Mathematics. — *"On the Light Path in the General Theory of Relativity."* By Prof. W. VAN DER WOUDE. (Communicated by Prof. H. A. LORENTZ.)

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In EINSTEIN'S theory the path of a ray of light is found by putting the condition that it is a geodesic null line in the four-dimensional time space [1]). If accordingly we represent the line element of this time space by

$$ds^2 = \sum_{i,k} g_{ik}\, dx_i\, dx_k \quad \ldots\ldots\ldots \quad (1)$$

the light path satisfies equally the equations of the geodesic as those of the null line

$$ds = 0 \quad \ldots\ldots\ldots\ldots \quad (2)$$

As far as we know the remarkable relation existing between these differential equations, has not yet been pointed out. We shall prove that this may be expressed in the following way:

a geodesic having one element, i.e. one point with the tangent at that point, in common with a null line, is itself a null line.

In order to prove this we shall first give the equations of the geodesic a form different from the usual one (§ 1), as on account of (2) it is not desirable to take s for the independent variable. With a view to an application which we shall give later on, we take one of the coordinates of the time space for the independent variable.

We shall conclude by pointing out the (evident) physical meaning of the theorem.

§ 1. If the line element is represented by

$$ds^2 = \sum_{i,k} g_{ik}\, dx_i\, dx_k\,,$$

the equations of the geodesic are

$$\frac{d^2 x_\nu}{ds^2} + \sum_{\lambda,\mu} \begin{Bmatrix} \lambda\,\mu \\ \nu \end{Bmatrix} \frac{dx_\lambda}{ds}\frac{dx_\mu}{ds} = 0 \quad \ldots\ldots \quad (3)$$

[1]) From this there follows for the statical field (g_{ik} independent of the time-coordinate x_0 and $g_{0l} = 0$ for $l \neq 0$) the principle of FERMAT for the minimum time of light in three dimensional space.

CHRISTOFFEL's symbol $\left\{ \begin{matrix} \lambda\ \mu \\ \nu \end{matrix} \right\}$ has here the meaning:

$$\left\{ \begin{matrix} \lambda\ \mu \\ \nu \end{matrix} \right\} = \sum_{\tau} g^{\nu\tau} \left[\begin{matrix} \lambda\ \mu \\ \tau \end{matrix} \right],$$

where $g^{\nu\tau}$ is the algebraical minor of $g_{\nu\tau}$ in the g-determinant divided by this determinant, and

$$\left[\begin{matrix} \lambda\ \mu \\ \nu \end{matrix} \right] = \frac{1}{2}\left(\frac{\partial g_{\lambda\nu}}{\partial x_{\mu}} + \frac{\partial g_{\mu\nu}}{\partial x_{\mu}} - \frac{\partial g_{\lambda\mu}}{\partial x_{\nu}}\right)$$

As independent variable we chose one of the coordinates, e.g. x_0. In this case

$$\frac{dx_\nu}{ds} \times \frac{ds}{dx_0} = \frac{dx_\nu}{dx_0}\ ;\ \frac{d^2x_\nu}{ds^2}\left(\frac{ds}{dx_0}\right)^2 + \frac{dx_\nu}{ds}\frac{d^2s}{dx_0{}^2} = \frac{d_2x_\nu}{dx_0{}^2}\ ; \quad . \quad . \qquad (4)$$

especially for $x_\nu = x_0$

$$\frac{d^2x_0}{ds^2}\left(\frac{ds}{dx_0}\right)^2 = -\frac{dx_0}{ds}\frac{d^2s}{dx_0{}^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4')$$

If therefore we multiply the former of the equations

$$\frac{d^2x_\nu}{ds^2} + \sum_{\lambda,\mu} \left\{ \begin{matrix} \lambda\ \mu \\ \nu \end{matrix} \right\} \frac{dx_\lambda}{ds}\frac{dx_\mu}{ds} = 0$$

$$\frac{d^2x_0}{ds^2} + \sum_{\lambda,\mu} \left\{ \begin{matrix} \lambda\ \mu \\ 0 \end{matrix} \right\} \frac{dx_\lambda}{ds}\frac{dx_\mu}{ds} = 0$$

by $\left(\frac{ds}{dx_0}\right)^2$, the latter by $\left(\frac{ds}{dx_0}\right)^2 \frac{dx_\nu}{dx_0}$, we find after subtraction by the aid of (4) and (4′)

$$\frac{d^2x_\nu}{dx_0{}^2} + \sum_{\lambda,\mu}\left[\left\{ \begin{matrix} \lambda\ \mu \\ \nu \end{matrix} \right\} - \left\{ \begin{matrix} \lambda\ \mu \\ 0 \end{matrix} \right\}\frac{dx_\nu}{dx_0}\right]\frac{dx_\lambda}{dx_0}\frac{dx_\mu}{dx_0} = 0. \quad . \quad . \quad . \qquad (5)$$

