Mathematics. — *"On the Plane Pencils Containing Three Straight Lines of a given Algebraical Congruence of Rays"*. By Dr. G. SCHAAKE. (Communicated by Prof. HENDRIK DE VRIES).

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§ 1. In his „*Kalkül der Abzählenden Geometrie*", p. 331, SCHUBERT finds that the vertices of the plane pencils containing three straight lines of the congruence which two complexes of rays of the orders m and m' have in common, form a surface of the order:

$$\tfrac{1}{3}\, mm' \,(mm'—2)\,(2mm'—3m—3m' + 4),$$

and the planes of these pencils envelop a surface of the same class. In this paper we shall examine what these results become for an arbitrary algebraic congruence of rays. With a view to this we make use of the representation of a special linear complex C on a linear three-dimensional space R_3 which is described in STURM: „*Liniengeometrie*", I, on p. 269. First, however, we shall give a derivation of this representation which differs from the one l. c.

§ 2. If we associate to a straight line l with coordinates $p_1, \ldots p_6$ the point P in a linear five-dimensional space R of which the six above mentioned quantities are the homogeneous coordinates, a special linear complex C is represented on the intersection of a variety V with the equation

$$p_1\,p_4 + p_2\,p_5 + p_3\,p_6 = 0$$

and one of its four-dimensional tangent spaces R_4.

This intersection is a quadratic hypercone K that has its vertex T in the point where R touches the variety V. As the generatrices of K intersect an arbitrary three-dimensional space in the points of a quadratic surface, K contains two systems of planes each of which projects one of the scrolls of the surface in question out of T. Two planes of the same system have only the vertex T in common, two planes of different systems a generatrix of K. The planes V_p of one system are the representation of the stars of rays of the complex C, which have therefore their vertices on the axis a of C, and the fields of C the planes of which pass through a, are associated to the planes V_v of the other system. The axis a of C and the plane pencils of this complex containing a, correspond resp. to the vertex T of K and the generatrices of this hypercone. A straight line of K in a plane V_p represents a plane pencil of C the vertex of which lies on a, and a plane pencil of C of which the plane passes through a, is associated to a straight line of a plane V_v.

Now we assume on K a point S and in the four-dimensional space R_4 a three-dimensional space R_3. The representation mentioned in § 1 arises, when we associate to each straight line l the projection L of P out of S on R_3, if P is the point on K corresponding to l.[1])

§ 3. The straight line s of C of which S is the image point on K, is a *singular straight line* of the second order for the correspondence $(l—L)$. For all the points of the plane ϱ that the three-dimensional tangent space R of K at S, lying in R_4, has in common with R_3, are associated in R_3 to this straight line.

In R there lie the two planes V_p^1 and V_v^1 of K of which the intersection is the generatrix b_1 of K through S. To these planes there correspond resp. the star of C, that has its vertex in the point of intersection A of s and a, and the field of C consisting of the rays of the plane α that passes through s and a. The star A and the field α have in common the plane pencil (A, α) to which the straight line b_1 on K is associated.

The planes V_p^1 and V_v^1 cut ϱ resp. along the straight lines p_1 and v_1 each consisting of points that are singular for the correspondence $(l—L)$. For to each point L of p_1 there corresponds on K a straight line of V_p^1 through S, hence in C a plane pencil containing s, with vertex in A. Likewise a plane pencil in α containing s, is associated to each point L of v_1. The point of intersection B_1 of p_1 and v_1 is the image point L for all rays l of the plane pencil (A, α). In this way the ∞^2 straight lines of the star A correspond to the ∞^1 points of p_1, the ∞^2 rays of the field α to the ∞^1 points of v_1.

