Hydrodynamics. — *"On the resistance experienced by a fluid in turbulent motion".* By J. M. BURGERS. (Communicated by Prof. P. EHRENFEST).

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§ 1. *Introductory remarks.*

The problem which is discussed in the following lines is to search for a method to calculate the resistance experienced by a fluid in turbulent motion. A definite solution has not been arrived at; a first attempt only is given.

As is generally known, in most cases the motion of a fluid through a straight cylindrical tube or channel is not in parallel lines with a constant velocity along each line. On the contrary it is usually very irregular: the velocity of a particle changes its value and its direction continually, and particles situated very near to each other have very different velocities, whereas there seems to be no definite law governing these deviations. This type of motion is called *sinuous* or *turbulent,* as distinguished from the *streamline* or *laminar* motion, which occurs at low velocities only. In studying turbulent flow the conception of the *mean motion* or *principal motion* has been introduced by various authors. This mean motion is obtained if in every point of the space occupied by the fluid the mean value of the true velocity with respect to time is determined, and then the steady motion is imagined the velocities of which are equal to these mean values. The true motion may be described as the resultant of the mean motion and of a fluctuating *relative motion.* The mean velocity of the latter is zero [1]).

A turbulent flow usually experiences a high resistance, which is approximately proportional to the second power of the velocity of the mean motion. If the law of resistance is written:

$$\text{loss of pressure per unit of length } J = C\,\frac{\varrho V^2}{d},$$

[1]) In connection with the distinction between mean motion and relative motion the reader is referred to: H. A. LORENTZ, Turbulente Flüssigkeitsbewegung und Strömung durch Röhren, Abhandl. über theoretische Physik I (1907), p. 58—60.

in which formula V represents the mean velocity (i.e. the volume of fluid which in unit of time flows through a section of the tube, divided by the area of that section), d the diameter of the tube, and ϱ the density of the fluid, then C is called the coefficient of the resistance, and appears to be a function of the characteristic number introduced by REYNOLDS: $R = \frac{Vd\varrho}{\mu}$ (μ is the coefficient of viscosity of the fluid). The value of C for different cases is given in textbooks; as an example may be mentioned:

a. for rough walled tubes C is approximately independent of R; however, it is a function of the roughness;

b. for very smooth tubes of circular diameter:

$$C = 0{,}1582\ R^{-\frac{1}{4}}\ ^{1)}.$$

The greater part of the theoretical investigations on the turbulent motion treat the problem: how does it originate? [2]) An explanation of the increase of resistance which accompanies the appearance of the turbulent state of flow has been given by REYNOLDS and LORENTZ [3]). More than once it has been remarked that this problem is one of statistical nature [4]). The resistance experienced by the fluid and indicated by our measuring apparatus is a mean value. It is possible that such a mean value may be calculated sufficiently approximate without an exact knowledge of the fluctuating and never exactly returning relative motions.

In the following lines a preliminary attempt is made to determine the value of the resistance and to explain the quadratic law. In the first part (paragraphs 2 and 3) two equations given by REYNOLDS and LORENTZ are discussed and put into such a form that immediately appears what quantities are wanted in order to calculate the resistance. In the second part (paragraphs 4 and 5) a simple idealized "model" of the turbulent flow is constructed which allows these quantities to be determined.

Instead of the flow through a tube or channel a more simple

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[1]) Comp. f i. R. VON MISES, Elemente der technischen Hydromechanik I (1914) p. 57 and H. BLASIUS, Mitt. über Forschungsarbeiten, herausgeg. vom V. D. I., Heft 131 (1913).

[2]) Cf. F. NOETHER, ZS. für angew. Math. u. Mechanik 1, p. 125, 1921.

[3]) O. REYNOLDS, Scientific Papers II, p. 575—577;
H. A. LORENTZ, l.c. p. 66—71.

[4]) Among others by TH. VON KÁRMÁN at a lecture at the "Versammlung der Mathematiker und Physiker" in Jena 1921; comp. a remark in the ZS. für angew. Math. u. Mechanik 1, p. 250, 1921.

type has been chosen: the motion of a fluid between two parallel walls, one of which has a translational motion in its own plane with the velocity V with respect to the other, while the distance between the two walls has the constant value l (comp. fig. 1). To ensure this motion forces of magnitude S per unit of area must be applied

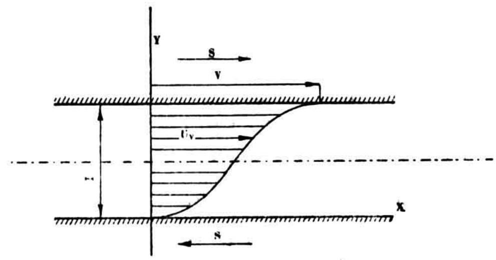

Fig. 1.

to the walls in opposite directions. The tangential force between any two adjacent layers of the fluid has the same value S. The law of resistance will be written:

$$S = C \varrho V^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The coefficient C is a function of REYNOLDS' number:

$$R = \frac{V l \varrho}{\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For small values of R the motion is laminar, and the value of C is easily seen to be:

$$C = \frac{1}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the value of R is high, the motion becomes turbulent, and C decreases much slower. There do not exist any direct measurements for this case of motion; however, the arrangement of the experiments made by COUETTE comes very near to it[1]). According to this author we may expect a formula of the following type:

$$C = c_1 + c_2 R^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

Investigations by VON KÁRMÁN on the law of decrease of the mean motion in the neighbourhood of a smooth wall[2]) point to:

$$C = 0.008 R^{-1/4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4b)$$

[1]) M. COUETTE, Ann. de Chim. et de Phys. (6) **21**, p. 457, 1890.