These are the equations of the geodesic which we had in view. Taken as the equations of the geodesic of a two-dimensional space (a surface in the usual meaning), they give

$$\frac{d^2v}{du^2} - \left\{ \begin{matrix} 2\ 2 \\ 1 \end{matrix} \right\}\left(\frac{dv}{du}\right)^3 + \left(\left\{ \begin{matrix} 2\ 2 \\ 2 \end{matrix} \right\} - 2\left\{ \begin{matrix} 1\ 2 \\ 1 \end{matrix} \right\}\right)\left(\frac{dv}{du}\right)^2 +$$

$$+\left(2\left\{ \begin{matrix} 1\ 2 \\ 1 \end{matrix} \right\} - \left\{ \begin{matrix} 1\ 1 \\ 1 \end{matrix} \right\}\right)\frac{dv}{du} + \left\{ \begin{matrix} 1\ 1 \\ 2 \end{matrix} \right\} = 0,$$

a well known form, which is often taken as the starting point for the discussion of the properties of this line.

§ 2. We multiply (5) by $g_{\nu\rho}\dfrac{dx_\rho}{dx_0}$ and sum with respect to ν and ρ; the equation thus found

$$\sum_{\nu,\rho} g_{\nu\rho}\left(\frac{d^2x_\nu}{dx_0^{\,2}}+\sum_{\lambda,\mu,\nu,\rho} g_{\nu\rho}\left[\begin{Bmatrix}\lambda\ \mu\\ \nu\end{Bmatrix}-\begin{Bmatrix}\lambda\ \mu\\ 0\end{Bmatrix}\frac{dx_\nu}{dx_0}\right]\frac{dx_\lambda}{dx_0}\frac{dx_\mu}{dx_0}\right)\frac{dx_\rho}{dx_0}=0 \quad . \qquad (6)$$

may be reduced to a different form.

Let us consider the first term:

$$\sum_{\nu,\rho} g_{\nu\rho}\frac{d^2x_\nu}{dx_0^{\,2}}\frac{dx_\rho}{dx_0};$$

As $g_{\nu\rho}=g_{\rho\nu}$ we may also write this

$$\frac{1}{2}\sum_{\nu,\rho} g_{\nu\rho}\left(\frac{d^2x_\nu}{dx_0^{\,2}}\frac{dx_\rho}{dx_0}+\frac{d^2x_\rho}{dx_0}\frac{dx_\nu}{dx_0}\right)=\frac{1}{2}\sum_{\nu,\rho} g_{\nu\rho}\frac{d}{dx_0}\left(\frac{dx_\nu}{dx_0}\frac{dx_\rho}{dx_0}\right).$$

In the second term

$$\sum_{\lambda,\mu,\nu,\rho} g_{\nu\rho}\begin{Bmatrix}\lambda\ \mu\\ \nu\end{Bmatrix}\frac{dx_\lambda}{dx_0}\frac{dx_\mu}{dx_0}\frac{dx_\rho}{dx_0}$$

we replace $\begin{Bmatrix}\lambda\ \mu\\ \nu\end{Bmatrix}$ by its expression between the square brackets and apply a reduction

$$\sum_{\lambda,\mu,\nu,\rho,\tau} g_{\nu\rho}\,g^{\nu\tau}\begin{bmatrix}\lambda\ \mu\\ \tau\end{bmatrix}\frac{dx_\lambda}{dx_0}\frac{dx_\mu}{dx_0}\frac{dx_\rho}{dx_0}=\sum_{\lambda,\mu,\tau}\left(\begin{bmatrix}\lambda\ \mu\\ \tau\end{bmatrix}\frac{dx_\lambda}{dx_0}\frac{dx_\mu}{dx_0}\sum_{\nu,\rho} g^{\nu\tau}g_{\nu\rho}\frac{dx_\rho}{dx_0}\right)=$$

$$=\sum_{\lambda,\mu,\tau}\begin{bmatrix}\lambda\ \mu\\ \tau\end{bmatrix}\frac{dx_\lambda}{dx_0}\frac{dx_\mu}{dx_0}\frac{dx_\tau}{dx_0},$$

as

$$\sum_{\nu} g^{\nu\tau}g_{\nu\rho}\begin{cases}1 & (\text{for } \rho=\tau)\\ 0 & (\text{for } \rho\neq\tau)\end{cases}.$$

