Mathematics. — *"An application of the theory of integral equations on the determination of the elastic curve of a beam, elastically supported on its whole length".* By Prof. C. B. BIEZENO. (Communicated by Prof. J. C. KLUIJVER).

(Communicated at the meeting of March 24, 1923).

In his well-known treatise „Vorlesungen über Technische Mechanik" (Vol. III, § 48) FÖPPL describes a construction, by which the elastic curve of a beam, elastically supported on his whole length, might been approximated.

If in the differential equation of this elastic curve

$$E\,I\,y'''' + k\,y = q$$

($E\,I$ = coefficient of stiffness of the beam, k = coefficient of stiffness of the supporting ground, q = specific continuous loading) the function y where known, it would be possible to refind this function by integrating four times the expression $\frac{q-ky}{E\,I}$.

This integration would graphically correspond to the construction of the elastic curve of a beam, which carries only well-known forces.

It is obvious, therefore, first to make a supposition about the elastic curve — in such a way, of course, that the reaction-forces of the supporting ground will be in equilibrium with the external forces of the beam —, then to integrate graphically the expression $\frac{q-ky}{EI}$, and finally to controll, if the before-mentioned accordance takes place.

„Im allgemeinen — such is the opinion of FÖPPL —wird man zunächst eine starke Abweichung in der Gestalt beider Kurven finden. Dann ändert man die zuerst gezeichnete Belastungsfläche so ab, dasz sich die Lastverteilung jetzt der Gestalt der gefundenen elastischen Linie nähert und wiederholt das Verfahren für diese zweite Annahme. Die Uebereinstimmung zwischen Belastungsfläche und zugehöriger elastischen Linie wird jetzt besser werden und nach mehrmaliger Wiederholung findet man mit hinreichender Genauigkeit die wirkliche Druckverteilung."

Certainly it will be possible, — under favourable conditions — to find in this way technical sufficient accordance between the supposed curve and the one, derivated from it; but generally the convergency of the described process is uncertain.

In the following paper a convergent process will be given.

2. The equation

$$E\,I\,y'''' + k\,y = q$$

is transformed in

$$y'''' + k'\,y = q'$$

if $\frac{k}{EI} = k', \frac{q}{EI} = q'$.

Putting $y'''' = \varphi(x)$ it becomes:

$$\varphi(x) + k' \int_0^x \varphi(x)\,dx^4 = q' + Ax^3 + Bx^2 + Cx + D$$

or, using the well-known relation

$$\int_0^x \varphi(x)\,dx^4 = \int_0^x \frac{(x-s)^3}{3\,!}\,\varphi(s)\,ds$$

$$\varphi(x) + k' \int_0^x \frac{(x-s)^3}{3\,!}\,\varphi(s)\,ds = q' + Ax^3 + Bx^2 + Cx + D.$$

A, B, C and D are constants of integration, which enable us to satisfy the following conditions:

1°. $y'' = 0, \quad y''' = 0$ at $x = 0.$

2°. $y'' = 0, \quad y''' = 0$ at $x = l.$

The former conditions imply, as is seen from the relation

$$y = \int_0^x \varphi(x)\,dx^4 - \frac{Ax^3 + Bx^2 + Cx + D}{k'}$$

that the coefficients A and B are zero. The coefficients C and D are determinated by the latter conditions.

3. According to Volterra the solution of the integralequation

$$\varphi(x) + k' \int_0^x \frac{(x-s)^3}{3\,!}\,\varphi(s)\,ds = q' + Cx + D$$

may be written as:

$$\varphi(x) = \varphi_0(x) + k'\varphi_1(x) + k'^2\varphi_2(x) + k'^3\varphi_3(x) + \ldots.$$

where

$$\varphi_0(x) = q' + Cx + D$$

$$\varphi_1(x) = -\int_0^x \frac{(x-s)^3}{3!}\varphi_0(s)\,ds$$

$$\varphi_2(x) = -\int_0^x \frac{(x-s)^3}{3!}\varphi_1(s)\,ds$$

$$\vdots$$

$$\varphi_n(x) = -\int_0^x \frac{(x-s)^3}{3!}\varphi_{n-1}(s)\,ds.$$

This solution however can only graphically be used, if the coefficients C and D are known. Nevertheless this coefficients depend on the second and first integral of $\varphi(x)$ in a point which is different from zero. Therefore we cannot find them a priori.

