as yet; either because the integrability of the equations of motion required a simple — although still plausible — law of attraction, so that the accurate law had to be purposely set aside for one of a simpler form; or because in the derivation of the required law the influence of the said motion was (consciously or unconsciously) eliminated

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<sup>1)</sup> These Proc. Vol. XXI, N<sup>o</sup>. 9, p. 1184, and These Proc. Vol. XXIII, N<sup>o</sup>. 6, p. 887.

by taking averages, as e.g. DEBIJE did in his paper on the VAN DER WAALS' cohesion forces <sup>1)</sup>.

In our first paper (see p. 1188) we assumed for the attractive force  $F = f \times 2x$ , in which  $x$  represents the distance of the moving molecule from the neutral point; and for the repulsive force on collision  $2\varepsilon \times y$ , in which  $y$  represents the compression of the molecule.

Later on (p. 897) I substituted for the two separate laws of force one single law of the form (cf. the cited paper for the meaning of the different letters)

$$F = f \cdot 2x \left[ 1 - \frac{(q-s)(\sigma-s)}{(l-s)^2 - x^2} \right],$$

which rendered the solution of the problem raised there still just possible by the aid of elliptical functions.

Though these two laws of attraction by no means represent reality quantitatively accurately, yet at low temperatures we found a relation between  $E$  and  $T$ , which is analogous with PLANCK'S well-known expression — which certainly proves that the essential part of our considerations (viz. our observance of the *time-average*) rests on solid foundations. The exact form of the law of attraction seems here to a certain extent to be of minor importance, and according to the results of the two papers to have influence only on some numerical coefficients.

<sup>1)</sup> Phys. Zeitschr. 21, 178 (1920). In 1908 VAN DER WAALS JR. already treated a similar problem, but he still considered the atoms (molecules) as electric double points which, like DEBIJE, he besides supposed *far enough* apart to simplify the problem. He found that the force decreased more rapidly than  $r^{-7}$ .

Later also KEESOM wrote a paper in connection with the said paper by DEBIJE (These Proc. Vol. XXIII, N<sup>o</sup>. 6, p. 939 and 943; also Phys. Zeitschr. 22, 129 (1921) and Mededeelingen Utrecht N<sup>o</sup>. 6) on the question of the forces of attraction. There he demonstrates that for H<sub>2</sub>, O<sub>2</sub> and N<sub>2</sub> DEBIJE'S quadrupoles yield a too large value; further that — at least for the said gases — the quadrupole-attraction has considerably more influence on the second virial-coefficient  $B$  than the so-called "induced" attraction (unless the temperature is very high), and that the VAN DER WAALS cohesion-forces can chiefly be attributed to forces which the molecules exert on each other in virtue of their quadrupole-momenta.

BURGERS (Dissertatie, Leiden 1918, p. 186) calculated the quadrupole-momentum of the H<sub>2</sub>-molecule according to the model constructed by BOHR and DEBIJE (to which, however, there are several objections), and found a remarkable agreement with the value  $2,03 \cdot 10^{-26}$  (electrostatic units  $\times$  cm<sup>2</sup>.; uncorrected for the polarisation of the molecules in each others' electric field), derived by KEESOM (Comm. Leiden, Suppl. 39a, p. 15) from the 2<sup>nd</sup> virial-coefficient, viz.  $2,05 \cdot 10^{-26}$ .

But, as has been said, a true insight — especially as far as the relation between  $\Delta$  and  $h\nu$  is concerned — cannot be obtained until the law of attraction is accurately known; then the quantities  $h$  and  $\nu$  will also appear automatically in the law of the action between two atoms (molecules).

We shall see in what follows that the real active force is by no means merely an exclusively *attractive* action increasing in intensity, which is only transformed into a *repulsive* force at collision or at a very short distance — but that from the very beginning the action has been **periodically** attractive and repulsive, which only becomes stronger and stronger on approach of the atoms. Several known phenomena can now be explained more easily, not only in the sphere of the solid bodies (and of the liquids), but also in that of the *gases* — particularly as concerns the so-called “*gas-degeneration*” at very low temperatures.

### § 9. Derivation of the Elementary Law of Force.

To simplify the calculations, we shall again place ourselves at the standpoint of the problem of one dimension. In connection with this the circular motion of the electrons round the positive nucleus must be transformed into a motion to and fro rectilinearly, viz. the projection of the circular motion on the direction of the joining line of the two nuclei, so that the electrons always move (fictitiously) to and fro through the nucleus.

Let  $r$  be the distance of the two nuclei  $A$  and  $B$ ,  $a$  the radius of the orbit of the electrons (thought perfectly circular), so that the deviations  $x$  and  $y$  of the electrons from the centre are represented by

$x = a \sin 2\pi \frac{t}{T}$ ,  $y = a \sin 2\pi \frac{t'}{T}$ . We then have together for the repulsive and attractive forces (see Fig. 1):

$$F = e^2 \left[ -\frac{1}{r^2} - \frac{1}{(r+x-y)^2} + \frac{1}{(r+x)^2} + \frac{1}{(r-y)^2} \right],$$

when for the present we confine ourselves in our considerations to mon-atomic substances, while only *one* electron moves round the nucleus. (H-atoms).

When an atom  $M$  moves between two other atoms  $P$  and  $Q$ , the total action (taken positive when  $M$  is drawn to the right ( $P$ ))

becomes evidently (see fig. 2) — on the supposition that the atoms

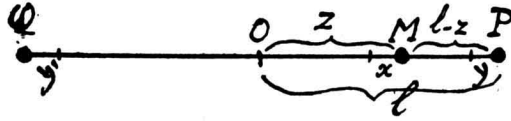


Fig. 2.

$P$  and  $Q$  are on an average at rest, and the mutual polarizing action of the atoms may be neglected:

$$\frac{F}{e^2} = \left. \begin{aligned} & -\frac{1}{(l-z)^2} - \frac{1}{(l-z+x-y)^2} + \frac{1}{(l-z+x)^2} + \frac{1}{(l-z-y)^2} \\ & + \frac{1}{(l+z)^2} + \frac{1}{(l+z-x-y')^2} - \frac{1}{(l+z-x)^2} - \frac{1}{(l+z-y')^2} \end{aligned} \right\},$$

when  $l$  is the mean distance of the atoms, and  $z$  the distance of the moving atom on the right from the mean position of equilibrium (neutral point)  $O$ . If therefore  $F$  is positive, the force of  $M$  is directed towards  $P$ .

