Mathematics. — "*A theorem concerning power-series in an infinite number of variables, with an application to* DIRICHLET'S [1]) *series.*" By H. D. KLOOSTERMAN. (Communicated by Prof. J. C. KLUYVER.)

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§ 1. An important relation between the theory of DIRICHLET'S series and the theory of power-series in an infinite number of variables (for abbreviation we shall write: power-series in an i. n. of v.) has been discovered by H. BOHR [2]). Let

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad , \quad s = \sigma + it \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

be an ordinary DIRICHLET'S series. Put $x_1 = \frac{1}{2^s}$, $x_2 = \frac{1}{3^s}$, ... $x_m = \frac{1}{p_m^s}$, ... (where p_m is the m-th prime-number, and let $n = p_{n_1}^{\nu_1} p_{n_2}^{\nu_2} \ldots p_{n_r}^{\nu_r}$, where $p_{n_1}, p_{n_2}, \ldots p_{n_r}$ are the different primes which divide n. Then the series (1) can *formally* be written as a power-series in an i. n. of v., thus:

$$P(x_1, x_2, \ldots x_m, \ldots) = \sum_{n=1}^{\infty} a_n x_{n_1}^{\nu_1} x_{n_2}^{\nu_2} \ldots x_{nr}^{\nu_r} =$$

$$c + \sum_{\alpha=1,2,\ldots} c_\alpha x_\alpha + \sum_{\substack{\alpha,\beta=1,2,\ldots \\ \alpha \leq \beta}} c_{\alpha,\beta} x_\alpha x_\beta + \sum_{\substack{\alpha,\beta,\gamma=1,2,\ldots \\ \alpha \leq \beta \leq \gamma}} c_{\alpha,\beta,\gamma} x_\alpha x_\beta x_\gamma + \ldots$$

This relation has been applied by BOHR to the so-called absolute-convergence-problem for DIRICHLET'S series, that is to say the determination of the abscissa of absolute convergence of (1) (the lower bound of all numbers β, such that the series (1) converges for $\sigma \geqq \beta$, in terms of (preferably as simple as possible) analytic properties of the function represented by (1). Let B be the abscissa of absolute convergence of (1), and D the lower limit of all numbers α, such that $f(s)$ is *regular* and *bounded* for $\sigma \geqq \alpha$. The absolute-convergence-problem will be solved, if the difference $B - D$ is known. BOHR proves that $B = D$ for any DIRICHLET'S series that can be *formally* represented in one of the following forms:

1) A more detailed proof of the theorem will be published elsewhere.

2) Göttinger Nachrichten, 1913.

$$f(s) = \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \frac{a_{p_m^l}}{(p_m^l)^s}$$

or

$$f(s) = \prod_{m=1}^{\infty} \left(1 + \sum_{l=1}^{\infty} \frac{a_{p_m^l}}{(p_m^l)^s}\right),$$

or, what comes to the same thing, for any DIRICHLET's series for which the connected power-series in an i. n. of v. has one of the forms

$$P(x_1, x_2, \ldots x_m, \ldots) = \sum_{n=1}^{\infty} Q_n(x_n) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or

$$P(x_1, x_2, \ldots x_m, \ldots) = \prod_{n=1}^{\infty} (1 + Q_n(x_n)) \quad . \quad . \quad . \quad (3)$$

where $Q_n(x_n)$ $(n = 1, 2, \ldots .)$ is a power-series in x_n without a constant term. The equality $B = D$ is a consequence of the theorem:

If: *a.* The series is *bounded* [1]) for $|x_n| \leq G_n$ $(n = 1, 2, \ldots .)$, then

b. it is absolutely convergent for $|x_n| \leq \theta G_n$, where θ is an arbitrary positive number in the interval $0 < \theta < 1$ [2]).

Now, if we consider the power-series (2) and (3), we see that the variables x_n occur to some extent separated from one another. This led BOHR to the conjecture, that the equality $B = D$ would hold for any DIRICHLET's series, for which the variables in the connected power-series in an i. n. of v. do not occur too much mixed up. Confirmation of this conjecture is the purpose of the present com-

[1]) According to HILBERT (Wesen und Ziele einer Analysis der unendlich vielen unabhängigen Variabeln, Palermo Rendiconti, vol. 27, p. 67) a power-series in an i. n. of v. is defined to be *bounded* if:

1°. The power-series $P_m(x_1, x_2, \ldots x_m)$ (*Abschnitte*), that may be obtained from the power-series in an i. n. of v. by putting $x_{m+1} = x_{m+2} = \ldots = 0$, are, for all values of m, absolutely convergent in the region $|x_1| \leq G_1$, $|x_2| \leq G_2, \ldots . |x_m| \leq G_m$.