[2]) TH. VON KÁRMÁN, ZS. für angew. Math. u. Mechanik, l.c.

In order to simplify the mathematical treatment it has been assumed that the motion is confined to a plane.

Finally in paragraph 7 some results are given for the flow between two fixed parallel walls.

§ 2. *The principal equation.*

In the following lines the mean or principal motion of the fluid will be denoted by U. It is a function of the variable y only; at the wall $y=0$ it is equal to 0, at the wall $y=l$ it takes the value V. The components of the velocity of the relative motion are written u and v; the vorticity of the relative motion is written:

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{dy} \quad \ldots \ldots \ldots \ldots (5)$$

These latter quantities are functions of the variables x, y and t. The velocities u and v are subjected to the boundary conditions:

$$u = 0, v = 0 \text{ for } y = 0 \text{ and for } y = l \quad \ldots \ldots (6)$$

and to the equation of continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \ldots \ldots \ldots \ldots (7)$$

Now both REYNOLDS and LORENTZ have shown that the peculiar character of turbulent motion is caused by the action of an apparent frictional force, influencing the principal motion, and due to the existence of the relative motions. This is expressed by the formula:

$$\mu \frac{dU}{dy} - \varrho\, \overline{uv} = S \quad \ldots \ldots \ldots \ldots (8)$$

The bar over uv indicates that the mean value of this quantity is meant, taken at a certain point during a certain lapse of time, or taken at a certain moment along a line parallel to the axis of x. This mean value is a function of the variable y only (the same remark applies to $\overline{\zeta^2}$ in formula (9)). The quantity $\overline{uv}$ is *negative*, and $S > \mu \frac{dU}{dy}$.

The relative motions, however, are not independent of the mean motion. In order that the relative motions may always retain the same energy, it is necessary that the following equation is fulfilled:

$$-\int_0^l dy\, \varrho\, \overline{uv} \frac{dU}{dy} = \int_0^l dy\, \mu\, \overline{\zeta^2} \quad \ldots \ldots \ldots (9)$$

The equations (8) and (9) are substantially the same as the formulae (36) and (46) from LORENTZ' paper l.c. above, only simplified according to the conditions of the problem before us.

Now firstly $\frac{dU}{dy}$ will be eliminated from eq. (9) by the aid of (8):

$$-S\int_0^l dy\, \varrho\, \overline{uv} = \int_0^l dy \{ \varrho^2 (\overline{uv})^2 + \mu^2 \overline{\zeta^2} \} \quad . \quad . \quad . \quad . \quad (10)$$

Secondly by integrating (8):

$$\mu V = Sl + \int_0^l dy\, \varrho\, \overline{uv} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This equation allows the elimination of S from (10):

$$\left.\frac{\mu V}{l} = \frac{\int_0^l dy \{ \varrho^2 (\overline{uv})^2 + \mu^2 \overline{\zeta^2} \} - \frac{1}{l}\left(\int_0^l dy\, \varrho\, \overline{uv}\right)^2}{-\int_0^l dy\, \varrho\, \overline{uv}}\right\} \quad . \quad . \quad (12)$$

In order to simplify the equations we may introduce undimensioned variables by means of the formulae:

$$x = l\,x',\ y = ly'\,;\ u = Vu',\ v = Vv'\,;\ \zeta = \frac{V}{l}\zeta' \quad . \quad . \quad (13)$$

If now in the following equations the accents are omitted again, we obtain:

$$\left.\frac{1}{R} = \frac{\int_0^1 dy\, (\overline{uv})^2 - \left(\int_0^1 dy\, \overline{uv}\right)^2}{-\int_0^1 dy\, \overline{uv}} + \frac{1}{R^2}\, \frac{\int_0^1 dy\, \overline{\zeta^2}}{-\int_0^1 dy\, \overline{uv}}\right\} \quad . \quad . \quad (14)$$

and by the same substitutions, from (11):

$$\frac{S}{\varrho V^2} = -\int_0^1 dy\, \overline{uv} + \frac{1}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The equations take a very simple form if the following abbreviations are used:

$$\left.\begin{aligned} -\int_0^1 dy\, \overline{uv} &= \sigma \\ \int_0^1 dy\, (\overline{uv})^2 &= (1+\tau)\,\sigma^2 \\ \int_0^1 dy\, \overline{\zeta^2} &= \varkappa\sigma \end{aligned}\right\} \quad \ldots \ldots \quad (16)$$

It will be easily recognized that the three quantities σ, τ and $\varkappa$ are all of them essentially *positive*.