Now the motion of the electrons round  $P$  and  $Q$  will exhibit *phase-difference* with that of the electron round  $M$ , so that we shall have to calculate the mean value for different values of  $y$  and  $y'$ , retaining the value of  $x$ , that varies periodically with the time in the considered molecule  $M$ , which we shall, accordingly, not eliminate by taking averages. The integral

$$I = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\omega}{((l-z)+x-y)^2} = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\omega}{\left[ (l-z) + a \sin 2\pi \frac{t}{T} - a \sin \left( 2\pi \frac{t}{T} + \omega \right) \right]^2}$$

has the form

$$I = \frac{1}{2\pi} \int_{\varphi}^{\varphi+2\pi} \frac{d\varphi}{(p-a \sin \varphi)^2},$$

when  $2\pi \frac{t}{T} + \omega = \varphi$  is put. ( $\omega$  is the phase-difference between  $P$  and  $M$ ). With  $a = 90 + \varphi$  this becomes:

$$I = \frac{1}{2\pi} \int_{\alpha}^{\alpha+2\pi} \frac{d\alpha}{(p+a \cos \alpha)^2} = \frac{p}{(p^2-a^2)^{3/2}},$$

as is easy to derive. Hence we get, performing the same thing with the integrals in which  $y'$  occurs:

$$\frac{F}{e^2} = \left. \begin{aligned} & -\frac{1}{(l-z)^2} - \frac{l-z+x}{((l-z+x)^2-a^2)^{3/2}} + \frac{1}{(l-z+x)^2} + \frac{l-z}{((l-z)^2-a^2)^{3/2}} \\ & + \frac{1}{(l+z)^2} + \frac{l+z-x}{((l+z-x)^2-a^2)^{3/2}} - \frac{1}{(l+z-x)^2} - \frac{l+z}{((l+z)^2-a^2)^{3/2}} \end{aligned} \right\}, \quad (1)$$

and this will be the law of action between  $M$  and the two adjacent atoms  $P$  and  $Q$ . The first four terms refer to  $P$ , towards which  $M$  e.g. moves; the last four to  $Q$ , from which  $M$  then moves away.

The expression (1) yields in the equation of motion perfectly unintegrable forms, and we shall try to find for them an approximate expression, when  $z$  and  $x$  are not too great with respect to  $l$ . At any rate it appears at once that  $F$  contains the factor  $x$ , so that the law of action becomes a **purely periodical** one.

Whenever  $x$  becomes  $= 0$ , i.e. the electron (fictitiously) moves through the nucleus,  $F$  will also become  $= 0$ , and the total force change from *positive* (at  $x$  positive), i.e. directed towards the *right*, into *negative* (at  $x$  negative), i.e. directed towards the *left*, and vice versa. In reality for  $x = 0$  both the first part of the second member of (1) becomes  $= 0$ , and the second part.

When in his cited article (p. 179 righthand side) DEBIJE states that the potential of a sphere with charge  $+e$  in its centre and  $-e$  on its circumference is on an *average*  $= 0$ , he is, of course, right. But in the first place I object to this view of the problem, since it will depend on the mutual position of the electrons in their orbits round the two centres and on their phase-difference, what action will result; which also renders it doubtful whether all orientations of the two electrons on the two sphere-surfaces will, indeed, be equivalent — even on an “average”. And in the second place it seems to me that his method — in order to find still a positive value (i.e. attractive action) for the resulting force — of taking the action into account which one atom exerts on the electric moment of the other, is open to doubt. For according to DEBIJE himself the electric field of this one atom will be on an average  $= 0$  (see a few lines lower). How can then this field, which is  $= 0$  on an average, exert an appreciable polarizing action<sup>1)</sup> on the other atom?

That the attractive action with the periodicity found by us, is

<sup>1)</sup> Even apart from the fact that the polarization will certainly always be very small, because in my opinion the exceedingly great velocity of the electrons in their orbits excludes an appreciable deformation. DEBIJE finds finally for the attractive action proportionality as  $r^{-9}$  (for so-called dipole gases on the other hand  $r^{-7}$ , cf. note 1 on page 183), as against VAN DER WAALS Jr. as  $r^{-(7+\delta)}$ .

not quite symmetrical with the repulsive action, is clear from formula (1). More-over, also for an estimation of the relative order of magnitude of these forces, we shall give some numerical considerations in the following paragraph.

### § 10. Some Numerical Calculations.

Since the centres of the atoms cannot get nearer to each other than  $2a$ , a mean distance of  $l = 3a$  is, indeed, an extreme value for solid bodies and liquids, sooner too great than too small. For when it is considered that for many liquids  $(v-b):v$  is  $\approx 1/14$  in the neighbourhood of the point of solidification, then  $l = (1 + 1/14) 2a$ . But as the quantity  $b$  in the equation of state will very certainly not be equal to the real volume of the molecules, but larger, in reality  $l$  will be  $> 2,05 a$ . Even at the absolute zero-point  $l$  will probably not be smaller than  $2,1 a$ . Let us now first put

$$l = 3a.$$

1.  $z = 0$ . The moving molecule is then exactly in  $O$ , halfway between the two others.

We can now write for (1):

$$\frac{F}{e^2} = \left[ \left\{ \frac{l-z}{((l-z)^2 - a^2)^{3/2}} - \frac{1}{(l-z)^2} \right\} - \left\{ \frac{l-z+x}{(l-z+x)^2 - a^2)^{3/2}} - \frac{1}{(l-z+x)^2} \right\} \right] - \left. \right\} \text{ (1a)}$$

$$- [\text{id. with } +z \text{ and } -x]$$

in which we get for the case  $z = 0$ :

$$\frac{F}{e^2} = \left[ \left\{ \frac{l}{(l^2 - a^2)^{3/2}} - \frac{1}{l^2} \right\} - \left\{ \frac{l+x}{((l+x)^2 - a^2)^{3/2}} - \frac{1}{(l+x)^2} \right\} \right] -$$

$$- \left[ \left\{ \frac{l}{(l^2 - a^2)^{3/2}} - \frac{1}{l^2} \right\} - \left\{ \frac{l-x}{(l-x)^2 - a^2)^{3/2}} - \frac{1}{(l-x)^2} \right\} \right],$$

in which the first and the third part cancel each other. In order, however, to get to know something of the mutual order of magnitude of the different parts, we have not omitted these terms.

For  $x = 0$  all the 4 terms are equal to each other; i.e.  $\approx (3:8^{3/2} - 1:9):a^2 = (0,1326 - 0,1111):a^2 = 0,0215:a^2$ , and we have

$$a^2 F:e^2 = (0,0215 - 0,0215) - (0,0215 - 0,0215) = 0 - 0 = 0.$$

For  $x = +a$  (extreme deviation of the electron towards the side of  $Q$  (see fig. 2) is found with  $4:15^{3/2} - (1:16) = 0,0689 - 0,0625 = 0,0064$ , and  $2:3^{3/2} - 1:4 = 0,3849 - 0,25 = 0,1349$ :

$$a^2 F:e^2 = (0,0215 - 0,0064) - (0,0215 - 0,1349) = 0,0151 - (-0,1134) = 0,1285.$$

A force, therefore, directed towards the right, chiefly originating from the repulsive action exerted by  $Q$  on  $M$ .

For  $x = -a$  (greatest deviation to the side of  $P$ ) everything is just the opposite, and we have:

$$a^3 F : e^3 = (-0,1134) - 0,0151 = \underline{-0,1285}.$$

Now the repulsive force predominates, which  $P$  exerts on  $M$ .

