Physics. — "*On* WHITTAKER'S *Quantum mechanism in the atom*". By Prof. H. A. LORENTZ.

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§ 1. Some months ago WHITTAKER [1]) has proposed an interesting model by means of which the quantum properties of the atom can be accounted for to a certain extent, the model showing in the first place how it may be that, in the collision of an electron against an atom, the former loses either no energy at all, or just a definite amount of it. In what follows I shall offer some remarks about the action between an atom and an electron, as it would be according to WHITTAKER'S views.

WHITTAKER supposes that, when an electron approaches an atom, a "magnetic current" is set up in this particle, comparable with the electric current that is excited in a diamagnetic particle by the approach of a magnetic pole. In this latter case the induced current makes the particle repel the pole (LENZ'S law) and similarly in the former case the magnetic current gives rise to a force tending to stop the motion of the electron.

The theory takes the simplest form when it is assumed that there are not only "electric charges", but also "magnetic" ones, accumulations of positive or negative magnetism. By the introduction of these into the fundamental equations, the parallelism between the electric and the magnetic quantities can be clearly brought out.

§ 2. Let ϱ be the density of the electric charge, $\mathbf{v}$ the velocity of one of its points, and similarly μ the density of magnetic charge, $\mathbf{w}$ its velocity; further $\mathbf{d}$ the electric force or the dielectric displacement in the aether, and $\mathbf{h}$ the magnetic force or magnetic induction. Then we have the fundamental equations

$$div\, \mathbf{d} = \varrho, \quad \text{(1)}$$

$$div\, \mathbf{h} = \mu, \quad \text{(2)}$$

$$rot\, \mathbf{h} = \frac{1}{c}(\dot{\mathbf{d}} + \varrho\, \mathbf{v}), \quad \text{(3)}$$

$$rot\, \mathbf{d} = -\frac{1}{c}(\dot{\mathbf{h}} + \mu\, \mathbf{w}) \quad \text{(4)}$$

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[1]) E. T. WHITTAKER, On the quantum mechanism in the atom, Proc. Royal Society Edinburgh 42 (1922), p. 129.

The force with which the field acts on unit of electric charge is given by

$$\mathbf{f} = \mathbf{d} + \frac{1}{c}[\mathbf{v}\,.\,\mathbf{h}] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and there is a corresponding force

$$\mathbf{g} = \mathbf{h} - \frac{1}{c}[\mathbf{w}\cdot\mathbf{d}] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

acting on unit of magnetic charge.

Remarks on the fundamental equations.

1. In order to simplify the mathematical treatment all quantities occurring in the equations are considered as continuous functions of the coordinates.

2. We shall suppose that, while points of an element of volume move with the velocity $\mathbf{v}$ varying from point to point, the electric charge of the element remains constant, so that the density ϱ changes in the inverse ratio as the size of the element. We shall make a similar assumption concerning the magnetic charge. By these assumptions the distributions, both of the electric current $\dot{\mathbf{d}} + \varrho\,\mathbf{v}$ and of the magnetic current $\dot{\mathbf{h}} + \mu\,\mathbf{w}$ are made to be solenoidal, as they must be if equations (3) and (4) shall be true.

3. For the sake of generality we have introduced different symbols $\mathbf{v}$ and $\mathbf{w}$ for the velocities of the electric and the magnetic charges. These charges may be imagined as penetrating each other and having independent motions.

§ 3. The fundamental equations form a consistent system and are in good agreement with ideas and theorems which physicists would be very unwilling to give up.

The force acting on the electric and the magnetic charges contained in an element of volume, taken per unit of volume, is given by

$$\varrho\,\mathbf{f} + \mu\,\mathbf{g} = \varrho\,\mathbf{d} + \mu\,\mathbf{h} + \frac{1}{c}[\varrho\,\mathbf{v}\,.\,\mathbf{h}] - \frac{1}{c}[\mu\,\mathbf{w}\cdot\mathbf{d}]$$

and for the x-component of this force one finds after some transformations

$$\varrho\,\mathbf{f}_x + \mu\,\mathbf{g}_x = \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} - \frac{\partial \mathbf{G}_x}{\partial t}$$

where

$$X_x = \tfrac{1}{2}(\mathbf{d}_x^2 - \mathbf{d}_y^2 - \mathbf{d}_z^2) + \tfrac{1}{2}(\mathbf{h}_x^2 - \mathbf{h}_y^2 - \mathbf{h}_z^2),$$
$$X_y = \mathbf{d}_x \mathbf{d}_y + \mathbf{h}_x \mathbf{h}_y, \qquad X_z = \mathbf{d}_x \mathbf{d}_z + \mathbf{h}_x \mathbf{h}_z, \text{ etc.},$$
$$\mathbf{G} = \frac{1}{c}[\mathbf{d} \,.\, \mathbf{h}].$$

This shows that the ponderomotive forces can still be expressed by means of MAXWELL's stresses and of the electromagnetic momentum $\mathbf{G}$. It should be noticed that this is possible because we have the positive sign in (5) and the negative sign in (6).