The equations (14) and (15) now reduce to:

$$\sigma\tau + \frac{\varkappa}{R^2} = \frac{1}{R} \quad \ldots \ldots \ldots \quad (17)$$

and:

$$\frac{S}{\varrho V^2} = C = \sigma + \frac{1}{R} \quad \ldots \ldots \ldots \quad (18)$$

Formula (17) will be denoted as the *principal equation.*

§ 3. *Discussion of the principal equation.*

Equation (17) shows first of all that an increase of the velocity V of the mean motion cannot be accompanied by a proportional change of the relative motion: in this case σ, τ and $\varkappa$ would remain constants, whereas R increases, which would violate equation (17).

If the value of R is given, (17) gives a condition to be fulfilled by the relative motion. If a certain type of relative motion, fulfilling this condition, accompanies the mean motion, the latter will experience a resistance determined by the value of C, calculated from (18). Now the problem arises: can we find admissible values of the quantities τ and $\varkappa$, without an exact knowledge of the true relative motion? If τ and $\varkappa$ are known, (17) gives σ (i.e. in some measure the relative intensity of the relative motions), and (18) gives the resistance coefficient. If we look at the application of statistical methods in the dynamical theory of gases, we should expect that for high values of R (which mean a fully developed state of turbulence), it may be possible to calculate τ and $\varkappa$ in the following manner: firstly we determine all kinds of relative motions which fulfil eqq. (6) and (7); secondly we admit that all these motions may be present independently of each other, their weights being governed by some law of probability, or by a maximum- or minimum-condition. Then the mean values are calculated for this assembly.

Prof. VON KÁRMÁN from Aix-la-Chapelle pointed out to me that before trying to find a condition governing the weight of the different types of motions, it would be advisable at first to search for the *maximum value* of S, or of σ. In this way a higher limit for the resistance of turbulent flow would be found.

That a maximum value exists may be shown thus:

From (17) it is deduced that σ may become great (i.e. especially: great as compared to $\frac{1}{R}$) only if $\varkappa < R$ and if τ becomes small. The value of τ is determined by the distribution of the values of $\overline{uv}$ over the interval $0 < y < 1$. Only if $\overline{uv}$ assumes a constant value throughout this interval, τ can attain its minimum value 0. However, $\overline{uv}$ cannot be a constant everywhere, as u and v decrease to 0 in the neighbourhood of the walls. Hence we will obtain the smallest possible value of τ if $\overline{uv}$ has a constant value throughout the whole region with the exception of two very thin layers along the walls, in which layers $|\overline{uv}|$ decreases to zero. If the thickness of these "boundary" layers is represented by ε, τ will be of the same order of magnitude as ε, hence with a numerical constant c_1:

$$\tau = c_1 \varepsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

In the boundary layers $\frac{\partial u}{\partial y}$ and ζ will be of the order of magnitude ε^{-1}, and so $\overline{\zeta^2}$ will be proportional to ε^{-2}. Hence if this intensive vorticity occurs in the boundary layers only:

$$\varkappa = c_2 \varepsilon^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Now equation (17) gives:

$$\sigma = \frac{1}{c_1 \varepsilon R} - \frac{c_2}{c_1 \varepsilon^2 R^2}$$

This expression attains a maximum value if:

$$\varepsilon = \frac{2c_2}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The thickness of the boundary layer appears to be inversely proportional to R. The value of σ becomes:

$$\sigma_{max} = \frac{1}{4 c_1 c_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

It appears that σ takes a value which is independent of R;

according to (18) C approximates to the same constant value, and thus according to (1) the *quadratic law* of resistance is obtained.

This reasoning is in many respects vague, and it does not admit of a determination of the values of c_1 and c_2. It only shows that the particles of fluid with high values of the vorticity $|\zeta|$ must be concentrated along the walls. To get a more definite result it is necessary to develop a picture of the structure of the turbulent motion. Two ways may be followed: we may try to analyze the possible motions into a sum of elementary functions (goniometrical or others) in a manner analogous to a series of FOURIER; or we may imagine the motion to be built up from an assembly of individual vortices (vortex filaments with their axes perpendicular to the plane of x-y), distributed in some way or other throughout the fluid. In the calculation of the critical value of R (i.e. the value at which the turbulence occurs for the first time) analogous methods have been used: REYNOLDS, ORR and other writers have directed their attention to disturbances which are propagated in a periodic way through the whole fluid; LORENTZ at the other hand has studied the disturbance caused by a single vortex [1]).

The statistical treatment of such an assembly of elementary motions is very difficult on account of the circumstance that every elementary motion is damped by the action of the viscosity. At the other side the mutual actions between the elementary motions (brought forth by the quadratic terms in the equations of hydrodynamics) and the influence of the mean motion continually generate new motions. From the formula given by LORENTZ it follows that types of motion for which $\iint dx\, dy\, uv$ is negative, are intensified by the action of the mean motion. Hence a mean stationary state can exist, in which every elementary motion changes continually its intensity and its phase (or its position, if it is an individual vortex), but in which every one of these motions has a constant mean intensity. It is obvious that for the greater part, if not exclusive, these will be types of motion for which $\iint dx\, dy\, uv < 0$.