2.  $z = \frac{1}{2}a$ . This is a mean position of  $M$  between  $O$  and the perfect contact at  $z = a$ .

For  $x = 0$  we have here, since  $2^{1/2} : (5^{1/4})^{3/2} - (1 : 6^{1/4}) = 0,2078 - 0,16 = 0,0478$ , and  $3^{1/2} : (11^{1/4})^{3/2} - (1 : 12^{1/4}) = 0,09276 - 0,08163 = 0,0111$ :

$$a^3 F : e^3 = (0,0478 - 0,0478) - (0,0111 - 0,0111) = 0 - 0 = \underline{0}.$$

For  $x = a$  we find:

$$a^3 F : e^3 = (0,0478 - 0,0111) - (0,0111 - 0,0478) = 0,0366 - (-0,0366) = \underline{0,0732}.$$

Here the attractive force of  $P$  supports the repulsive force of  $Q$  (which happens to have the same value).

Finally  $x = -a$  yields with  $1^{1/2} : (1^{1/4})^{3/2} - (1 : 2^{1/4}) = 1,0733 - 0,4444 = 0,6289$ , and  $4^{1/2} : (19^{1/4})^{3/2} - (1 : 20^{1/4}) = 0,0533 - 0,0494 = 0,0039$ :

$$a^3 F : e^3 = (0,0478 - 0,6289) - (0,0111 - 0,0039) = (-0,5811) - 0,0072 = \underline{-0,5883}.$$

It will be seen that the action is now quite asymmetric: that towards the right at  $x = +a$  is much weaker than that towards the left at  $x = -a$ . This is, of course, owing to the fact that in this latter position the electron is much nearer  $P$  than it is to  $Q$  in the case  $x = +a$ .

3.  $z = a$ . This is the extreme position of  $M$  close to  $P$  (distance of the centres =  $2a$ ), in which we shall now find an infinitely great repulsive force at  $x = -a$ .

In the case  $x = 0$  we get:

$$a^3 F : e^3 = (0,1349 - 0,1349) - (0,0064 - 0,0064) = 0 - 0 = \underline{0}.$$

For  $x = +a$  we find:

$$a^3 F : e^3 = (0,1349 - 0,0215) - (0,0064 - 0,0215) = 0,1134 - (-0,0151) = \underline{0,1285}.$$

And  $x = -a$  yields with  $(1 : 0^{3/2}) - (1 : 1) = \infty - 1$ , and  $(5 : 24^{3/2}) - (1 : 25) = 0,0425 - 0,04 = 0,0025$ :

$$a^3 F : e^3 = (0,1349 - \infty) - (0,0064 - 0,0025) = (-\infty) - 0,0039 = \underline{-\infty}.$$

The above can be combined in the following survey (values of  $a^3 F : e^3$ ).

		$s = 0$	$s = \frac{1}{2}a$	$s = a$
	$x = 0$	0	0	0
$(l = 3a)$	$x = +a$	0,1285	0,0732	0,1285
	$x = -a$	-0,1285	-0,5883	$-\infty$

Let us now repeat these calculations for the case

$$l = 2,1a$$

1.  $z = 0$ .

For  $x = +a$  is found with  $3,1 : (8,61)^{3/2} - (1 : 9,61) = 0,12 - 0,10 = 0,02$ , and  $1,1 : (0,21)^{3/2} - (1 : 1,21) = 11,43 - 0,83 = 10,60$ :

$$a^2 F : e^2 = -0,02 + 10,60 = 10,58;$$

while for  $x = -a$  of course  $-10,58$  will be found.

2.  $z = 0,05a$ .

With  $2,05 : (3,2025)^{3/2} - (1 : 4,2025) = 0,36 - 0,24 = 0,12$ ;  $2,15 : (3,6225)^{3/2} - (1 : 4,6225) = 0,31^2 - 0,21^2 = 0,10$ ;  $3,05 : (8,3025)^{3/2} - (1 : 9,3025) = 0,13 - 0,11 = 0,02$ ;  $1,15 : (0,3225)^{3/2} - (1 : 1,3225) = 6,28 - 0,76 = 5,52$  we get for  $x = +a$ :

$$a^2 F : e^2 = (0,12 - 0,02) - (0,10 - 5,52) = 5,52,$$

while  $x = -a$ , with  $3,15 : (8,9225)^{3/2} - (1 : 9,9225) = 0,12 - 0,10 = 0,02$ ;  $1,05 : (0,1025)^{3/2} - (1 : 1,1025) = 32,00 - 0,91 = 31,09$ , gives:

$$a^2 F : e^2 = (0,12 - 31,09) - (0,10 - 0,02) = -31,05.$$

3.  $z = 1,1a$ .

With  $(2 : 3^{3/2}) - (1 : 4) = 0,38^2 - 0,25 = 0,13^2$ ;  $2,2 : (3,84)^{3/2} - (1 : 4,84) = 0,29^2 - 0,20^2 = 0,08^2$ ;  $(3 : 8^{3/2}) - (1 : 9) = 0,13 - 0,11 = 0,02$ ;  $(1,2 : 0,44^{3/2}) - (1 : 1,44) = 4,11 - 0,69 = 3,42$  we get for  $x = +a$ :

$$a^2 F : e^2 = (0,13^2 - 0,02) - (0,08^2 - 3,42) = 3,45.$$

The value  $-\infty$  will again be found for  $x = -a$ . Hence we have now for  $a^2 F : e^2$ :

		$s = 0$	$s = 0,05a$	$s = 0,1a$
	$x = 0$	0	0	0
$(l = 2,1a)$	$x = +a$	10,58	5,52	3,45
	$x = -a$	-10,58	-31,05	$-\infty$



In consequence of the so much smaller distance of the molecules, the action has in many cases become as much as 80-times greater.

It appears from the above calculations that at *low* temperatures, in which case  $l$  approaches to  $2a$ , the terms without  $x$  can very well be omitted, so that we then might write, putting  $z$  everywhere  $= 0$ :

$$\left( \begin{smallmatrix} \text{low temp.} \\ l \rightarrow 2a \end{smallmatrix} \right) \frac{F}{e^2} = - \left[ \frac{l+x}{((l+x)^2 - a^2)^{3/2}} - \frac{1}{(l+x)^2} \right] + \left[ \frac{l-x}{((l-x)^2 - a^2)^{3/2}} - \frac{1}{(l+x)^2} \right], \quad (1b)$$

in which the 1<sup>st</sup> term predominates at  $x = -a$ , the 2<sup>nd</sup> at  $x = +a$ .

