Physics. — "*A remarkable case of quantization.*" By Prof. P. EHRENFEST and G. BREIT.

(Communicated at the meeting of January 28, 1922).

1. It is possible to indicate simple mechanical systems for which a formal application of the quantum rules gives well defined and yed apparently unreasonable stationary motions. BOHR's Principle of Correspondence [1]) offers an essentially new viewpoint for the treatment of these cases and will probably contribute to their complete solution. It will suffice to discuss a special case which is so chosen as to minimize the mathematical analysis. [2])

2. A rigid electric dipole having a moment of inertia I is free to rotate in the X, Y plane about its own midpoint.

Let us suppose that by means of a suitable kinematical arrangement the rotating dipole is thrown back elastically as soon as the angle φ, which the dipole makes with the axis of X, reaches the boundaries of the interval

$$-f\,.\,2\pi \leq \varphi \leq +f\,.\,2\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where f is a large, in general an irrational number. Let an angular velocity ω be given to the dipole. Its angular momentum is then $p = I\omega$ and it executes a periodic motion with the *period*

$$T = 4f\,.\,\frac{2\pi}{\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

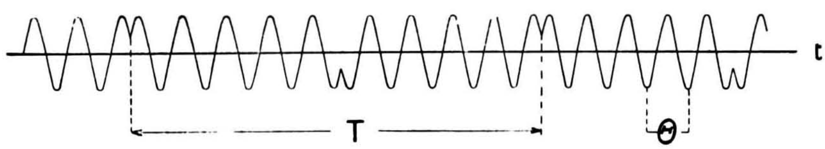
During the motion the dipole traverses the interval (1) making in a period $2f$ complete revolutions to the right followed by the same number of revolutions to the left. In the motion the "*quasi-periode*"

1) N. BOHR, Quantum theory of line-spectra I, II Kopenhagen 1918. H. KRAMERS, Intensities of spectral lines. Kopenhagen 1919.

2) A case which differs slightly from the one discussed in § 2, namely the case of a rigid dipole torsionally suspended by an elastic thread of small rigidity one of us submitted to EINSTEIN for consideration as early as 1912 (with reference to the problem of quantization of H_2 molecules — P. EHRENFEST. Verh. d. D. Phys. Ges. **15**, 451, 1913). It was impossible however to settle the difffculty here discussed by the means which were then available.

$$\Theta = \frac{2\pi}{\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

becomes noticeable. This period is a $4f^{th}$ part of T and is equal to the time taken by the dipole to make a complete revolution through the angle 2π. The projection of the moment of the dipole on a line in the plane X-Y-say on the axis of X depends on the time in the manner shown on the figure (for the sake of economy the "large number" f is here taken as being approximately 2).



3. The quantum relation for our system is

$$\int p dq = nh \qquad (n = 0, 1, 2 \ldots) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where the coordinate q is the angle φ, p is the corresponding momentum $I\,\omega$ and the integral is taken over a complete perid T. This gives in our case

$$4f \,.\, 2\pi p = nh \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

or

$$p = n \frac{h}{8f\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If now the restricting boundary of the interval (1) is so chosen as to make f very large, then the differences between consecutive values of p (see (6)) (and therefore also between consecutive values of the energy) are very small.

4. This result appears to be unacceptable. In fact if we pass to the limit of $f = \infty$ i.e. if the restriction of the boundaries on the dipole disappears then equation (4) gives certainly

$$p = m \frac{h}{2\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

for now θ is the period. Here (Equ. (7)) p changes by finite steps whereas if the previous consideration be applied (Equ. (6)) the steps become infinitesimal for $f = \infty$. This is the contradiction to be discussed.

5. Bohr's principle of correspondence offers a new point of view for the treatment of this case. As before let f be a very large

number and suppose that the permissible values of p are truly given by Equ. (6). We want to know the requirements made by the principle of correspondence as to the "probability of a transition" from the state $n = n_1$ to the state $n = n_2$ (say as the result of absorption in a field of radition). The Principle of correspondence regards the probability of the transitions as analogous to the amplitudes of "corresponding" harmonics in a Fourier series expansion of the function represented graphically on the figure. This function represents the dependence on the time of the X or Y component of the dipole's moment. The Fourier expansion of the function may be put into the form

$$X = \sum_1^\infty A_s \cos\left(s\frac{2\pi}{T}\right) \quad . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

The harmonics "corresponding" to the transition $n_1 \rightarrow n_2$ are given by:

$$s = n_2 - n_1 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

From an inspection of the figure or by means of a short calculation it becomes apparent that for a large value of f the amplitudes of all the harmonics are small with the exception of those harmonics whose period is nearly equal to the "quasiperiod" θ i.e. with the exception of those for which

$$\frac{T}{s} \doteqdot \Theta \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (10)$$

or

$$s \doteqdot 4f \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (11)$$

Therefore if f is large all the transitions have a very small probability with the exception of those for which very nearly

$$n_2 - n_1 \doteqdot 4f \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (12)$$

and therefore (in virtue of (6))

$$p_2 - p_1 = (n_2 - n_1)\frac{h}{4f\,.\,2\pi} \doteqdot \frac{h}{2\pi} \quad . \; . \; . \; . \; . \quad (13)$$

which is the same as the interval between consecutive values of p prescribed by (7) for infinitely large values of f.

6. If therefore we should take a collection of identical samples of our system having all the same *very large value of f*, being all at rest i.e. in the state $p = 0$ at the time $t = 0$ and if we should subject each sample independently to the action of a black body radiation — then we should find at a later time t that:

A. Out of the very dense succession of the p levels which are permitted by (6) only those are occupied by an appreciable number of the systems which nearly coïncide with the levels of p given by (7).

B. The transitions which take place have almost without exception the magnitude $\frac{h}{2\pi}$ (and not a multiple of it) (See (13)). This is again in good agreement with the fact that for $f = \infty$ the FOURIER expansion of the x (or y) component contains only the fundamental and no higher harmonics so that for this case the Principle of Correspondence allows only the transitions (see (7)) for which $m_2 - m_1 = \pm 1$.

7. A question must now be mentioned the precise explanation of which would be of value. For the discussion of thermal equilibrium in our complex we must know the "weights" (the *a priori* probability) to be ascribed to each p level. For $f \pm \infty$ it would appear that the same weight should be given to every stop of (6) — independently of the value of f and independently of the density with which the levels follow each other. On the other hand for $f = \infty$ only the levels given by (7) are to have a weight (the same for all). A closer examination of this case will probably make it necessary to consider the fact that we are concerned here with a double limit viz. *lim* $t = \infty$ (the lapse of an infinitely long time for the establishment of thermal equilibrium) and *lim* $f = \infty$; our dissatisfaction is really based on an unconscious demand that the result should be independent of the order in which the two limits are approached.

The junior author of the paper (G. BREIT) is Fellow of the National Research council, United States of America.

The University, Leiden.