The statistical problem will not be attacked here. On the contrary a simple type of turbulent motion will be studied in the following paragraphs, built up from an assembly of elliptic vortices, all of them having the same configuration, but having different dimensions.

[1]) O. REYNOLDS, l.c. p. 570;
W. MC. F. ORR, Proc. Roy. Irish Acad. **27**, p. 124—128, 1907;
H. A. LORENTZ, l.c. p. **48**.

If they are distributed over the fluid in a certain way, with an appropriate distribution of intensities, it will appear that it is possible to make τ very small, without making the value of $\varkappa$ surpass that of R. It further appears that in the choice of the dimensions of the vortices an element remains arbitrarious, which element may be adjusted in such a way that σ takes a maximum value.

§ 4. LORENTZ' *elliptic vortex*.

It has been shown by LORENTZ that we can obtain a simple type of motion which obeys the conditions (6) and (7), and for which $\iint dx\, dy\, uv < 0$, by considering a vortex in which the particles of the fluid describe elliptic paths [1]). Geometrically this motion can be deduced from that in a circular vortex by a lateral compression. In the circular vortex the fluid moves in concentric orbits with the angular velocity ω, which is a function of the radius r of the orbit. At the outer boundary of the vortex ω has the value zero, whereas in its centre ω and $\frac{d\omega}{dr}$ have finite values. LORENTZ takes for ω a BESSEL function of r; in order to obtain simpler formulae in this paper an algebraic function will be taken.

The construction of the elliptic vortex is shown in figure 2. The

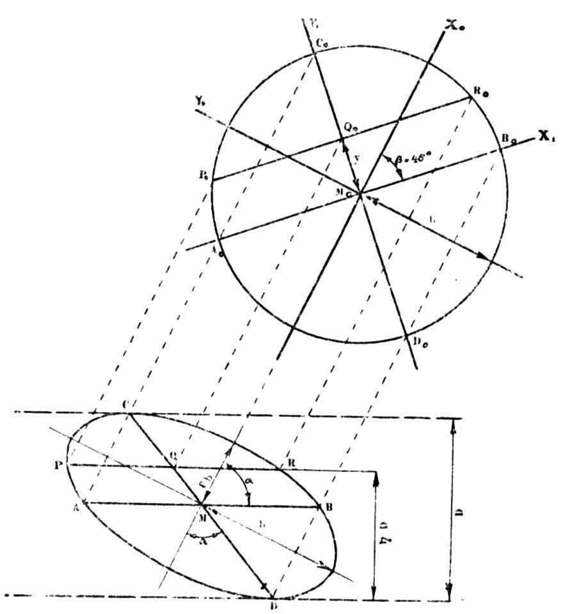

Fig. 2.

[1]) H. A. LORENTZ, l.c. p. 48—52.

axes of the ellipse have the lengths $2b$ and $2\varepsilon b$, in which expression ε has the value $\frac{1}{3}(\sqrt{15}-\sqrt{6})=0{,}475$; the smaller one makes the angle $arctg\,\frac{1}{\varepsilon}=\alpha$ with the direction of the mean motion. The conjugated diameters $A\,B$ and $C\,D$ correspond to the diameters of the circle $A_0\,B_0$ and $C_0\,D_0$, which make angles of 45° with the directions of the axes of the ellipse. Besides the system of coordinates $x_0\,y_0$ used by LORENTZ, the system $x_1\,y_1$ along $M_0\,B_0$ and $M_0\,C_0$ will be introduced.

From the formulae given by LORENTZ at page 49 we deduce the following expression for the value of uv in a point of the vortex, corresponding to the point $x_0\,y_0$ of the circle:

$$\left.\begin{aligned} M_0 = -\,uv &= \frac{1}{2}(x_0^2-\varepsilon^2 y_0^2)\,\omega^2 \sin 2\alpha + \varepsilon x_0 y_0\,\omega^2 \cos 2\alpha = \\ &= \frac{\varepsilon}{1+\varepsilon^2}\,\omega^2\,\{x_1^2(1-\varepsilon^2)+x_1 y_1(1+\varepsilon^2)\}\end{aligned}\right\} \qquad (23)$$

For the determination of the mean value of uv along a line parallel to the axis of x, it is necessary to calculate the integral of M_0 along a line $P\,R$ which is parallel to the same axis. This line corresponds to the line $P_0\,R_0$ of the circle; the lengths of these lines are in the constant proportion:

$$\frac{AB}{A_0B_0}=\frac{1}{\sqrt{2}\,sin\,\alpha}=\sqrt{\frac{1+\varepsilon^2}{2}}$$

Hence this integral takes the value:

$$M_1=\int_{-\sqrt{b^2-y_1^2}}^{+\sqrt{b^2-y_1^2}} dx_1\,\frac{\varepsilon\omega^2}{\sqrt{2(1+\varepsilon^2)}}\{x_1^2(1-\varepsilon^2)+x_1y_1(1+\varepsilon^2)\}. \quad . \quad (24)$$