The well known expressions for the electric and the magnetic energy and for the flow of energy likewise remain unchanged. Indeed, starting from the fundamental equations, one finds for the work, per unit of time and unit of volume, of the forces exerted by the field

$$(\varrho\, \mathbf{f} \,.\, \mathbf{v}) + (\mu\, \mathbf{g} \,.\, \mathbf{w}) = -\frac{\partial E}{\partial t} - div\, \mathbf{S},$$
$$E = \tfrac{1}{2}(\mathbf{d}^2 + \mathbf{h}^2), \qquad \mathbf{S} = c\,[\mathbf{d} \,.\, \mathbf{h}].$$

§ 4. If the distribution and the motion of the charges are known, the field can be calculated by means of two scalar potentials φ, χ and two vector potentials $\mathbf{a}, \mathbf{b}$. These functions are given by the formulae

$$\varphi = \frac{1}{4\pi}\int \frac{[\varrho]}{r}\, dS, \qquad \chi = \frac{1}{4\pi}\int \frac{[\mu]}{r}\, dS,$$
$$\mathbf{a} = \frac{1}{4\pi c}\int \frac{[\varrho\, \mathbf{v}]}{r}\, dS, \qquad \mathbf{b} = \frac{1}{4\pi c}\int \frac{[\mu\, \mathbf{w}]}{r}\, dS,$$

in which the integrations have to be extended over all space. The distance from the point for which one wants to determine the potentials for the time t is denoted by r and the meaning of the square brackets is that the quantities ϱ, etc. have to be taken such as they are at the time $t - \frac{r}{c}$.

In terms of the potentials we have for the field

$$\mathbf{d} = -\frac{1}{c}\dot{\mathbf{a}} - grad\, \varphi - rot\, \mathbf{b},$$

$$\mathbf{h} = -\frac{1}{c}\dot{\mathbf{b}} - grad\, \chi + rot\, \mathbf{a}.$$

§ 5. We shall now suppose, following WHITTAKER, that in the atom there is a circular ring R, over which magnetism is uniformly distributed. We shall consider it as very thin, so that we may speak of a "line", and we shall denote by a the radius and by k the amount of magnetism per unit of length. Let the centre O be taken as origin of coordinates, the axes OY and OZ being in the plane of the circle, and let s be the distance from a fixed point, measured along the circle. The positive direction of s will be determined by the rotation $OY \rightarrow OZ$, and will therefore correspond, as we shall say, to the direction of OX. We shall finally suppose the ring to be a rigid body that can only rotate about OX, and we shall in the first place calculate the couple acting on it when an electron with charge e moves in the neighbourhood.

The force on an element ds is $k\mathbf{g}\,ds$ and its moment with respect to OX $ak\mathbf{g}_s\,ds = ak\mathbf{h}_s\,ds$. Thus the resultant couple is $ak\int \mathbf{h}_s\,ds$, where the value of the integral may be deduced from (3). For this purpose we imagine some stationary surface σ having the circle R for its boundary and the normal n to which is drawn in a direction corresponding to the positive direction of s. Then, if this surface does not intersect the electron,

$$\int \mathbf{h}_s\,ds = \frac{1}{c}\int \dot{\mathbf{d}}_n\,d\sigma = \frac{1}{c}\frac{d}{dt}\int \mathbf{d}_n\,d\sigma \quad . \quad . \quad . \quad . \quad . \quad (7)$$

We shall suppose the motion of the electron to be so slow and to change so slowly that it may be said, in any of its positions P, to be surrounded by the electric field that would exist if the electron were at rest in that position. Then the last integral in (7) has the value $\frac{e}{4\pi}\omega$, if ω is the solid angle subtended at P by the ring R, the sign of ω depending on the direction, towards the positive or the negative side, in which straight lines drawn from P pass through the surface. Hence, the equation of motion of the ring will be (ϑ angular velocity, Q moment of inertia)

$$Q\frac{d\vartheta}{dt} = \frac{ake}{4\pi c}\frac{d\omega}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If this equation is to hold for a certain lapse of time, the surface σ must be chosen in such a way as not to be traversed by the electron during that interval.