To a plane pencil with vertex on a a straight line on K in a plane V_p, which accordingly intersects V_{v_1}, is associated; consequently to such a plane pencil in R_3 corresponds a straight line cutting v_1. Inversely the plane through S and a straight line of R_3 cutting v_1, intersects the hypercone K along a straight line in V_{v_1} through S, to which there corresponds the plane pencil of C that is associated

[1]) The method applied here, has been indicated for the rays of space by FELIX KLEIN. Cf. Mathem. Annalen, Bd. 5, p. 257.

to the singular point of intersection of the chosen straight line with v_1, and along a straight line cutting V_{v_1}, which lies therefore in a plane V_p and corresponds to a plane pencil of C the vertex of which lies on a. In the same way it is evident that the pencils of C in planes through a, are represented on the straight lines of R_3 which cut p_1, and that the plane pencils containing a are associated to the straight lines through the point of intersection B_1 of p_1 and v_1 (for a plane through SB_1 cuts the hypercone K outside SB_1 along a generatrix).

To a star of C, the vertex of which lies consequently on a, there corresponds on K a plane V_p that cuts V_{v_1} along a straight line and the projection of which on R_3 passes accordingly through v_1. Hence a plane through v_1 is associated to a star of C in R_3. It is easily seen that also the reverse holds good and that the fields of C, the planes of which pass through a, are represented on the planes of R_3 through p_1.

§ 4. A congruence $\Gamma(\alpha, \beta)$ of the order α and the class β has in common with C a scroll Ω of the order $\alpha + \beta$ that has a as an α-fold directrix. If further Γ has the rank r, there are r plane pencils through a containing two straight lines of Ω.

The curve γ in R_3 on which Ω is represented, cuts p_1 in the α points that are associated to the α generatrices of Ω which pass through A, and v_1 in the β points that correspond to the β generatrices of Ω in the plane (a, s). A plane through p_1 cuts γ outside p_1 in the β image points of the straight lines which the corresponding field of C has in common with Ω, and it appears in the same way that a plane through v_1 intersects the curve γ outside v_1 in α points. Hence the order of γ is $\alpha + \beta$.

To the r plane pencils through a that contain two straight lines of Ω, there correspond in R_3 as many bisecants of the curve γ through B_1. Besides the lines p_1 and v_1 which cut γ resp. α and β times pass through B_1. The number of apparent double points of γ is accordingly:

$$r + \tfrac{1}{2}\,\alpha\,(\alpha-1) + \tfrac{1}{2}\,\beta\,(\beta-1).$$

We shall just mention an application that STURM gives on p. 271 of his book quoted in § 1. The order of the focal surface of the congruence Γ is equal to the number of sheaves with vertices on a containing two straight lines of Γ, hence also of Ω, that are infinitely near to each other. These are represented on the planes through v_1 touching γ outside v_1. Hence the order of the focal surface of Γ is equal to the number of points of intersection outside γ of v_1 with the surface of the tangents of γ. The order of the latter surface, that has γ as a double curve (cuspidal curve), is equal to

$$2(\alpha\beta - r).$$

We find this by substituting in the formula $n(n-1)-2h$ for n the order $\alpha+\beta$ of γ and for h the above mentioned number of apparent double points of this curve. As v_1 cuts the surface under consideration on the double curve γ in β points, we find for the number of points of intersection outside γ, i. e. the order of the focal surface of the congruence Γ:

$$2\beta(\alpha-1)-2r.$$

The class of the focal surface of Γ is equal to the number of planes through a containing two straight lines of Γ, hence also of Ω, that are infinitely near to each other, or equal to the number of planes through p_1 touching γ outside p_1. As p_1 cuts the curve γ in α points, we find for the class in question:

$$2\alpha(\beta-1)-2r.$$

§ 5. In order to find the order of the surface formed by the vertices of the plane pencils containing three generatrices of Γ, we try to find the number of these plane pencils that have their vertices on a. These belong to C and are represented on the trisecants of γ that cut v_1 outside this curve.