2°. There exists a number K, independent of m, such that, for every m, the inequality

$$|\, P_m(x_1, x_2, \ldots x_m) \,| < K$$

holds in the region $|x_1| \leq G_1$, $|x_2| \leq G_2, \ldots . |x_m| \leq G_m$.

[2]) It is well known, that b follows from a for *any* power-series in a *finite* number of variables. Originally HILBERT had assumed this also, as being self-evident, for an i. n. of v. But BOHR showed that this could not be true by constructing an example to the contrary.

munication. In fact it can be proved that $B = D$ holds for any DIRICHLET's series that can *formally* be written in the form

$$f(s) = \varphi\left(\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\frac{a_{p_m^l}}{(p_m^l)^s}\right),$$

where φ is an arbitrary (non-constant) [1]) integral function. As a consequence of the relation, already mentioned above several times, the following theorem concerning power-series in an i. n. of v. is equivalent to this statement.

Theorem. If φ is an integral function and $Q_n(x_n)$ $(n = 1, 2, \ldots)$ a *formal* [2]) power-series in x_n, without a constant term, and if the power-series in an i. n. of v. $P(x_1, x_2, \ldots. x_m, \ldots.) = \varphi(Q_1(x_1) + Q_2(x_2) + \ldots. + Q_m(x_m) + \ldots.)$ is bounded for $|x_n| \leqslant G_n$ $(n = 1, 2, \ldots.)$, then it is absolutely convergent for $|x_n| \leqslant \theta G_n$, if $0 < \theta < 1$.

In the following pages an outline of the proof of this theorem will be given.

§ 2. For the sake of simplicity we take $G_1 = G_2 = \ldots. = G_n = G > 1$, but $\theta G < 1$.

Because the given power-series in an i. n. of v. is bounded, there exists a number K, not depending on m, such that

$$|\varphi(Q_1(x_1) + Q_2(x_2) + \ldots + Q_m(x_m))| < K. \quad . \quad . \quad (4)$$

The first part of the proof of the theorem of § 1 discusses the power-series $Q_n(x_n)$ $(n = 1, 2, \ldots.)$. It is proved that it follows from (4) that all these power-series possess a certain region of convergence. Further research shows that two cases may occur:

1°. The functions $Q_n(x_n)$ are all regular for $|x_n| < G$. This is the general case.

2°. If the integral function $\varphi(y)$ has the form $V\left(e^{\frac{y}{M}}\right)$ (where V is again an integral function), then it is only possible to conclude that the functions $Q_n(x_n)$ are logarithms of functions regular for $|x_n| < G$, namely that they have the form $Q_n(x_n) = \log(1 + R_n(x_n))$, where $R_n(x_n)$ is regular for $|x_n| < G$, and $R_n(0) = 0$ [3]).

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[1]) If φ is a constant, the theorem is trivial.

[2]) That is to say, the existence of a region of convergence is not assumed, but will appear to be a consequence of the other assumptions.

[3]) It is interesting to observe, that obviously the series (2), with $\varphi(y) = y$, falls under the first case, and the series (3), with $\varphi(y) = e^y$, $V(z) = z$, under the second case.

For shortness' sake we confine ourselves to the first case. (The proof in the second case is not essentially different, though in details more intricate). Then the functions $Q_n(x_n)$ are, because $G > 1$, all regular in their resp. circles $|x_n| \leqq 1$.

For any function $f(z)$, regular for $|z| \leqq 1$, and for which $f(0) = 0$, we now define a number r as follows: r is the radius of the largest circle, of which all points represent numbers assumed by $f(z)$ in the circle $|z| \leqq 1$. Let r_n $(n = 1, 2, \ldots)$ be the corresponding quantity for $Q_n(x_n)$. Then we first prove, that the series $\sum\limits_{n=1}^{\infty} r_n$ converges.