As has been mentioned already above, ω is a function of $r_0=\sqrt{x_0^2+y_0^2}=\sqrt{x_1^2+y_1^2}$; this function will be taken to be:

$$\omega = c\,(b^2-r_0^2)^{5/4} = c\,(b_2-x_1^2-y_1^2)^{5/4}\ ^{1)} \quad . \quad . \quad . \quad . \quad (25)$$

The second term of the integral vanishes on account of the symmetry of ω; the first term gives:

$$M_1=\frac{\sqrt{2}\,\varepsilon(1-\varepsilon^2)}{\sqrt{1+\varepsilon^2}}\,c^2\int_0^{\sqrt{b^2-y_1^2}} dx_1\,x_1^2\,(b^2-x_1^2-y_1^2)^{5/2}$$

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[1]) In the formulae below everywhere c^2 occurs; the sign of c is of no importance.

or, using the substitution:

$$x_1 = \sqrt{b^2 - y_1^2}\,\sin\chi,$$

$$\left.\begin{aligned} M_1 &= \frac{\sqrt{2}\,\varepsilon\,(1-\varepsilon^2)}{\sqrt{1+\varepsilon^2}}\,c^2\,(b^2-y_1^2)^4 \int\limits_0^{\pi/2} d\chi\,\sin^2\chi\,\cos^6\chi = \\ &= \frac{5\pi}{256}\,\frac{\sqrt{2}\,\varepsilon\,(1-\varepsilon^2)}{\sqrt{1+\varepsilon^2}}\,c^2\,(b^2-y_1^2)^4 \end{aligned}\right\} \quad . \qquad (26)$$

Formula (25) was chosen with a view of obtaining this latter result for M_1, which facilitates the further calculations. If a new variable η is introduced, determined by the formula:

$$\eta = \frac{y_1 + b}{2\,b}$$

(it appears from this formula that η has the value 0 on the tangent at the ellipse at the point D, and takes the value 1 on the tangent at C), then equation (26) can be written:

$$M_1 = A\eta^4\,(1-\eta)^4 = A\varphi\,(\eta) \;. \;\;.\;\;.\;\;.\;\;.\;\;. \quad (27)$$

Here A is a factor independent of the variable η.

If we imagine a great number of these vortices to be present, all of them having the same dimensions and lying between the same tangents parallel to the axis of x, (comp. fig. 3), the amount contributed by them to the value of $\overline{uv}$ will be proportional to the function represented by (27)[1].

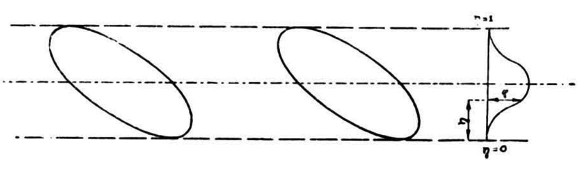

Fig. 3.

The integral of the quantity M_0 taken over the entire area of the vortex amounts to:

$$M_2 = \frac{2\pi}{63}\,\frac{\varepsilon^2(1-\varepsilon^2)}{1+\varepsilon^2}\,c^2\,b^9 . \;\;.\;\;.\;\;.\;\;.\;\;.\;\;. \quad (28)$$

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[1]) Other types of motion may lead to the same form of the function determining M_1; for instance we may take the motion defined by the current function

$$\Psi = \eta^2\,(1-\eta^2)\,(e^{1-\eta}\cos ax - e^{\eta}\sin ax)$$

for values of η between 0 and 1, so that the components of the velocity have the values:

$$u = -\,\partial\Psi/\partial x,\; v = \partial\Psi/\partial y .$$

The integral of the square of the vorticity $N_2 = \iint dxdy\,\zeta^2$ extended over the same area becomes according to the formula given by LORENTZ:

$$N_2 = \frac{\pi}{4\varepsilon}(3 + 2\varepsilon^2 + 3\varepsilon^4)\int_0^b dr_0\, r_0^3 \left(\frac{d\omega}{dr_0}\right)^2 = \frac{5\pi}{42}\frac{3 + 2\varepsilon^2 + 3\varepsilon^4}{\varepsilon} c^2 b^7 \quad . \quad (29)$$

From (28) and (29) we deduce:

$$\frac{N_2}{M_2} = \frac{15}{4}\frac{(3 + 2\varepsilon^2 + 3\varepsilon^4)(1+\varepsilon^2)}{\varepsilon^2(1-\varepsilon^2)}\frac{1}{b^2},$$

or, introducing the "thickness" D of the vortex (cf. fig. 2), so that:

$$b = D\frac{\sqrt{2(1+\varepsilon^2)}}{4\varepsilon},$$

we get:

$$\frac{N_2}{M_2} = 30\frac{3 + 2\varepsilon^2 + 3\varepsilon^4}{\varepsilon(1-\varepsilon^2)}\frac{1}{D^2} = \frac{294}{D^2} \quad . \quad . \quad . \quad . \quad . \quad (30)$$