Though we may now write:

$$\frac{d^2z}{dt^2} = \frac{e^2}{m} \left[ \frac{d}{dx} \left\{ \frac{1}{\sqrt{(l+x)^2 - a^2}} - \frac{1}{l+x} \right\} + \frac{d}{dx} \left\{ \frac{1}{\sqrt{(l-x)^2 - a^2}} - \frac{1}{l-x} \right\} \right] = \frac{e^2}{m} \frac{dL}{dx},$$

the direct integration of this equation is impossible. For, since  $x = a \sin \varphi$  (in which  $\varphi$  is in general  $= 2\pi \frac{t}{T} + \theta$ , because at the beginning of the motion of  $M$  through  $O$  towards  $P$  ( $t=0$ , when  $M$  in  $O$ ) the electron need not necessarily at the same time be in the position  $x=0$ ), we get:

$$\frac{d^2z}{dt^2} = \frac{e^2}{m} \frac{dL}{dt} \frac{dt}{dx} = \frac{e^2}{m} \frac{dL}{dt} \frac{T : 2\pi}{a \cos(2\pi t/T + \theta)} = \frac{c}{a} \frac{dL}{\cos \varphi dt},$$

and evidently nothing can be done with this differential equation — on account of the complicated form of  $L$ , while  $\sqrt{a^2 - x^2}$  can be substituted for  $a \cos \varphi$ .

Hence nothing remains but making the expression (1<sup>b</sup>) or (1<sup>a</sup>) integrable by expansion into series.

### § 11. Expansion into Series for F.

Instead of e.g.  $\frac{l-z}{((l-z)^2 - a^2)^{3/2}} - \frac{1}{(l-z)^2}$  we shall expand

$$\frac{1}{\sqrt{(l-z)^2 - a^2}} - \frac{1}{l-z},$$

and then differentiate the result with respect to  $z$ , which is easier. We then get:

$$\frac{1}{(l-z+a)^2 (l-z-a)^{1/2}} - \frac{1}{l-z} = \frac{1}{\sqrt{l^2 - a^2}} \left\{ \left(1 - \frac{z}{l+a}\right)^{-1/2} \left(1 - \frac{z}{l-a}\right)^{-1/2} \right\} - \frac{1}{l} \left(1 - \frac{z}{l}\right)^{-1}.$$

For the expression between {} we may write by joining corresponding terms:

$$\begin{aligned}
 & 1 + \frac{1}{2} 2z \frac{l}{l^2 - a^2} + \frac{1.3}{2.4} 2z^2 \frac{l^2 + a^2}{(l^2 - a^2)^2} + \frac{1.3.5}{2.4.6} 2z^3 \frac{l^3 + 3la^2}{(l^2 - a^2)^3} + \\
 & \qquad \qquad \qquad + \frac{1.3.5.7}{2.4.6.8} 2z^4 \frac{l^4 + 6l^2a^2 + a^4}{(l^2 - a^2)^4} + \text{etc.} \\
 & + \frac{1}{2} \cdot \frac{1}{2} z^2 \frac{1}{l^2 - a^2} + \frac{1}{2} \cdot \frac{1.3}{2.4} 2z^3 \frac{l}{(l^2 - a^2)^2} + \frac{1}{2} \cdot \frac{1.3.5}{2.4.6} 2z^4 \frac{l^2 + a^2}{(l^2 - a^2)^3} + \text{etc.} \\
 & \qquad \qquad \qquad + \frac{1.3}{2.4} \cdot \frac{1.3}{2.4} z^4 \frac{1}{(l^2 - a^2)^2} + \text{etc.,}
 \end{aligned}$$

the structure of which is clear. Hence, after multiplication by  $(l^2 - a^2)^{-1/2}$  and expansion of  $\left(1 - \frac{z}{l}\right)^{-1}$ , the original form becomes:

$$\begin{aligned}
 & \left( \frac{1}{\sqrt{l^2 - a^2}} - \frac{1}{l} \right) + z \left( \frac{\frac{1}{2} 2l}{(l^2 - a^2)^{3/2}} - \frac{1}{l^2} \right) + z^2 \left( \frac{\frac{1.3}{2.4} 2(l^2 + a^2) + \frac{1}{2} \cdot \frac{1}{2} (l^2 - a^2)}{(l^2 - a^2)^{5/2}} - \frac{1}{l^3} \right) + \\
 & \qquad \qquad \qquad + z^3 \left( \frac{\frac{1.3.5}{2.4.6} 2(l^3 + 3la^2) + \frac{1}{2} \cdot \frac{1.3}{2.4} 2l(l^2 - a^2)}{(l^2 - a^2)^{7/2}} - \frac{1}{l^4} \right) + \\
 & + z^4 \left( \frac{\frac{1.3.5.7}{2.4.6.8} 2(l^4 + 6l^2a^2 + a^4) + \frac{1}{2} \cdot \frac{1.3.5}{2.4.6} 2(l^2 + a^2)(l^2 - a^2) + \frac{1.3}{2.4} \cdot \frac{1.3}{2.4} (l^2 - a^2)^2}{(l^2 - a^2)^{9/2}} - \frac{1}{l^5} \right) + \text{etc.,}
 \end{aligned}$$

i.e.

$$A + P_0 z + P_1 z^2 + P_2 z^3 + P_3 z^4 + \dots,$$

in which  $A, P_0$ , etc. contain only  $l$  and  $a$ . Through differentiation now arises:

$$\frac{l - z}{((l - z)^2 - a^2)^{3/2}} - \frac{1}{(l - z)^2} = P_0 + 2P_1 z + 3P_2 z^2 + 4P_3 z^3 + \dots,$$

and likewise:

$$\frac{l + z}{((l + z)^2 - a^2)^{3/2}} - \frac{1}{(l + z)^2} = P_0 - 2P_1 z + 3P_2 z^2 + 4P_3 z^3 + \dots,$$

so that the difference of these two expressions (c.f. equation (1a)) is represented by

$$f(z) = 4P_1 z + 8P_2 z^2 + 12P_3 z^3 + \dots$$

After substitution of  $z - x$  for  $z$ , we have also:

$$f(z - x) = 4P_1 (z - x) + 8P_2 (z - x)^2 + 12P_3 (z - x)^3 + \dots,$$

and finally the following equation is obtained:

$$\begin{aligned}
 F: e^2 = f(z) - f(z - x) = & \underline{4P_1 x + 8P_2 (3z^2 x - 3zx^2 + x^3) +} \\
 & \underline{+ 12P_3 (5z^4 x - 10z^3 x^2 + \dots + x^5) + \text{etc.,} \dots} \quad (2)
 \end{aligned}$$

so that  $F$  clearly contains the factor  $x$  (see § 9).