Now, two cases must be distinguished, the electron passing or not passing across the circular plane within the ring, or, as we shall say, through the ring. In the latter case, σ may be made to coincide with the circular plane and we shall have, both before and after the encounter, if the electron is at a great distance,

$\omega = 0$. In the former case this will not be true. Let us suppose that the electron goes through the ring once, in the positive direction, and let A and B be two positions, before and after the encounter, both far away from the ring. Then, whatever be these positions, provided only that they do not coincide, we can choose the surface σ in such a way that it is not intersected by the path of the particle from A to B, and that $\omega = 0$ at the point A. It is easily seen that then the final value will be $\omega = 4\pi$.

Bij integration of (8) one finds

$$\vartheta = \vartheta_0 + \frac{a k e}{4 \pi c Q} \omega, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

if ϑ_0 is the angular velocity which the ring may have had before the encounter.

§ 6. We have next to consider the motion of the electron. The rotation of the ring constitutes a magnetic current

$$i = a k \vartheta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

giving rise to an electric field that is easily determined if we suppose it not to differ appreciably from the field that would exist if i were constant. The calculation, exactly similar to that of the magnetic field due to an electric current (the vector potential $\mathbf{b}$ is first determined and then $\mathbf{d} = -\operatorname{rot} \mathbf{b}$) leads to the result

$$\mathbf{d}_x = -\frac{i}{4 \pi c} \frac{\partial \omega}{\partial x}, \quad \mathbf{d}_y = -\frac{i}{4 \pi c} \frac{\partial \omega}{\partial y}, \quad \mathbf{d}_z = -\frac{i}{4 \pi c} \frac{\partial \omega}{\partial z}, \quad . \quad (11)$$

from which, combined with (10) and (9), we can deduce that the force $e\mathbf{d}$ acting on the electron depends on a potential

$$\psi = \frac{a k e}{4 \pi c} \vartheta_0 \omega + \frac{a^2 k^2 e^2}{32 \pi^2 c^2 Q} \omega^2 \quad . \quad . \quad . \quad . \quad . \quad (12)$$

If we wanted exactly to determine the motion we should also have to take into account the force with which, owing to its velocity, the electron is acted on by the magnetic field that is due to the ring and to stationary magnetic charges eventually existing in the atom, and so the problem would become very difficult. Since, however, the latter force does no work, we can write down the equation of energy

$$\tfrac{1}{2} m v^2 = \tfrac{1}{2} m v_0^2 - \psi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

(v_0 the initial velocity at a point where $\omega = 0$) and this is sufficient for some interesting conclusions.

Indeed, if the electron has not passed through the ring, we shall have finally $\omega = 0$, $\psi = 0$, so that at the end of the encounter the angular velocity of the ring and the velocity of the electron will again have their initial values ϑ_0, v_0. This will also be the case if the electron goes twice through the ring, first in the positive and then in the negative direction.

If, however, it goes through the ring no more than once, the final value of ω will be 4π and according to (12) and (13) the electron will have lost an amount of energy

$$\frac{ake}{c}\vartheta_0 + \frac{a^2k^2e^2}{2c^2Q}.$$

The ring will have gained just as much. This follows directly from (9) and also from the remark that, as may be seen by (9) and (13),

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}Q\vartheta^2$$

remains constant during the motion.

In the case $\vartheta_0 = 0$ the energy that is imparted to the ring by an "effective" encounter is given by

$$\frac{a^2k^2e^2}{2c^2Q} \quad \text{. . . . .} \quad (14)$$

This agrees with WHITTAKER's result. In his calculations he has confined himself to a motion of the electron along the axis of the ring, but the preceding considerations show that the theory can easily be generalized. However, it is also seen that, if in an effective encounter the ring is to receive the amount of energy represented by (14), the rotation which may have been imparted to it by a previons encounter, must first have disappeared in one way or another.