This fraction surpasses only by a small amount its minimum value, calculated by LORENTZ:

$$14,68\,\frac{2(3 + 2\varepsilon^2 + 3\varepsilon^4)}{\varepsilon(1-\varepsilon^2)}\frac{1}{D^2} = \frac{288}{D^2}\ ^{1)}$$

§ 5. *Distribution of the vortices over the fluid.*

It has already been remarked in § 1 and 3 that our object in this paragraph is not to analyse the true distribution of the vorticity of the fluid, but that we will construct an ideal case only, a "model", which affords us an admissible image of the behaviour of the quantities $\overline{uv}$ and $\overline{\zeta^2}$. This model is obtained by distributing a number of elliptic vortices, of the type studied in the foregoing paragraph, over the mean current $U(y)$. In doing this we do not want to pay any attention to the abscissae of the centra of the vortices, if only their mean distribution along lines parallel to the axis of x be uniform. Positively and negatively rotating vortices are distributed uniformly through each other. If two or more vortices may happen to overlap, they may as well strengthen as enfeeble their respective fields; hence in calculating the mean values $\overline{uv}$ and $\overline{\zeta^2}$ it is unnecessary to take account of these overlappings, and the contributions of the different vortices may be simply summed.

If for a moment we direct our attention to a special class of

[1]) Comp. a remark made by LORENTZ, l.c. p. 54/55. The function defined by eq. (25) above fulfils the condition: $d\omega/ds = 0$ for $s = 1$ $(s = r_0/b)$.

vortices, the thickness D of which lies between the limits D and $D + dD$, and the lower tangents of which (i.e. the tangent at the point D in fig. 2) are enclosed between the limits $y = \xi$ and $y = \xi + d\xi$, then we may say that all of them are lying between the same lines parallel to the axis of x, and by what has been remarked above all of them will give proportional contributions to the field of $\overline{uv}$-values. As the integral $-\int dx\, uv$ extended over a section PR of a single vortex has been calculated in (26) and (27), we may write the contribution of the whole class:

$$b\,(D, \xi)\, \eta^4\, (1-\eta)^4\, dD\, d\xi = b\varphi\,(\eta)\, dD\, d\xi.$$

In this expression: $\eta = (y-\xi)/D$, and the factor $b\,(D,\xi)\, dD\, d\xi$ represents the product of the number of these vortices contained in a strip of unit length parallel to the axis of x, their mean intensity (i.e. the mean of c^2), and the other factors which are contained in the letter A of formula (27). If the function $b\,(D, \xi)$ is given, the distribution of $\overline{uv}$ can be calculated.

It is not necessary to know the value of the quantity $\overline{\zeta^2}$ at every point of the current, its integral only over the whole breadth being wanted, which integral can be found as the sum of the integrals of ζ^2 over all vortices contained in a strip of the full breadth, and of unit length. With the aid of formula (30) we find as the contribution of the considered class of vortices:

$$\left.\begin{aligned} &\iint dx\, dy\, \zeta^2 = -\frac{294}{D^2}\iint dx\, dy\, uv = \\ &= \frac{294}{D^2}\, b\, dD\, d\xi \int_{\xi}^{\xi+D} dy\, \varphi\!\left(\frac{y-\xi}{D}\right) = \frac{294}{630}\frac{b\, dD\, d\xi}{D} \end{aligned}\right\} \quad . \; . \; . \quad (31)$$

A simplification further arises from the fact that the second and third equations (16) which determine τ and $\varkappa$ are homogeneous as regards to the intensity of the vortices. In using these equations it is allowed to multiply b with an arbitrary factor. The true value of σ is found from the principal equation (17). It would be possible to calculate the true value of b afterwards, but this is of no use.

The problem put in paragraph 3: to make σ as great as possible, obliges us to search for a function $b\,(D, \xi)$ which gives a value of $-\overline{uv}$ as nearly constant as possible. Two rather simple types of functions will be discussed.

I. We will begin with an investigation of what can be reached if all vortices have the same thickness D. In that case in order to obtain a constant value of $-\overline{uv}$, it is necessary to make b independent of ξ, in other words to distribute the vortices uniformly over the breadth of the current. However, it is obvious that the vortices cannot pass through the walls of the channel; hence we must take:

$$\left.\begin{array}{ll} b = \mathit{constans}, & \text{if } 0 < \xi < 1 - D \\ b = 0 & , \text{ if } \xi < 0 \text{ or } \xi > 1 - D \end{array}\right\} \quad . \quad . \quad . \quad (32)$$

Consequently the quantity $-\overline{uv}$ will have a constant value in the region defined by: $D < y < 1 - D$ only; in the two remaining strips it decreases to zero.

