Mathematics. — *"On the Points of Continuity of Functions"*. By Prof. J. WOLFF. (Communicated by Prof. HENDRIK DE VRIES).

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Let $f(P)$ be a function of the coordinates of a point P in a space with an arbitrary number of dimensions. The points where f is continuous, form an *inner limiting set*, i.e. the intersection of an enumerable set of open sets of points Ω_n, where we may assume that Ω_{n+1} is a part of Ω_n for any n. For the points, where the function oscillates less than $\frac{1}{n}$, form an open set Ω_n because the oscillation is an upper semi-continuous function. The set of the points of continuity is the intersection of all Ω_n, $n = 1, 2, 3, \ldots$. YOUNG [1]) has shown that to any inner limiting set E given in a linear interval, there belongs a function in that interval which is continuous in the points of E and discontinuous in any other point. We shall give here a simple proof, which is directly valid for spaces of any number of dimensions.

1. Let a set of points E be given as the intersection of an enumerable set of open sets Ω_n, where Ω_{n+1} is a part of (or coincides with) Ω_n.

We define $f(P)$ for any point of space in the following way: in the first place $f(P) = 0$ if P lies in E. Now let P be a point not lying in E, n_P the least value of n for which Ω_n does not contain the point P.

We put

$$f(P) = \frac{\psi(P)}{n_P} \quad \ldots\ldots \quad (1)$$

where $\psi(P)$ is the function which in the points of space of which all the coordinates are rational, is equal to 1, in any other point of space equal to -1.

We may say that (1) holds also good for the points of E, if there we assume $n_P = \infty$.

2. Now we shall show, that $f(P)$ is continuous in the points of E and discontinuous outside them.

[1]) W. H. YOUNG. Wiener Sitzungsber., vol 112, Abt. IIª, p. 1307.

Let us first assume that P belongs to E. In this case $f(P) = 0$. If ε be an arbitrary positive number, we may choose the natural number ν in such a way that

$$\frac{1}{\nu} < \varepsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

As P lies in Ω_ν and Ω_ν is open, there exists a region U round P which lies also in Ω_ν. For any point Q of U we have therefore $n_Q > \nu$, so that according to (1) and (2)

$$|f(Q)| < \varepsilon,$$

Hence f is continuous in any point of E.

Let us now assume P to lie in the complement of E. If P is not an limiting-point of Ω_{n_P}, it has a neighbourhood U which has no point in common with Ω_{n_P} and which lies in $\Omega_{n_P - 1}$. For any point Q of U we have in this case $n_Q = n_P$. Hence

$$|f(Q)| = |f(P)|$$

As the points where f is positive as well as the points where f is negative, lie everywhere dense on U, the oscillation of f in P is equal to $2\,|f(P)|$.

If however P is an limiting-point of Ω_{n_P}, every neighbourhood U of P contains a part of Ω_{n_P}. For any point of that part $n_P > n_Q$, hence

$$\left| f(Q) - f(P) \right| \geqq \frac{1}{n_P} - \frac{1}{n_P + 1} \quad . \quad . \quad . \quad (3)$$

As the points Q for which the inequality (3) holds good, have P for a limiting-point, P is a point of discontinuity of f. Herewith the theorem has been entirely proved.