The order of the surface Δ of the trisecants of γ is found by substituting in the formula:

$$(n-2)\{h-\tfrac{1}{6}n(n-1)\},$$

given by CAYLEY, for n the order $\alpha+\beta$ of γ and for h the number of apparent double points of this curve found in § 3. We find in this case:

$$(\alpha+\beta-2)\{r+\tfrac{1}{2}\alpha(\alpha-1)+\tfrac{1}{2}\beta(\beta-1)-\tfrac{1}{6}(\alpha+\beta)(\alpha+\beta-1)\}$$

or, after a simple reduction:

$$(\alpha+\beta-2)r+\tfrac{1}{3}\alpha(\alpha-1)(\alpha-2)+\tfrac{1}{3}\beta(\beta-1)(\beta-2).$$

In order to find the number of generatrices of Δ that cut v_1, we remark that these are the common straight lines of Δ and the special linear complex that has v_1 as axis. Now the axis of a special linear complex C may be considered as a double line of C. This follows in the first place from the representation of C on a hypercone K that has been described in § 2 and through which the axis of C is transformed into the vertex of K, but also from the well known property that $n-2$ generatrices of a scroll of the order n cut a straight line of this scroll. As further v_1 has β points in common with γ, it is apparently a $\frac{\beta(\beta-1)(\beta-2)}{6}$-fold generatrix of Δ. The number of generatrices of Δ cutting v_1, is therefore found by diminishing the order-number found above, by:

$$\tfrac{1}{3}\beta(\beta-1)(\beta-2).$$

Hence there are

$$(\alpha+\beta-2)r+\tfrac{1}{3}\alpha(\alpha-1)(\alpha-2)$$

straight lines of Δ which cut v_1.

In the first place the straight line p_1 must be counted $\frac{\alpha(\alpha-1)(\alpha-2)}{6}$ times, for as this line has β points in common with γ it is an $\frac{\alpha(\alpha-1)(\alpha-2)}{6}$-fold generatrix of Δ. Further the number found above has to be diminished by the number of trisecants of γ that cut v_1 on γ. This is the case in each of the β points that γ has in common with v_1. We find the number of trisecants of γ passing through such a point, by the aid of the property that through a point of a twisted curve of the order n with h apparent double points, there pass $h-n+2$ straight lines that contain two more points of the curve, if we take into account that in our case for each of the said β points v_1 counts $\frac{(\beta-1)(\beta-2)}{2}$ times among the trisecants of γ passing through them, as v_1 contains $\beta-1$ more points of γ outside the point under consideration. Consequently

$$\beta\{r+\tfrac{1}{2}\alpha(\alpha-1)+\tfrac{1}{2}\beta(\beta-1)-\alpha-\beta+2-\tfrac{1}{2}(\beta-1)(\beta-2)\}$$

or

$$\beta\{r+\tfrac{1}{2}\alpha(\alpha-1)(\alpha-2)\}$$

trisecants of γ that cut v_1 on γ, must be taken apart.

If we subtract these two numbers of straight lines from the aforesaid number of straight lines of Δ that cut v_1, we find that

$$\tfrac{1}{6}(\alpha-2)\{6r-(\alpha-1)(3\beta-1)\}$$

trisecants of γ intersect v_1 outside this curve.

According to the beginning of this § we arrive at the following theorem:

The locus of the vertices of the plane pencils that have three straight lines in common with a congruence $\{\alpha,\beta\}$ of the rank r, is a surface of the order:

$$\tfrac{1}{6}(\alpha-2)\{6r-(\alpha-1)(3\beta-\alpha)\}.$$

§ 6. In order to show that the result found in § 5, is in accordance with the result of SCHUBERT, mentioned in § 1, we have to know the rank of the congruence $\Gamma(mm', mm')$ that two complexes C_1 and C_2 of the orders m and m' have in common. It might suffice to refer to SCHUBERT, *Kalkül der Abzählenden Geometrie*, where there is found on p. 330 a derivation of this number. We shall however show that the order of Γ may also be found by the aid of the representation used in this paper.

The surface Ω consisting of the straight lines of Γ which cut the axis a of C, is of the order $2mm'$ and has a as an mm'-fold straight line. It is the intersection of the two congruences $\Sigma_1(m, m)$ and $\Sigma_2(m', m')$ consisting of the straight lines out of C_1 and C_2 that cut a.