For this purpose we consider (4), valid for all sets of values of $x_1, x_2, \ldots x_m$, satisfying $|x_n| \leqq G$ $(n = 1, 2, \ldots m)$, and, *a fortiori*, for all satisfying $|x_n| \leqq 1$. Because $\varphi(y)$ is an integral function, it is possible to choose a number L so large, that the maximum value of $|\varphi(y)|$, on the circle $|y| = L$, is $> K$. Now suppose that, for some value of m, $r_1 + r_2 + \ldots + r_m > L$. Then the maximum value of $|\varphi(y)|$ on the circle $|y| = r_1 + r_2 + \ldots + r_m$ would be $> K$. Now if we let the variables x_n $(n = 1, 2, \ldots m)$ describe their resp. circles $|x_n| \leqq 1$, then $Q_n(x_n)$ assumes all values satisfying $|Q_n(x_n)| = r_n$. Hence $y = Q_1(x) + Q_2(x_2) + \ldots + Q_m(x_m)$ assumes all values satisfying $|y| = r_1 + r_2 + \ldots + r_m$. Therefore it would be possible to find a set of values $x'_1, x'_2, \ldots x'_m$ such that

$$y = Q_1(x'_1) + Q_2(x'_2) + \ldots + Q_m(x'_m) = (r_1 + r_2 + \ldots + r_m)e^{i\psi},$$

where $(r_1 + r_2 + \ldots + r_m)e^{i\psi}$ represents that point of the circle $|y| = r_1 + r_2 + \ldots + r_m$ where $|\varphi(y)|$ assumes its maximum value. Therefore we should have

$$|\varphi(Q_1(x'_1) + Q_2(x'_2) + \ldots + Q_m(x'_m))| > K,$$

contradictory to (4). Therefore the supposition $r_1 + r_2 + \ldots + r_m > L$ can not be true. Since L is independent of m, this proves the convergence of $\sum\limits_{n=1}^{\infty} r_n$.

We now apply the following theorem of BOHR [1]):

Let the function $f(z) = \sum\limits_{n=1}^{\infty} a_n z^n$ $(f(0) = 0)$ be regular for $|z| \leqq 1$. Let $M(\varrho)$ be the maximum value of $|f(z)|$ on the circle $|z| = \varrho$ $(0 < \varrho < 1)$. Then, if r is the quantity defined above, we have $r \geqq k\,M(\varrho)$, where k is a number which depends on ϱ only (k is

[1]) Not yet published.

therefore the same for all functions satisfying the assumptions of the theorem).

Hence, if $M_n(\varrho)$ is the maximum value of $|Q_n(x_n)|$ on the circle $|x_n| = \varrho$ $(n = 1, 2, \ldots)$, we have $r_n \geqq k\, M_n(\varrho)$. Since we have proved that $\sum_{n=1}^{\infty} r_n$ is convergent, it now follows that the series $\sum_{n=1}^{\infty} M_n(\varrho)$ converges also (for $\varrho < 1$). From this fact the theorem of § 1 can be easily deduced.

For let $Q_n(x_n) = \sum_{p=1}^{\infty} a_p^{(n)} x_n^p$ $(n = 1, 2, \ldots)$. Then

$$|a_p^{(n)}| \leq \frac{M_n(\varrho)}{\varrho^p} \binom{n = 1, 2, \ldots}{p = 1, 2, \ldots} (\varrho < 1).$$

If $\Theta = \theta\, G$ (where θ is the constant of § 1), then it follows that, if $\Theta < \varrho < 1$, $\left(\text{we take for example } \varrho = \frac{1 + \Theta}{2}\right)$,

$$\sum_{p=1}^{\infty} |a_p^{(n)}|\, \Theta^p \leq \frac{2\, \Theta\, M_n(\varrho)}{1 - \Theta}.$$

Hence the series

$$\sum_{n=1}^{\infty} \sum_{p=1}^{\infty} |a_p^{(n)}|\, \Theta^p,$$

is also convergent. This proves a *fortiori* the convergence of the given power-series in an i. n. of v. for $|x_n| \leqq \Theta = \theta\, G\, (n = 1, 2 \ldots)$.

It cannot be denied that the assumption, that φ is an integral function, is somewhat unaesthetic. However, the author has not succeeded in dealing with the more general problem, where φ is an arbitrary (purely formal) power-series. In any case the method described does not give the required result in the more general case.

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