§ 7. If, in the case $\vartheta_0 = 0$, the electron is to pass through the ring for good and all, it must initially have at least the amount of energy (14). If it has less, it can by no means get beyond a point, where

$$\psi = \tfrac{1}{2}mv_0^2 \quad , \quad \omega = \frac{4\pi c v_0}{ake}\sqrt{mQ} \quad \text{. . . .} \quad (15)$$

Such a point is really reached, the electron returning after having got to it, when the motion is along the axis. In general, however, the problem is less simple. The locus of the points which satisfy the condition (15) is a surface limited by the circle R and having, for a somewhat high value of v_0, the shape of a wide bag lying on the positive side of the circle, which forms its opening. An electron that flies into this bag can never leave it across the surface which it will perhaps not reach at all. Indeed, it may be that, before the velocity is exhausted, its direction comes to be tangential to a surface $\omega = \text{const.}$, characterized by a value of ω smaller than the one given by (15). It seems probable that in such a case the electron, after having moved in the bag for a certain length of time, will leave it through the opening, but it is difficult to make sure of this. [1])

§ 8. In WHITTAKER's model the ring R is made up of the poles, of equal signs, of a number of magnets arranged along radii of the circle and having their opposite poles at or near the centre. It might seem at first sight that in a structure of this kind the magnets can be replaced by perfectly conducting solenoids carrying pre-existent electric currents, so that we can do without magnetic charges.

In reality, however, no satisfactory model can be obtained in this way. This is seen most easily when the electron is supposed to move along the axis OX. In the magnetic field due to this motion the lines of force are circles around the axis, and therefore the force acting on an element of current at a point P, is directed along a line lying in the plane POX. For such a force the moment with respect to OX is zero; consequently, neither a solenoid nor a system of solenoids can be acted on by a couple tending to produce a rotation about OX.

Thus it would seem that the hypothesis of "magnetism" existing independently of electric currents is quite essential in WHITTAKER's model. I need not speak at length of the reasons for which such an assumption is not to be readily admitted. Let it be remarked only that the equations (1)—(6), though forming a consistent system, do not allow us to establish variation theorems of the kind of HAMILTON's principle. In this principle we are concerned with the *difference* between the potential and the kinetic energy, so that, in the equations, the two energies do not occur in the same way. Now, if there are only electric charges, we can, as is well known, arrive at an equation of the Hamiltonian form, in which $\frac{1}{2}\mathbf{d}^2$ takes the place of the potential and $\frac{1}{2}\mathbf{h}^2$ that of the kinetic energy. If there are only magnetic charges, there is a similar formula, in which, however, the electric

[1]) An interesting discussion of this question has been given (Phil. Mag. **44**, 1922, p. 777) by Mr. B. B. BAKER, wo has considered the case of an electron not moving along the axis of the ring, without, however, taking into account the forces that may arise from the existence of a magnetic field.

and the magnetic energy have changed their parts. It is clear that it must be difficult to combine the two theorems into one.

I must not omit to say that WHITTAKER does not want to attach too great importance to the special form of his model. He aptly remarks that, after having obtained a satisfactory system of equations, we may discard the model by which we have been led to it. What is especially interesting in WHITTAKER's idea seems to me to be this, that it shows the possibility of a sharp criterion by means of which it can be decided whether an encounter is effective or otherwise. Such a criterion there must certainly be.

§ 9. Generalization of the model. Suppose that there is in the atom a definite closed circuit s, in which a magnetic current i may circulate, the energy being $\frac{1}{2} Li^2$. Then we have the differential equation

$$L\frac{di}{dt}=\int \mathbf{h}_s \, ds,$$

or, if an electron moves near the atom,

$$L\frac{di}{dt}=\frac{e}{4\pi c}\frac{d\omega}{dt}.$$

Take this instead of (8), and combine it with (11). The amount of energy that is transmitted in an effective encounter (initially $i=0$) is now found to be

$$\frac{e^2}{2c^2L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

In order to obtain a "vibrator" [1]) we can link the circuit s with another circuit s', in which an electric current can circulate (no resistance, energy $\frac{1}{2} L' i'^2$); indeed, we have

$$L\frac{di}{dt}=\frac{1}{c}i' \quad , \quad L'\frac{di'}{dt}=-\frac{1}{c}i.$$

The frequency is given by

$$\nu=\frac{1}{2\pi c\sqrt{LL'}}.$$

If now an electron passes through the circuit s in a time that is short in comparison with the period, the vibrator receives the amount of energy (16) and this amount will subsequently be radiated. It will be equal to $h\nu$ if

$$\frac{\pi e^2}{c}\sqrt{\frac{L'}{L}}=h.$$

[1]) Cf. WHITTAKER, l.c. § 5, p. 139.

One can also try to illustrate other phenomena by means of the model. In its passage from one stationary state of motion to another an electron may be imagined to go through the circuit s of a vibrator, so that the energy which it loses is first imparted to the vibrator and then radiated by it. Conversely, after having taken in some way from a beam of incident light the energy $h\nu$, the vibrator could give this energy to an electron that passes through it at the right moment. But in all this we are confronted with very serious difficulties.