After calculation of the numerators (to which also  $P_4$  has been added), we have:

$$\left. \begin{aligned} P_1 &= \frac{l^2 + \frac{1}{2}a^2}{(l^2 - a^2)^{3/2}} - \frac{1}{l^2} ; & P_2 &= \frac{l^2 + 3l^2a^2 + \frac{3}{8}a^4}{(l^2 - a^2)^{5/2}} - \frac{1}{l^4} \\ P_3 &= \frac{l^2 + \frac{15}{2}l^4a^2 + \frac{45}{8}l^2a^4 + \frac{5}{16}a^6}{(l^2 - a^2)^{7/2}} - \frac{1}{l^6} ; & \text{etc.} \end{aligned} \right\} \quad (3)$$

For small values of  $z$   $F : e^z$  approaches:

$$(z=0) \quad F : e^z = 4P_1x + 8P_2x^3 + 12P_3x^5 + \text{etc.} ; \quad x = a \left( \sin \frac{2\pi t}{T} + \theta \right) \quad (4)$$

**Remark.** That the coefficients of  $l^2, l^4, l^6$ , etc. in the expressions for  $P_1, P_2, P_3$ , etc. become every time  $= 1$ , is not surprising. For they are resp.  $2 \left( \frac{1.3}{2.4} \right) + \left( \frac{1}{2} \right)^2$ ,  $2 \left( \frac{1.3.5.7}{2.4.6.8} \right) + 2 \left( \frac{1}{2} \cdot \frac{1.3.5}{2.4.6} \right) + \left( \frac{1.3}{2.4} \right)^2$ , etc., being the coefficients of the expansion of  $(1-y)^{-1/2} (1-y)^{-1/2}$ , i.e. of  $(1-y)^{-1}$ , which are all  $= 1$ . The coefficients of  $a^2, a^4, a^6$  etc., viz.  $\frac{1}{2}, \frac{3}{8}, \frac{5}{16}$ , etc. are evidently those of the expansion ( $l=0$ ) of  $\left( 1 - \frac{z}{a} \right)^{-1/2} \left( 1 + \frac{z}{a} \right)^{-1/2} = \left( 1 - \frac{z^2}{a^2} \right)^{-1/2}$ , i.e.  $\frac{1}{2}, \frac{1.3}{2.4}, \frac{1.3.5}{2.4.6}$ , etc. According to what follows, the coefficients of the second terms, viz.  $\frac{1}{2}, 3, \frac{15}{2}, 14$ , etc. are represented for  $P_1, P_2, \dots, P_n$  by  $\frac{1}{4} n(n+1)$ , i.e. by  $\frac{1 \times 2}{4}, \frac{3 \times 4}{4}, \frac{7 \times 8}{4}$ , etc. [When we add to this resp.  $\frac{5}{2}, \frac{9}{2}$ , etc. ( $= \frac{1}{4}(2n+3)$ ) of the exponents in the denominators, we get the coefficients of the limiting values of  $P_1, P_2$ , etc. for  $l = \infty$ , mentioned farther on viz.  $\frac{1}{4}(n+2)(n+3) = \frac{3 \times 4}{4}, \frac{5 \times 6}{4}, \frac{7 \times 8}{4}$ , etc.].

Indeed, a somewhat different way of expansion into series of

$$\begin{aligned} \frac{l-z}{(l-z)^2 - a^2} - \frac{1}{(l-z)^2} &\text{ leads to} \\ \frac{1}{(l-z)^2} \left[ \frac{3}{2} \frac{a^2}{(l-z)^2} + \frac{3.5}{2.4} \frac{a^4}{(l-z)^4} + \dots \right] &= \frac{3}{2} \frac{a^2}{(l-z)^4} + \frac{3.5}{2.4} \frac{a^4}{(l-z)^6} + \dots = \\ = \frac{3}{2} \frac{a^2}{l^4} \left( 1 - \frac{4z}{1l} + \frac{4.5z^2}{1.2l^2} + \dots \right) &+ \frac{3.5}{2.4} \frac{a^4}{l^6} \left( 1 + \frac{6z}{1l} + \frac{6.7z^2}{1.3l^2} + \dots \right) + \text{etc.} = \\ = \left( \frac{3}{2} \frac{a^2}{l^4} + \frac{3.5}{2.4} \frac{a^4}{l^6} + \dots \right) &+ z \left( \frac{3}{2} \cdot \frac{4a^2}{1l^5} + \frac{3.5}{2.4} \cdot \frac{6a^4}{1l^7} + \dots \right) + \\ &+ z^2 \left( \frac{3}{2} \cdot \frac{4.5a^2}{1.2l^6} + \frac{3.5}{2.4} \cdot \frac{6.7a^4}{1.2l^8} + \dots \right) + \text{etc.}, \end{aligned}$$

so that for the factor of  $z$  is found (see above):

$$2P_1 = \frac{a^2}{l^1} \left( \frac{3}{2} \cdot \frac{4}{1} + \frac{3.5}{2.4} \cdot \frac{6}{1} \frac{a^2}{l^2} + \dots \right); \text{ for that of } z^3 \text{ we have}$$

$$4P_3 = \frac{a^2}{l^3} \left( \frac{3}{2} \cdot \frac{4.5.6}{1.2.3} + \frac{3.5}{2.4} \cdot \frac{6.7.8}{1.2.3} \frac{a^2}{l^2} + \dots \right); \text{ we get further}$$

$$6P_5 = \frac{a^2}{l^5} \left( \frac{3}{2} \cdot \frac{4.5.6.7.8}{1.2.3.4.5} + \frac{3.5}{2.4} \cdot \frac{6.7.8.9.10}{1.2.3.4.5} \frac{a^2}{l^2} + \dots \right); \text{ etc., etc.}$$

On the one hand e.g.  $6P_5$  approaches for  $l = \infty$  to  $\frac{3}{2} \cdot \frac{4.5.6.7.8}{1.2.3.4.5} \frac{a^2}{l^5}$ , while on the other hand according to (3)  $P_5$  evidently becomes  $= \left( x_5 + \frac{13}{2} \right) \frac{a^2}{l^5}$ , if  $x_5$  represents the coefficient of  $l^4 a^2$ . In  $P_n$  we have therefore  $x_n = \frac{1}{n+1} \frac{3(n+1)(n+2)(n+3)}{2} - \frac{2n+3}{2} = \frac{(n+2)(n+3)}{4} - \frac{2n+3}{2} = \frac{n(n+1)}{4}$ , by which the above is proved.

According to (4) we can now write for  $F: e^2$ :

$$(z=0) \frac{F}{e^2} = 2 \frac{a^2}{l^1} \left( \frac{3}{2} \cdot \frac{4}{1} + \frac{3.5}{2.4} \cdot \frac{6}{1} \frac{a^2}{l^2} \right) x + 2 \frac{a^2}{l^3} \left( \frac{3}{2} \cdot \frac{4.5.6}{1.2.3} + \frac{3.5}{2.4} \cdot \frac{6.7.8}{1.2.3} \frac{a^2}{l^2} + \dots \right) x^3 + \\ + 2 \frac{a^2}{l^5} \left( \frac{3}{2} \cdot \frac{4.5.6.7.8}{1.2.3.4.5} + \frac{3.5}{2.4} \cdot \frac{6.7.8.9.10}{1.2.3.4.5} \frac{a^2}{l^2} + \dots \right) x^5 + \text{etc.},$$

or also:

$$(z=0) \frac{F}{e^2} = 12 \frac{a^2}{l^5} x \left[ \left\{ 1 + \frac{5}{4} \cdot \frac{6}{4} \frac{a^2}{l^2} + \frac{5.7}{4.6} \cdot \frac{8}{4} \frac{a^4}{l^4} + \frac{5.7.9}{4.6.8} \cdot \frac{10}{4} \frac{a^6}{l^6} + \dots \right\} + \right. \\ \left. + \frac{4.5.6}{2.3.4} \frac{x^2}{l^2} \left\{ 1 + \frac{5}{4} \cdot \frac{6.7.8}{4.5.6} \frac{a^2}{l^2} + \frac{5.7}{4.6} \cdot \frac{8.9.10}{4.5.6} \frac{a^4}{l^4} + \dots \right\} + \right. \\ \left. + \frac{6.7.8}{2.3.4} \frac{x^4}{l^4} \left\{ 1 + \frac{5}{4} \cdot \frac{6.7.8.9.10}{4.5.6.7.8} \frac{a^2}{l^2} + \frac{5.7}{4.6} \cdot \frac{8.9.10.11.12}{4.5.6.7.8} \frac{a^4}{l^4} + \dots \right\} + \text{etc.} \right],$$

i.e.

$$(z=0) \frac{F}{e^2} = 12 \frac{a^2}{l^5} x \left[ \left\{ 1 + \frac{15}{8} \frac{a^2}{l^2} + \frac{35}{12} \frac{a^4}{l^4} + \frac{525}{128} \frac{a^6}{l^6} + \dots \right\} + \right. \\ \left. + 5 \frac{x^2}{l^2} \left\{ 1 + \frac{7}{2} \frac{a^2}{l^2} + \frac{35}{4} \frac{a^4}{l^4} + \dots \right\} + 14 \frac{x^4}{l^4} \left\{ 1 + \frac{45}{8} \frac{a^2}{l^2} + \frac{165}{8} \frac{a^4}{l^4} + \dots \right\} + \text{etc.} \right],$$

for which may be written:

$$(z=0) \frac{F}{e^2} = 12 \frac{a^2}{l^5} x \left[ \varphi_1 + 5 \frac{x^2}{l^2} \varphi_3 + 14 \frac{x^4}{l^4} \varphi_5 + \text{etc.} \right] \quad (4a)$$

**Two limiting cases.**

1. If  $l$  is *large* with respect to  $a$  (gases), (4) or (4a) evidently approaches to

$$\frac{F}{e^2} = 12 \frac{a^2}{l^5} x, \quad (l \text{ large}), \dots \dots \dots (5)$$

and this not only for  $z=0$ , but also for finite values of  $z$ , provided they are not large with respect to  $l$ .

This is, therefore, the limiting value of the action, when the moving molecule, with comparatively large distances of the molecules, is between  $P$  and  $Q$ , if not too close to one of them. The action is in inverse ratio to the **fifth** power of  $l$ , and directly proportional to the square of the radius  $a$  of the orbit of the electrons. It remains pretty well unchanged at the same value of  $x$ , when  $M$  moves from  $O$  slightly to the left or to the right, but it is on the contrary proportional to the deviation  $x$  of the electron in its orbit; hence it is **purely periodical**. For  $x=0$  the action is  $=0$ , but not so for  $x = +a$  or  $x = -a$ , even if  $M$  is in the neutral point  $O$ .

As regards the action between  $M$  and  $P$ , resp.  $Q$  separately, we have according to (1a) and what was found above:

$$F_1 : e^2 = (P_0 + 2P_1 z + 3P_1 z^2 + \dots) - (P_0 + 2P_1 (z-x) + 3P_1 (z-x)^2),$$

i.e.

$$\left. \begin{aligned} F_1 : e^2 &= 2P_1 x + 3P_1 (2zx - x^2) + 4P_1 (3z^2 x - 3zx^2 + x^3) + \dots \\ \text{and likewise} \\ F_2 : e^2 &= -2P_1 x + 3P_1 (2zx - x^2) - 4P_1 (3z^2 x - 3zx^2 + x^3) + \dots \end{aligned} \right\}$$

so that about half of the total action  $(F_1 - F_2) : e^2 = 4P_1 x + \text{etc.}$  comes from the atom  $P$ ; the other part, as the action of a force in the opposite direction, from the atom  $Q$ . If e.g.  $x = +a$ , the electron is as far as possible in the direction of  $Q$ , and  $M$  will be attracted by  $P$  with a force  $=$  about  $2P_1 a e^2$ , and repelled by  $Q$  with an almost equal force. (Here  $P_1 = 3a^2 : l^5$ ).

When the gas-molecule  $M$  moves towards  $P$ , its velocity  $u$  will, accordingly, also undergo *periodical* modifications, till it has approached  $P$  so closely, that at  $l - z =$  about  $2a$ , the attractive force being still finite at  $x = +a$ , the repulsive force begins already to approach infinity at  $x = -a$ . (see § 10); hence the velocity is reduced to 0, after which  $M$  moves back (collision).

But with *very small values* of  $u$  it may occur that this return already takes place long before  $P$  has been reached. We shall revert to this in the next paragraph; it is the well-known case of **gas-degeneration**.

2. If  $l$  is *small* with regard to  $a$  (*liquids and solid bodies*), then  $l \ll 2a$ ,  $z \ll 0$  may be put, and (4) holds therefore again, in which now, however,  $P_1, P_2$ , etc. approach according to (3) to the following values: <sup>1)</sup>

$$\left. \begin{aligned} P_1 &= \frac{1}{a^3} \left( \frac{1}{6} \sqrt{3} - \frac{1}{8} \right) = \frac{0,1637}{a^3}, \text{ hence } 4P_1 = \frac{0,6548}{a^3} \\ P_2 &= \frac{1}{a^5} \left( \frac{227}{1944} \sqrt{3} - \frac{1}{23} \right) = \frac{0,1710}{a^5}, \text{ ,, } 8P_2 = \frac{1,3680}{a^5} \\ P_3 &= \frac{1}{a^7} \left( \frac{1103}{11664} \sqrt{3} - \frac{1}{128} \right) = \frac{0,1560}{a^7}, \text{ ,, } 12P_3 = \frac{1,8718}{a^7} \end{aligned} \right\}$$

so that now  $F: e^2$  will approach

$$\frac{F}{e^2} = \frac{0,655}{a^3} x \left( 1 + 2,089 \frac{x^2}{a^2} + 2,859 \frac{x^4}{a^4} + \dots \right) \dots \quad (6)$$

It is self-evident that this expansion into series is now only valid for *small values of  $x$*  with respect to  $a$ . For  $x = \pm a$  of course  $F$  becomes  $= \pm \infty$ . I will just point out here, that when in  $x = a \sin \varphi$  means are taken over all values of  $\varphi$  between 0 and  $2\pi$ , according to (2) and (4) the total force would become  $= 0$  only at  $z = 0$ . But when  $z$  is not  $= 0$ , hence when  $M$  is no longer halfway between  $P$  and  $Q$ , this is evidently no longer the case (*even powers of  $x$* ). And according to the above separate expressions of  $F_1$  and  $F_2$  they do not become  $= 0$  at  $z = 0$  even when averaged. The separate forces "averaged" with respect to  $x$ , and also the mean total force will always be *repulsive* (excepted at  $z = 0$  in the last case), because the terms with even powers of  $x$  have all of them the  $-$  sign. <sup>2)</sup> And this refutes DEBIJE's assertion (see § 8), that without special suppositions (polarisation of the molecules in each other's electric field) the resulting action would always be  $= 0$  according to a well-known electric theorem. It is possible to verify by calculation that this is also true for the problem in three-dimensional space.