With the omission of a constant factor, the following expressions for $-\overline{uv}$ are found:

$$\left.\begin{array}{l}
a) \text{ if } y < D: \\
-\overline{uv} = \displaystyle\int_0^y d\xi\, \varphi\left(\frac{y-\xi}{D}\right) = D\int_0^{y/D} d\eta\, \varphi(\eta) = \\
= \dfrac{D}{630}\left\{126\left(\dfrac{y}{D}\right)^5 - 420\left(\dfrac{y}{D}\right)^6 + 540\left(\dfrac{y}{D}\right)^7 - 315\left(\dfrac{y}{D}\right)^8 + 70\left(\dfrac{y}{D}\right)^9\right\} \\
b) \text{ if } D < y < 1 - D: \\
-\overline{uv} = \displaystyle\int_{y-D}^{y} d\xi\, \varphi\left(\frac{y-\xi}{D}\right) = D\int_0^1 d\eta\, \varphi(\eta) = \frac{D}{630} \\
c) \text{ if } y > 1 - D\text{: in the expression given under } a)\ y \text{ has to} \\
\text{be replaced by } 1 - y.
\end{array}\right\} \quad (33)$$

By means of these formulae we find:

$$-\int_0^1 dy\, \overline{uv} = \frac{D}{630}(1 - D)$$

$$\int_0^1 dy\, (\overline{uv})^2 = \left(\frac{D}{630}\right)^2 (1 - 1{,}172\, D).$$

hence:

$$\tau = 0{,}828\, D + \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

All vortices being of the same dimensions, equation (30) gives immediately:

$$\varkappa = \frac{294}{D^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Inserting these values into equation (17):

$$\sigma = \frac{1}{0{,}828\, DR} - \frac{294}{0{,}828\, D^3 R^2} - \ldots \quad . \quad . \quad . \quad . \quad (36)$$

(if the terms of the highest order only are written down). This formula gives a maximum value for σ if the thickness D of the vortices is determined by:

$$D = \frac{29{,}7}{\sqrt{R}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

which gives:

$$\sigma = \frac{0{,}027}{\sqrt{R}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

The coefficient C of the resistance formula (1) now becomes, according to (18):

$$C = \frac{S}{\varrho V^2} = \frac{0{,}027}{\sqrt{R}} + \text{ terms of the order } \frac{1}{R} \quad . \quad . \quad (39)$$

C diminishes proportionally to $\frac{1}{\sqrt{R}}$; hence we do not obtain the quadratic law of resistance, but the resistance appears to be proportional to the $1\frac{1}{2}$-power of the velocity. This does not conform to the result of paragraph 3. In the latter paragraph, however, it was assumed that the most intensive vorticity was concentrated in the neighbourhood of the walls only, whereas in the model considered above it is distributed uniformly over the whole breadth. If all vortices have the same dimensions, it is not possible to distribute them otherwise, without disturbing the field of $\overline{uv}$-values. Hence we must try to obtain a better result by using vortices of different dimensions.

II. If we take vortices of different dimensions, say with thicknesses ranging from $D = 1$ to a lower limit D_0 (to be determined later on), the thickness of the boundary layers in the most favourable case will be of the same order of magnitude as D_0. The same applies to the quantity τ. If now the contribution of the vortices of thickness D to the integral $\int \overline{\zeta^2}\, dy$ becomes asymptotically proportional to $\frac{dD}{D^2}$ for small values of D, the value of this integral will become of the order of magnitude of $\frac{1}{D_0}$. In this case we shall be in the circumstances considered in the deduction of equations (19) and (20). Paying attention to equation (31), it is necessary that $B = \int b\, d\xi$ shall be proportional to $\frac{1}{D}$ for small values of D.

Now it appears that a distribution of vortices fulfilling these conditions can be found, if all vortices are put against the walls. If this be done, it is of course unnecessary to use the variable ξ introduced in the beginning of this paragraph, as the positions of all vortices are fixed. Only a determination of the function $B(D)$ is wanted. The following form of this function gives the right distribution of $\overline{uv}$-values:

1. the class of vortices whose thicknesses lie between the limits D and $D+dD$ have a total intensity proportional to $B\,dD = 2\dfrac{dD}{D}$; these vortices are divided into two equal groups, each of them situated along one of the walls;

2. besides the vortices mentioned under 1), there is a number of vortices of thickness $D=1$, which have the total intensity $1/4$ (in same unit as used above).

With this determination of $B(D)$, the value of $-\overline{uv}$ appears to be, if $D_0 < y < 1-D_0$:

$$\left.\begin{aligned} -\overline{uv} &= \int\limits_y^1 \frac{dD}{D}\,\varphi\left(\frac{y}{D}\right) + \int\limits_{1-y}^1 \frac{dD}{D}\,\varphi\left(\frac{1-y}{D}\right) + \frac{1}{4}\,\varphi(y) = \\ &= \int\limits_y^1 \frac{d\eta}{\eta}\,\varphi(\eta) + \int\limits_{1-y}^1 \frac{d\eta}{\eta}\,\varphi(\eta) + \frac{1}{4}\,\varphi(y) = \\ &= \frac{1}{280} \end{aligned}\right\} \quad (40)$$

The first term represents the contribution of the vortices lying along the wall $y=0$; of these vortices only those are of importance for which $D>y$. The second term represents the contribution of the vortices situated at the other side; here only those for which $D>1-y$ are of importance. The third term represents the contribution of the group of vortices whose thickness D is equal to 1 [1]).

