Mathematics. — *"Numbers of Circles Touching Plane Curves Defined by Representation on Point Space."* By L. J. SMID Jr. (Communicated by Prof. HENDRIK DE VRIES),

(Communicated at the meeting of June 24, 1922).

The circles of a plane (degenerations included) may be represented without exception through a one-one representation on the points of a projective space. (R. MEHMKE, Zeitschrift für Mathematik und Physik 24 (1879)). We can arrive at it among others in the following way:

Let W be an umbilical point of a quadric O^2 and let w be the tangent plane at that point, B a plane parallel to w. A plane section of O^2 with its pole relative to O^2 is projected out of W on B as a circle with its centre, and inversely. We consider this pole as the image of the circle.

As a special case we may take for O^2 a quadric of revolution of which W is a vertex. If moreover O^2 is a sphere, we get the representation of Prof. JAN DE VRIES (Verhandelingen 29); if W moves to infinity it becomes the representation of Dr. K. W. WALSTRA (Verhandelingen 25).

Prof. HK. DE VRIES has studied cyclographically the circles touching a curve C in B of the order μ, the class ν, passing ε times through both the circle points (with ε different tangents in finite space which cut C at those points in $\varepsilon+1$ points), touching the line g_∞ singly in σ different points and having further no other singularities than δ nodes, $\varkappa$ cusps, τ bi-tangents and ι inflexional tangents (Verhandelingen 8).

We arrive at the same results through the above mentioned representation. We shall only consider the principal ones.

The curve C is projected out of W on O^2 as a curve consisting of the two generatrices through W, counted ε times, and a curve k of the order $n = 2\mu - 2\varepsilon$ passing $(\mu-2\varepsilon)$-times through W. σ pairs of tangents at W coincide, because the parabolic branches of C give rise to cusps of k in W. Further k has δ nodes, $\varkappa$ cusps and $(\mu-\varepsilon)(\mu-\varepsilon-1)$ apparent nodes and no stationary tangents. By means of PLÜCKER's formulas we find other numbers characteristic of k.

From the nature of the representation there follows that the points of the surface L of the tangents of k represent the circles cutting C at right angles. The tangent planes to O^2 at the points of k

envelop a developable surface K the points of which represent the circles touching C and the points of the cuspidal curve l of K represent the osculation circles of C. There exists a polar relation between the points, tangents, and planes of osculation of k and the planes of osculation, tangents, and points of l. Out of the characteristic numbers of k and L we find accordingly through dualisation the characteristic numbers of l and K, for instance:

Order of l: $m = \iota + 3\mu - 6\varepsilon - 2\sigma$

Order of K: $r = 2\mu + \nu - 4\varepsilon - \sigma$

Cusps of l: $\beta = 5\mu - 3\nu + 3\iota - 8\varepsilon - 3\sigma$

Order of the nodal curve of K: $x = \frac{1}{2}\{(2\mu + \nu - 4\varepsilon - \sigma)^2 - 13\mu - \nu - 3\iota + 24\varepsilon + 7\sigma\}$.

From this follows among others:

To a given pencil there belong r tangent circles of C, but to a concentric pencil only $r - (\mu - 2\varepsilon)$ in finite space (class of the evolute). If we have 3 curves C_1, C_2, C_3, the surfaces K_1, K_2, K_3, have in all $r_1 r_2 r_3$ points in common, of which however there lie $4(\mu_1 - 2\varepsilon_1)(\mu_2 - 2\varepsilon_2)(\mu_3 - 2\varepsilon_3)$ in W. The rest is the number of circles touching the 3 curves.

Through a given point there pass m osculating circles of C. The projection of l out of W on B is the evolute; l passes σ times through W, hence the order of the evolute is $m - \sigma$. The evolute has β cusps (vertices of C) in finite space and moreover $\mu - 2\varepsilon - 2\sigma$ at infinity, arising because $\mu - 2\varepsilon - 2\sigma$ tangents of l pass through W, lie in w and have their points of contact outside W.

Through a point there pass x circles touching C twice. The locus of the centres of these bi-tangent circles is the projection of the nodal curve of K. This curve however passes $s = (\mu - 2\varepsilon)(\mu - 2\varepsilon - 1) - \sigma$ times through W, so that the order of the projection is only $x - s$.

The number of tangents to l cutting l once more, is $\gamma = rm + 12r - 14m - 6n$. Of these $2\sigma(\mu - 2\varepsilon - 2)$ lie in w through W. The rest gives the number of circles of curvature touching C once more.

The number of triple points of K is:

$$t = \frac{1}{6}\{r^3 - 3r(r + n + 3m) - 58r + 42n + 78m\}.$$

Of these however

$$4\,\frac{(\mu - 2\varepsilon)(\mu - 2\varepsilon - 1)(\mu - 2\varepsilon - 2)}{1\,.\quad 2\,.\quad 3} - 2\sigma(\mu - 2\varepsilon - 2)$$

lie in W. The rest gives the number of circles touching C thrice.

If we work out these formulas they get the same form as those of Prof. DE VRIES as is to be expected.