4. To meet this difficulty, we introduce the function

$$\chi_0(x) = q' + C_0 x + D_0;$$

C_0 and D_0 being two constants, determinated by:

$$\int_0^l \chi_0(x)\,dx = 0$$

$$\int_0^l \chi_0(x)\,.\,x\,dx = 0.$$

By choosing C_0 and D_0 in this manner, we reach that 1°. C_0 and D_0 can easily be graphically found, and 2°. that the function

$$\overline{\varphi}_1(x) = -\int_0^x \frac{(x-s)^3}{3!}\chi_0(s)\,ds$$

satisfies at the point $x = l$ the conditions

$$\overline{\varphi}_1''' = 0, \quad \overline{\varphi}_1'' = 0,$$

or the conditions

$$\int_0^l \chi_0(x)\,dx = 0, \quad \int_0^l dx\int_0^x \chi_0(x)\,dx = 0$$

For:

$$\overline{\varphi''}_1(x)_{x=l} = -\int_0^l dx \int_0^x \chi_0(x)\,dx = \Big\{ -x \int_0^x \chi_0(x)\,dx \Big\}_0^l + \int_0^l x\,\chi_0(x)\,dx = 0.$$

If we should deduce the function $\overline{\varphi}_2(x)$ from $\overline{\varphi}_1(x)$, in the manner which VOLTERRA indicates, the second and third derivates of $\overline{\varphi}_2(x)$ would not be zero at the point $x = l$. Therefore we define the function

$$\chi_1(x) = -\left[\int_0^x \frac{(x-s)^3}{3!}\chi_0(s)\,ds + C_1 x + D_1\right]$$

C_1 and D_1 being constants determinated by

$$\int_0^l \chi_1(x)\,dx = 0$$

$$\int_0^l \chi_1(x)\,.\,x\,dx = 0.$$

In this way, the second and third derivates of $\chi_1(x)$ take at the points $x = 0$ and $x = l$ the prescribed values; on the other hand fore-fold integration of $\chi_1(x)$ gives rise to a function, the second and third derivates of which are at the point $x = l$ also equal to zero.

This being stated, we are lead to define the series of functions

$$\chi_0(x) = q' + C_0 x + D_0$$

$$\chi_1(x) = -\left[\int_0^x \frac{(x-s)^3}{3!}\chi_0(s)\,ds + C_1 x + D_1\right]$$

$$\chi_2(x) = -\left[\int_0^x \frac{(x-s)^3}{3!}\chi_1(s)\,ds + C_2 x + D_2\right]$$

$$\vdots$$

$$\chi_n(x) = -\left[\int_0^x \frac{(x-s)^3}{3!}\chi_{n-1}(s)\,ds + C_n x + D_n\right]$$

where the coefficients C_i and D_i are bound by the conditions

$$\int_0^l \chi_i(x)\,dx = 0$$

$$\int_0^l \chi_i(x)\,.\,x\,dx = 0$$

and to put

$$\varphi = \chi_0(x) + k'\chi_1(x) + k'^2\chi_2(x) + \ldots .$$

This function satisfies formally the equation

$$\varphi(x) + k'\int_0^x \frac{(x-s)^3}{3!}\varphi(s)\,ds = q' + Cx + D$$

and the expression y, which follows from it:

$$y = \frac{q' - \varphi}{k'} = \frac{q' - (q' + C_0 x + D_0) - k'\chi_1(x) - k'^2\chi_2(x) - \ldots}{k'} =$$

$$= -\frac{C_0 x + D_0}{k'} - \chi_1(x) - k'\chi_2(x) - k'^2\chi_3(x)\ldots$$

satisfies formally the conditions, imposed at the ends $x = 0$ and $x = l$.

For, substituting the expression φ in the integral equation we obtain — provided that it be allowed to integrate term by term the series, which occurs under the sign of integration:

$$C_0 x + D_0 - k'(C_1 x + D_1) - k'^2(C_2 x + D_2) - \ldots = Cx + D.$$

If the series, which appears in the first member of this equation, converges, there can be disposed of the constants C and D in such a manner, that the equation becomes an identity.