Σ_1 and Σ_2 are represented resp. on two surfaces S_1 and S_2 in R_3. As C_1, hence also Σ_1, contains m generatrices of an arbitrary plane pencil of C, all points of p_1 and v_1 are m-fold points of S_1 and all straight lines cutting p_1 and v_1 have m more points in common with S_1. S_1 has accordingly the order $2m$ and p_1 and v_1 are m-fold straight lines of S_1. In the same way S_2 has the order $2m'$ and p_1 and v_1 are m'-fold straight lines of this surface. The intersection of S_1 and S_2 consists of the straight lines p_1 and v_1, each counted mm' times, and the curve γ on which Ω is represented. This curve has the order $2mm'$ and has mm' points in common with each of the straight lines p_1 and v_1. We first determine the number of apparent double points of γ.

The cone Λ projecting γ out of an arbitrary point L of R_3, is of the order $2mm'$ and has in common with S_1 besides γ a curve ϱ of the order $4m^2m' - 2mm' = 2mm'(2m-1)$. The curve ϱ has $(m-1)$-fold points in the $2mm'$ points where γ cuts the lines p_1 or v_1, because the entire intersection of Λ and S_1 must have there m-fold points. Further Λ cuts each of the lines p_1 and v_1 in mm' more points, that are m-fold points for ϱ. As all these points are m'-fold for S_2, ϱ has $4mm'^2(2m-1) - 2mm'^2(m-1) - 2m^2m'^2 =$ $= 2mm'^2(2m-1)$ points of intersection with S_2 outside p_1 and v_1. These belong to γ and lie partly in the points where a generatrix of Λ touches the surfaces S_1 on γ, hence in the points of intersection with γ outside p_1 and v_1 of the first polar surface of L relative to S_1. As this polar surface is of the order $2m-1$ and has $(m-1)$-fold straight lines in p_1 and v_1, it cuts γ outside p_1 and v_1 in $2mm'(2m-1) -$ $2mm'(m-1) = 2m^2m'$ points. The remaining $2mm'^2(2m-1) - 2m^2m' =$ $= 2mm'(2mm' - m - m')$ points where ϱ and γ cut each other outside p_1 and v_1, are points that the bisecants of γ through L have in common with this curve. The number of apparent double points of γ is therefore equal to $mm'(2mm'-m-m')$.

If we choose L in the point of intersection B_1 of p_1 and v_1, $\frac{mm'(mm'-1)}{2}$ of the chords of γ through this point coincide with each of the lines p_1 and v_1. Through B_1 there pass accordingly $mm'(m-1)(m'-1)$ bisecants of γ different from p_1 and v_1. According to § 3 these are the representation of as many plane pencils through a containing two straight lines of Ω, hence also of Γ. *The rank of the congruence Γ that two complexes of the orders m and m' have in common, is therefore equal to $mm'\,(m-1)\,(m'-1)$.*

If we substitute this number for r in the expression found in § 5, and if we make α and β equal to mm', we find indeed that the order of the surface formed by the vertices of the plane pencils containing three straight lines of the intersection of two complexes of rays of the orders m and m', is equal to:

$$\tfrac{1}{3}\, mm'\,(mm'-2)\,(2mm'-3m-3m'+4).$$

We get another check through the application of our formula to the congruence consisting of the straight lines passing through one of n given points. For this congruence $\alpha = n$ and $\beta = r = 0$. The locus of the vertices of the plane pencils which three straight lines have in common with this congruence, consists of the planes that may be passed through each triple of the given points. By the said substitutions in the formula of § 5, we find indeed the number of these planes, namely:

$$\tfrac{1}{6}\, n\,(n-1)\,(n-2).$$

To the theorem derived in § 5 there corresponds dually:

The planes of the plane pencils that have three straight lines in common with a congruence $\{\alpha, \beta\}$ of the rank r, envelop a surface of the class:

$$\tfrac{1}{6}\,(\beta-2)\,\{6r-(\beta-1)\,(3\alpha-\beta)\}.$$