<sup>1)</sup> For the quantities  $\varphi_1, \varphi_2$ , etc. in (4a) the values  $\varphi_1 = \frac{32}{12} \times 0,6548 = 1,7459$ ;  
 $\varphi_2 = \frac{128}{60} \times 1,3680 = 2,9184$ ;  $\varphi_3 = \frac{512}{168} \times 1,8718 = 5,7045$ ; etc. are easily found.

<sup>2)</sup> This has of course nothing to do with the question of the *Virial* of attraction and repulsion, as in the calculation of this the *time-average* plays a part. Indeed, in the equation of state for gases  $\frac{pv}{RT} = 1 + \frac{b}{v} - \frac{a/v}{RT}$  also the *Virial* of repulsion  $b/v$  predominates at high temperatures. This question must afterwards be treated separately.

§ 12. The Equations of Motion. Possibility of Multiple Orbits at Low Temperatures.

Since with small values of  $z$  we may write according to (4):

$$F = \gamma_1 x + \gamma_2 x^3 + \gamma_3 x^5 + \dots \quad (x = a \sin \varphi),$$

this may always be reduced to

$$F = \alpha_1 \sin \varphi + \alpha_3 \sin 3\varphi + \alpha_5 \sin 5\varphi + \dots,$$

after which integration is possible. But since the expansion into series is not practicable after all for large values of  $x$ , in the neighbourhood of  $+a$  or  $-a$ , it is better to retain only the 1<sup>st</sup> term in the original equation, or to write:

$$F : \gamma_1 x (1 + \lambda_2 x^2 + \lambda_4 x^4 + \dots),$$

and take the averages with respect to the factor between parentheses. Then we get:

$$F = \gamma_1 x \times \lambda_m = \gamma x,$$

in which  $\gamma$  will be  $> 0,655 e^2 : a^2$ , when  $l$  approaches  $2a$ . As now

$$\frac{1}{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \sin^2 \varphi \text{ is } = \frac{1}{2}, \text{ the same with } \sin^4 \varphi \text{ will be } = \frac{1.3}{2.4}, \text{ with } \sin^6 \varphi$$

is  $= \frac{1.3.5}{2.4.6}$ , etc.,  $F$  becomes for  $l = 2a$  according to (6):

$$F = \frac{0,655 e^2}{a^2} x (1 + 1,04 + 1,07 + \dots),$$

hence averaged many times greater than  $\gamma_1 x$ . Let us now, for the sake of orientation, integrate the simple equation

$$\frac{d^2 z}{dt^2} = \frac{F}{m} = \frac{\gamma}{m} x, \dots \dots \dots (7)$$

in which  $\gamma$  for small values of  $x$  will approach  $\gamma_1$ , when  $l = 2a$  (solid bodies and liquids), whereas for large values of  $l$  (gases)  $\gamma$  approaches  $12a^2 e^2 : l^5$  (see above).

Thus we find with  $\varphi = 2\pi \frac{t}{T} + \theta$ :

$$\frac{dz}{dt} = u = u_0 - \frac{T \gamma}{2\pi m} (a \sin \varphi - a \cos \theta), \dots \dots (8)$$

so that duly  $u$  becomes  $= u_0$  at  $t = 0$  (when  $M$  passes through the neutral point  $O$ ). Repeated integration yields for the path passed over:

$$z = \left( u_0 + \frac{T \gamma}{2\pi m} a \cos \theta \right) t - \left( \frac{T}{2\pi} \right)^2 \frac{\gamma}{m} (a \sin \varphi - a \sin \theta), \dots (9)$$

which yields  $z = 0$  at  $t = 0$ . It will now further entirely depend on the value of the phase-difference  $\theta$  (the difference of time between the fictitious passage of the electron through the nucleus, and of  $M$  through  $O$ ), what type of path of periodic movement will be obtained.

1. If  $\theta = 0$  (the electron passes (fictitiously) the nucleus from the right towards the left exactly when  $M$  moves in  $O$  towards the right in the direction of  $P$ ), we get:

$$u = u_0 + \frac{T}{2\pi m} \gamma (a - a \cos \varphi) \quad ; \quad \varphi = 2\pi \frac{t}{T}.$$

The value of  $u$  (see fig. 3) will now always be  $> u_0$ , so that there can only be question of its becoming 0 on collision ( $l - z = 2a$ ).

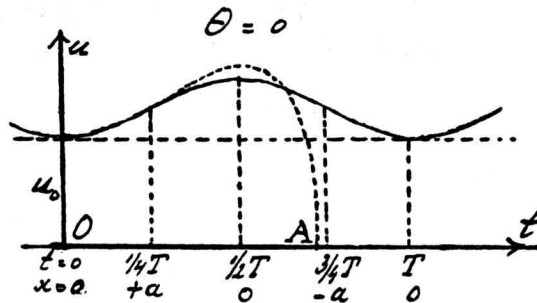


Fig. 3.

The molecule  $M$  will then approach  $P$  so closely till the electron has assumed the position close to  $x = -a$ , in consequence of which the repulsive force becomes very great. Then the velocity becomes  $= 0$  in an exceedingly short moment, and the molecule is thrown

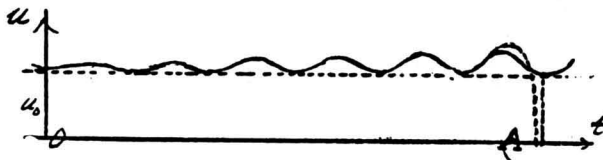


Fig. 3a. (Gases).

back (in  $A$ , close to  $t = 3/4 T$ ). When the molecules are far enough from each other (gases), several periods may pass before this collision at last sets in (Fig. 3a). The increasing values of the amplitudes in Fig. 3a must of course be attributed to the increasing influence of  $z$ , through which the action exerted becomes stronger and stronger (cf. also the calculations in § 10). This gives also rise to the deviations of the course, following from (8), close to the collision (repre-



sented by the dotted lines). Indeed, for the sake of simplicity we have so far always neglected the influence of  $z$ .

We still point out that the magnitude and the sign of the action exercised are always represented in the figures by the inclination of the tangents to the curve.

2.  $\theta = 180^\circ$ . Then the electron passes the nucleus (fictitiously) just from the *left* to the *right*, when  $M$  goes from  $O$  to  $P$ , and (8) becomes:

$$u = u_0 - \frac{T}{2\pi m} \gamma (a + a \cos \varphi) \quad ; \quad \varphi = 2\pi \frac{t}{T} + \pi.$$

Now the velocity  $u$  always remains below  $u_0$  (see Fig. 4). The case of "collision" has been drawn at two successively possible places, viz. at  $A$  and  $A'$ .