Of course it would now be necessary to examine the convergency of the described process of iteration.

For this investigation however we refer to the paper of Mr. J. DROSTE, which follows this. We will state here only, that convergency is sure, if $\frac{kl^4}{EI} < 500$, and go on to demonstrate in which manner the process can be graphically performed.

5. At the first place the system of forces, which loads the beam, is substituted by another load, changing linearly, ($q_0 = \alpha x + \beta$), and which is statically equivalent with the first.

This substitute load causes a sinking down of the beam, determinated by

$$y_0 = \frac{\alpha x + \beta}{k}.$$

This y_0 can be considered as the first approximation of the required y, and can be brought in relation with the expression $C_0 x + D_0$, which is defined in N°. 3.

Indeed, $\alpha x + \beta$ satisfies the equations

$$\int_0^l (\alpha x + \beta)\, dx = \int_0^l q\, dx$$

$$\int_0^l (\alpha x + \beta) \,.\, x\, dx = \int_0^l q \,.\, x\, dx$$

on the contrary $C_0 x + D_0$ is defined by

$$\int_0^l (C_0\, x + D_0)\, dx = -\int_0^l q' dx = -\int_0^l \frac{q}{EI} dx$$

$$\int_0^l (C_0\, x + D_0) \,.\, x\, dx = -\int_0^l q' \,.\, x\, dx = -\int_0^l \frac{q}{EI} \,.\, x\, dx.$$

It follows, that $\alpha x + \beta \equiv - EI (C_0 x + D_0)$, so that:

$$y_0 = \frac{\alpha\, x + \beta}{k} = -\frac{C_0\, x + D_0}{k'}.$$

The load which really charges the beam differs from the substitute load by:

$$q_1 = q - q_0 = q - (\alpha\, x + \beta) = EI\, (q' + C_0\, x + D_0) = EI\, \chi_0\, (x).$$

By adding this load (which is in equilibrium) to the load q_0, we would regain the real conditions of loading.

However, if we add the load q, the beam gets a deflexion y_1, determinated by:

$$E\, I\, y_1'''' = E\, I\, \chi_0\, (x)$$

Hence:

$$y_1 = \int_0^x \chi_0\, (x)\, dx^4 = \int_0^x \frac{(x-s)^3}{3\,!} \chi_0\, (s)\, ds + A_1\, x^3 + B_1\, x^2 + C_1\, x + D_1.$$

The second and third derivates of y_1 being zero for $x = 0$, it follows that $A_1 = 0$, $B_1 = 0$.

Choosing C_1 and D_1 so that:

$$\int_0^l y_1\, dx = 0$$

$$\int_0^l y_1 \,.\, x\, dx = 0.$$

we identify y_1 and $-\chi_1\, (x)$.

At the same time, the forces, defined by ky_1, are in equilibrium. If the elastic ground were loaded with ky_1, it would obtain the the deflexion y_1. In this case the beam and the ground would have the same shape. However the load on the ground can only arise from the beam. The deflexion y on the ground therefore involves necessarily a reaction-load $-ky_1$ on the beam.

This latter load gives rise to another deflexion y_2 of the beam, defined by:

$$E\,I\,y_2'''' = -\,k\,y_1 = k\,\chi_1\,(x)$$

Hence

$$y_2 = k' \left\{ \int_0^x \frac{(x-s)^3}{3\,!}\,\chi_1\,(s)\,ds + C_2\,x + D_2 \right\}.$$

If we require again that the load ky_2, which follows from y_2, is in equilibrium, we find that:

$$y_2 = -\,k'\,\chi_2\,(x).$$

From this, we deduce $y_3 = -\,k'^2\,\chi_3(x)$ and so on. Therefore, the terms of the series:

$$y = -\,\frac{C_0\,x + D_0}{k'} - \chi_1\,(x) - k'\,\chi_2\,(x) - k'^2\,\chi_3\,(x)\,\ldots$$

represent elastic curves of a beam, which is loaded in a well-defined manner.