According to the meaning of the symbols [ ], we may replace the expression thus found by

$$\frac{1}{2}\sum_{\lambda,\mu,\tau}\frac{\partial g_{\lambda\mu}}{\partial x_\tau}\frac{dx_\lambda}{dx_0}\frac{dx_\mu}{dx_0}\frac{dx_\tau}{dx_0}=\frac{1}{2}\sum_{\lambda,\mu}\frac{dx_\lambda}{dx_0}\frac{dx_\mu}{dx_0}\frac{dg_{\lambda\mu}}{dx_0}.$$

The two former terms of (6) may therefore be combined to:

$$\frac{1}{2}\frac{d}{dx_0}\sum_{\lambda,\mu} g_{\lambda\mu}\frac{dx_\lambda}{dx_0}\frac{dx_\mu}{dx_0}=\frac{1}{2}\frac{d}{dx_0}\left(\frac{ds}{dx_0}\right)^2.$$

We write the third term

$$\sum_{\lambda,\mu,\nu,\rho} g_{\nu\rho}\begin{Bmatrix}\lambda\ \mu\\ \nu\end{Bmatrix}\frac{dx_\nu}{dx_0}\frac{dx_\lambda}{dx_0}\frac{dx_\mu}{dx_0}\frac{dx_\rho}{dx_0}$$

as

$$\sum_{\lambda,\rho}\left(g_{\nu\rho}\frac{dx_\nu}{dx_0}\frac{dx_\rho}{dx_0}\sum_{\lambda,\mu}\begin{Bmatrix}\lambda & \mu\\ & 0\end{Bmatrix}\frac{dx_\lambda}{dx_0}\frac{dx_\mu}{dx_0}\right)=\left(\frac{ds}{dx_0}\right)^2\sum_{\lambda,\mu}\begin{Bmatrix}\lambda & \mu\\ & 0\end{Bmatrix}\frac{dx_\lambda}{dx_0}\frac{dx_\mu}{dx_0},$$

so that (6) is transformed into

$$\frac{1}{2}\frac{d}{dx_0}\left(\frac{ds}{dx_0}\right)^2+\left(\frac{ds}{dx_0}\right)^2\sum_{\lambda,\mu}\begin{Bmatrix}\lambda & \mu\\ & 0\end{Bmatrix}\frac{dx_\lambda}{dx_0}\frac{dx_\mu}{dx_0}=0 \quad . \quad . \quad . \quad (7)$$

§ 3. Let us now define a line in time space by

$$x_i=\varphi_i\,(x_0),$$

where we require of the functions φ:

1. that the line defined in this way satisfy the equations of the geodesic;

2. that in a definite point A

$$\left(\frac{ds}{dx_0}\right)^2_A\equiv\sum_{i,k}\left(g_{ik}\frac{dx_i}{dx_0}\frac{dx_k}{dx_0}\right)_A=0.$$

Of course we also suppose that the coordinates x_i are defined as uniform continuous functions of x_o and that also g_{ik} and its derivatives are uniform continuous functions of the coordinates, at least in the region in consideration.

We have in this way taken care that the line defined by (8) is a geodesic and has a null element in A. As it is a geodesic each of its points satisfies (7); each x_i being a function of x_o, we may conclude that

$$\frac{d}{dx_0}\left(\frac{ds}{dx_0}\right)^2-\left(\frac{ds}{dx_0}\right)^2\Phi\,(x_0)=0,$$

where Φ is a uniform continuous function of x_o.

Hence, along each geodesic

$$\left(\frac{ds}{dx_0}\right)^2_P=\left(\frac{ds}{dx_0}\right)^2_A e^{\int_{a_0}^{p_0}\Phi\,(x_0)\,dx_0}\;; \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

by a_0 and p_0 we understand the values which x_0 assumes at the starting point A and an arbitrary point P of the line.

However, we have also made the assumption that the geodesic in consideration has a null element in A. Accordingly here

$$\left(\frac{ds}{dx_0}\right)^2_A=0.$$

On the other hand there follows from (8) that along this line always

$$\left(\frac{ds}{dx_0}\right)^2=0,$$

in other words that the line in consideration is a geodesic null line, which was to be proved.

§ **4**. Let x_0 be the time coordinate. In three-dimensional space in a point A an arbitrary direction is defined by giving definite *ratios* to $\frac{dx_l}{dx_0}$ ($l = 1, 2, 3$). If inversely we assume these ratios as given, we can give such *values* to $\frac{dx_l}{dx_0}$ that the condition

$$\left(\frac{ds}{dx_0}\right)^2 \equiv \sum_{\lambda\mu} g_{\lambda\mu} \frac{dx_\lambda}{dx_0} \frac{dx_\mu}{dx_0} = 0 \qquad (\lambda, \mu = 0, 1, 2, 3).$$

is satisfied.

The theorem which we have proved, has therefore the meaning: *In three-dimensional space there passes a ray of light at any moment through any point in any direction.*