What distinguishes this case from the preceding one, is the possibility that  $u$  becomes  $= 0$  before the "collision", and the molecule

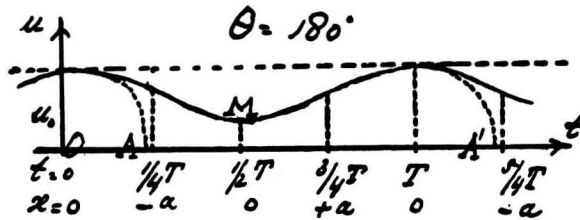


Fig. 4.

accordingly already "returns" before  $P$  has been reached. This will evidently take place as soon as  $u_0$  is so small that  $M$  lies low enough for the curve to intersect the  $t$ -axis ( $u = 0$ ) (Fig. 4a). This takes place e.g. in  $B$ . Transformed spatially this means that the molecules will move round the position of equilibrium  $O$  in **closed orbits**, as soon as we get below the point where the curve touches the  $t$ -axis for the first time (**melting point**)<sup>1)</sup>. In this case the

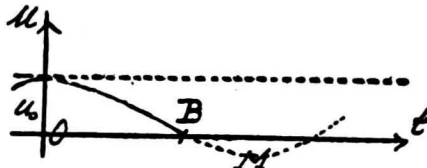


Fig. 4a.

<sup>1)</sup> I may be allowed to anticipate on what follows, and state here that the melting-point calculated in this way for H — if it were realisable — will lie at  $36^\circ,4$  abs. As this melting-point must lie higher than that of  $H_2$  (because the molecular attraction  $a$  that plays a part in it, is greater for H than for  $H_2$ ), this result is not impossible. (melting-point  $H_2$  lies at  $14^\circ$  abs.).

central force directed towards the centre ( $O$ ) will always correspond to the resulting repulsion of the surrounding molecules.

Also for *gases* can this take place, but as the distance of the molecules is then greater, hence the oscillations in the value of  $u$  much smaller, the limiting value of  $u_0$  (**point of degeneration**) lies much lower than the corresponding value of  $u_0$  (melting-point) for solid bodies. Besides — in accordance with the mutual distance

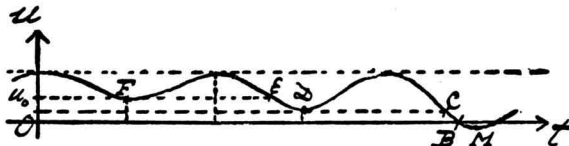


Fig. 4b. (Gases).

of the molecules — this transition may take place at different places, during the 1<sup>st</sup> period, the 2<sup>nd</sup>, the 3<sup>rd</sup> period etc. (Cf. Fig. 4b, where the return takes place at the third period).

To *given* mutual distance of the molecules (gas density) corresponds, therefore, a definite value of  $u_0$  for which the transition already takes place at the 1<sup>st</sup> period, (*degeneration point proper*), a value where the transition does not occur until the 2<sup>nd</sup> period, etc. etc. Here too the molecules will, therefore, revolve round the positions of equilibrium in closed, ever narrower orbits — as the temperature descends.

And thus the phenomenon of **gas-generation** has been explained in a natural way.

When at a given gas-density  $u_0$  becomes too great, then  $u$  does not become 0 before the molecule comes in contact with  $P$  (“collides”), as is drawn at  $A$  in Fig. 3 and 4. Hence no longer any closed orbits (solid bodies above the melting-point; gases above the degeneration point).

It is very remarkable in this, that when  $u_0$  (for gases) becomes gradually smaller and smaller (hence the temperature lower and lower), the place where  $u$  becomes zero suddenly *skips* from  $C$  to  $D$  (see Fig. 4b), from  $E$  to  $F$ , etc. — which corresponds to this that the corresponding *wider* orbit round the position of equilibrium abruptly, hence *discontinuously*, changes into a *narrower* orbit. The latter varies only between  $D$  and  $E$ , lying *close to each other* (the figure represents time-abscissae, but distance-abscissae of course correspond with them in a corresponding  $z-u$  diagram), after which it suddenly skips again to the *still narrower* orbit, corresponding to  $F$ . This is then the final orbit, which as  $u_0$  gets still lower, again gradually shrinks. It does not diminish to 0, however, but to a

limiting orbit, which will be discussed later (in connection with the *zero-point energy*)<sup>1)</sup>.

And thus an *analogue* has been obtained of the possible quantized orbits which a negative electron can describe round the positive nucleus. The points *D* and *E* lie in this latter case *exceedingly close* together, so that the discontinuity in the value of the radii of the possible orbits is almost complete.

But for this the assumption is required that also for electron and nucleus the force acts periodically, e. g. through this that the positive nucleus executes a *pulsating* movement (analogous to the motion studied by BJERKNES)<sup>2)</sup>. It may also be assumed that the nucleus always sucks in "ether" from its surroundings (which is led off to the 4<sup>th</sup> dimension), the electron expelling ether in the same way. When a rotation is assumed to take place of the electron round an axis coinciding with the direction of the motion, the known equations can be derived of the electro-magnetic field<sup>3)</sup>.

But this cannot yet be fully discussed here. One thing at least is certain, that *if* the electrons revolve round the nucleus in *definite* orbits (in which the quantity  $h$  plays a part in the determination of radius and velocity), that then necessarily, in consequence of our above considerations, this same quantity  $h$  must play a part in the movement of the molecules in closed orbits round positions of equilibrium — in consequence of which that quantity will naturally occur in the relation between *E* and *T* which we derived in our previous paper, as analogue of PLANCK's relation; while the quantity  $v$  will be in connection with the time of revolution of the molecules in their closed orbits, which in its turn will again be in relation with the time of revolution *T* of the electrons round the nucleus — as we saw above.

*Clarens, summer 1921.*

*To be continued.*

<sup>1)</sup> On decrease of temperature such an abrupt succession of some ever narrower orbits is perhaps also possible for *solid* bodies, and this may possibly be brought in connection with some *allotropic* states, which are met with in many elements and compounds.

<sup>2)</sup> Very suggestive in this respect is an old Paper, almost entirely forgotten, by VOIGT in the "Journ. f. reine v. angew. Mathematik", Band 89, on "Der leuchtende Punkt." VOIGT chiefly calculated the state of vibration close to this point, when either a periodic translatory movement, or a periodic rotatory movement was supposed. Later on KIRCHHOFF (Ibid 90, p. 34) considerably simplified VOIGT's derivation.

<sup>3)</sup> The assumption of expulsion of ether from the electron with *the velocity of light* would then also explain that the velocity of the electron can never exceed the velocity of light, and an idea can be obtained of the mechanics of relativity (factor  $1 - v^2/c^2$ ).