6. Fig. 1 illustrates the described construction in the case: $l = 200$ cm., b = breath of the beam = 25 cm., I = 5000 cm^4, E = 100000 kg/cm^2; $EI = 5 \times 10^8$ kg.cm^2, $\bar{k} = 5$ kg/cm^3, $k = b\bar{k} = 125$ kg/cm^2. The load diagram has a parabolic form; the specific load at the ends of the beam is $^1/_4$ of its value at the middle. The total load is 15000 kg. The scale of length in horizontal direction is $n = 5$ (1 cm $\longleftrightarrow$ means 5 cm $\longleftrightarrow$).

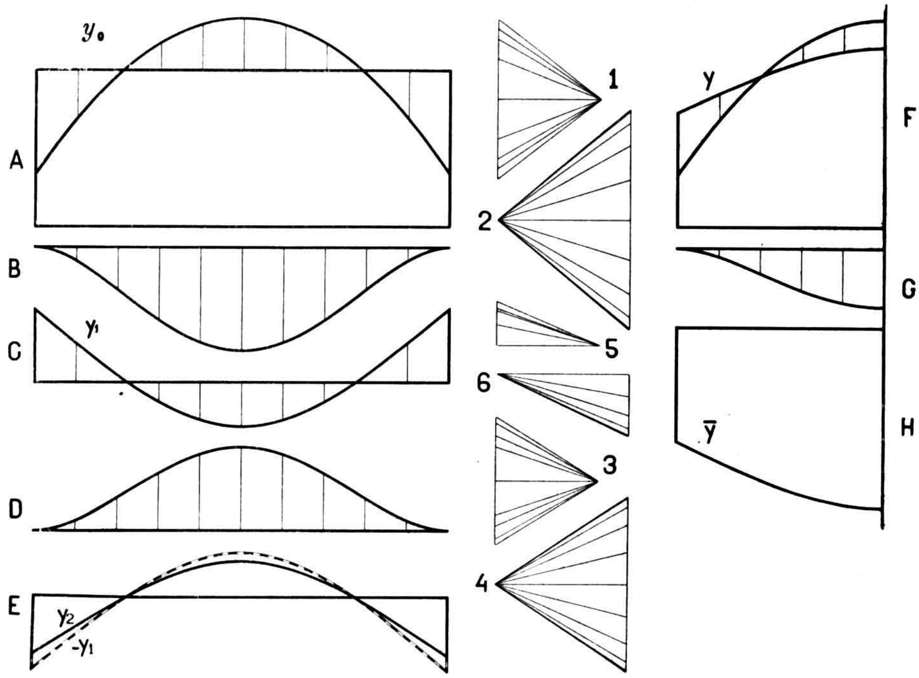
The deflexion are 25 times enlarged; 1 cm. $\updownarrow$ represents $^1/_{25}$ cm $\updownarrow$.

The linear load q_0, which is statically equivalent to the given load q, will give a sinking down to the beam, which is:

$$y_0 = \frac{15000\ \text{KG}}{125\ \text{KG/cm}^2 \times 200\ \text{cm}} = 0{,}6\ \text{cm}.$$

This sinking down is represented in figure $1a$ by $25 \times 0{,}6$ cm. = 15 cm.; and gives rise to the straight line y_0. This line also represents, when the scale is altered, the load q_0; in this case 1 cm.

$\updownarrow$ must be interpreted as $\dfrac{15000 \text{ kg.}}{200 \times 15 \text{ cm}} = 5$ kg/cm (say m_1 kg/cm).



On this scale the parabolic load q has been drawn in fig. $1a$, so that the load $q - q_0$, — which determines the elastic curve y_1 — is represented in fig. $1a$ by the hatched area.

In the well-known manner the elastic curve y_1, which corresponds to the load $q - q_0$, is constructed (see figures $1b$ and $1c$ with the corresponding pole figures 1 and 2).

To determine the situation of the pole in the second pole figure, we make the following remarks.

In figure $1a$ 1 cm. $\longleftrightarrow$ represents n cm. $\longleftrightarrow$; 1 cm. $\updownarrow$ represents m_1 kg/cm. Therefore 1 cm² of fig. $1a$ represents nm_1 kg.

Assuming now that in the first pole figure 1 cm. (whether $\longleftrightarrow$ or $\updownarrow$) will represent m_2 cm² of figure $1a$ (in the drawing m_2 is supposed to be 5) and that the first pole distance has a length of H_1 cm (in the drawing 10 cm), we see that H_1 represents $m_1 m_2 n H_1$ kg.

Hence 1 cm. $\updownarrow$ in fig. $1b$ represents $m_1 m_2 n^2 H_1$ kg. cm. Consequently the unity of area in fig. 1*b* means in the next integration $\frac{m_1 m_2 n^3 H_1}{EI}$ units.

The second pole distance H_2 therefore represents $\frac{m_1 m_2 m_3 n^4 H_1 H_2}{EI}$ units, if we suppose that 1 cm. of this distance represents m_3 cm² (in the drawing 10 cm²) of the area in fig. 1*b*.

From all this it follows finally that 1 cm. $\updownarrow$ in fig. 1*c* represents

$$\frac{m_1\, m_2\, m_3\, n^4\, H_1 H_2}{EI}\ cm.$$

Now the elastic curves y_1 and y_0 must been drawn on the same scale; hence:

$$\frac{m_1\, m_2\, m_3\, n^4\, H_1\, H_2}{EI} = {}^1/_{25}$$

$$H_2 = \frac{1}{25} \frac{EI}{m_1\, m_2\, m_3\, n^4\, H_1} = 12{,}8\ cm.$$

The elastic curve y_1 once found, the drawing process is to be repeated so many times, that the last approximations may be neglected. By adding the different curves $y_0, y_1, y_2 \ldots$ we obtain the elastic curve y. The final result can be controlled as follows. We load the beam at the one side by the well-known external forces, at the other side by the continuous load ky, which follows from the elastic curve y. Then we construct the elastic curve $\bar{y}$. If the result y were exact, the curves y and $\bar{y}$ must be identical. Fig. 1*f*, *g*, *h* shows, that a difference between the curves y and $\bar{y}$ cannot be observed.

7. Considering fig. 1, it appears that the ordinates of the curves y_2 and y_1 are proportional. If the factor of proportionality is called $-\mu$, so that $y_2 = -\mu y_1$, it is easily seen that the ordinates of the curve y_3 can be written as $-\mu y_2$ and so on.

The ordinates $y_1, y_2, \ldots y_n$ at any point can therefore been looked upon as terms of a geometrical series and the curve y can be obtained by adding y_0 to the sum of all the following approximations.

Not only when the factor of proportionality μ is < 1, but also when $\mu > 1$, it may occur that the described drawing process is useful to find the elastic curve.

Supposing that the load $-ky_n$ gives rise to the deflexion $-\mu y_n$ there can be found a factor ν, such that the function νy_n

satisfies the equation $EI\,y'''' + ky = -ky_n$. Using the relation $-EI\,\mu y_n'''' = -ky_n$, we find the condition:

$$\nu \frac{ky_n}{\mu} + k\nu y_n = -ky_n$$

whence:

$$\nu = \frac{-\mu}{\mu + 1}.$$

We therefore can obtain the deflexion y of the beam by adding $\frac{-\mu}{\mu+1} y_n$ to the sum of the curves $y_0, y_1 \ldots y_n$, or by adding $\left(1 + \frac{-\mu}{1+\mu}\right) y_n = \frac{1}{1+\mu} y_n$ to the sum $y_0 + y_1 + \ldots + y_{n-1}$.

Thus we can stop the drawing of curves, as soon as two consecutive ones y_n and y_{n+1} are found, the ordinates of which are proportional.

Though — generally — the above mentioned proportionality only appears exactly after an infinite number of iterations, it nevertheless will be approximately observed tolerably soon. Neglecting in such a case that part of the last found loading diagram which troubles the proportionality between its ordinates and those of the foregoing diagram, we can use the preceding remark, provided that 1° the neglected load diagram be insignificant, and 2° it gives no rise to following load diagrams which grow larger and larger.

The second condition is satisfied when $\frac{kl^4}{EI} < 14600$.

The justification of this latter statement can be given most naturally by the aid of the deductions, given by Mr. Droste. We therefore refer to his paper.