[1]) If we should take the quantity B proportional to D^{-n}, with $n<1$, the integral $\int \zeta^2\,dy$ would take a smaller value, but now the first term of equation (40) which gives the contribution of the vortices situated against the wall $y=0$, would become:

$$\int\limits_y^1 \frac{dD}{D^n}\,\varphi\left(\frac{y}{D}\right) = y^{1-n}\int\limits_y^1 d\eta\,\eta^{2+n}(1-\eta)^4 \quad \text{(for } y>D_0\text{)}$$

If y becomes small, this expression approaches to zero. Only if $n=1$ it approaches to a value independent of y, which is necessary in order that a constant value of $-\overline{uv}$ at all points outside of the boundary layer may be obtained.

In the boundary layer defined by $0 < y < D_0$, the value of $-\overline{uv}$ is found to be:

$$
\left.\begin{aligned}
-\overline{uv} &= \int\limits_{D_0}^{1} \frac{dD}{D}\varphi\left(\frac{y}{D}\right) + \int\limits_{1-y}^{1} \frac{dD}{D}\varphi\left(\frac{1-y}{D}\right) + \frac{1}{4}\varphi(y) = \\
&= \frac{1}{280} - \int\limits_{y}^{D_0} \frac{dD}{D}\varphi\left(\frac{y}{D}\right) = \\
&= \frac{1}{280}\left\{70\left(\frac{y}{D_0}\right)^4 - 224\left(\frac{y}{D_0}\right)^5 + 280\left(\frac{y}{D_0}\right)^6 - 160\left(\frac{y}{D_0}\right)^7 + 35\left(\frac{y}{D_0}\right)^8\right\}
\end{aligned}\right\} \quad (41)
$$

Using the formulae (40) and (41) we find:

$$-\int\limits_0^1 dy\,\overline{uv} = \frac{1}{280}(1 - 0{,}889\, D_0)$$

$$\int\limits_0^1 dy\,(\overline{uv})^2 = \left(\frac{1}{280}\right)^2 (1 - 1{,}068\, D_0)$$

and by means of the latter there results:

$$\tau = 0{,}710\, D_0 - \ldots\ldots \quad (42)$$

The value of $\varkappa$ can be calculated in the following way: The vortices having thicknesses between the limits D and $D + dD$ contribute to the integral $-\int dy\,\overline{uv}$ the amount:

$$2\frac{dD}{D}\int\limits_0^D dy\,\varphi\left(\frac{y}{D}\right) = \frac{dD}{315};$$

hence, according to (30), to the integral $\int dy\,\overline{\zeta^2}$:

$$\frac{294}{315}\frac{dD}{D^2}$$

To this must be added the contribution of the vortices with thickness 1, amounting to:

in $\quad -\int dy\,\overline{uv} \;:\; \frac{1}{2520};$

hence in $\quad \int dy\,\overline{\zeta^2} \;:\; \frac{294}{2520}.$

Adding all parts together, we get:

$$\int dy\,\overline{\zeta^2} = \frac{294}{315}\left(\frac{1}{D_0} - 1\right) + \frac{294}{2520} = \frac{294}{315}\left(\frac{1}{D_0} - \frac{7}{8}\right).$$

Finally the value of $\varkappa$ becomes:

$$\varkappa = \frac{261}{D_0} + \ldots \ldots \quad (43)$$

The values given by (42) and (43) are inserted into the principal equation (17); retaining the terms of the highest order only, we find:

$$\sigma = \frac{1}{0{,}710\, D_0\, R} - \frac{261}{0.710\, D_0^2\, R^2} - \ldots \ldots \quad (44)$$

σ attains its maximum value if the lower limit D_0 of the thickness of the vortices is determined by:

$$D_0 = \frac{522}{R} \quad \ldots \ldots \quad (45)$$

This is much below the value of D given by equation (37). Using (45) we find:

$$\underline{\sigma = 0{,}00135 + \ldots} \ldots \ldots \quad (46)$$

and the coefficient of the resistance formula becomes:

$$C = \frac{S}{\varrho V^2} = 0{,}00135 + \text{terms of the order } \frac{1}{R} \quad \ldots \quad (47)$$

So this arrangement of the vortices leads to the *quadratic* law of resistance.

§ 6. *Discussion.*

In paragraph 5 II we have found the value 0,00135, as a higher limit of the coefficient C of the resistance formula using an idealized model of the distribution of the vorticity in a turbulent current.

If it is possible to calculate C without the use of this special model, using equations (17) and (18) and conditions (6) and (7) only, a still higher limit will probably be found. At the other side if we compare the value of C obtained here to the value given by formula (4*b*), it appears that in the region which is of importance: $R = 10000$ to 1000000, the value of C is *too high.*[1])

Hence we may assert that the true resistance is not the highest possible resistance. In order to determine the true state of affairs, a further condition will be necessary.

From the result that the value of C appears to be too high, we may deduce that the distribution of the value of $-\overline{uv}$ over the current is too uniform. Paying attention to the results of measurements of the distribution of the velocity over the breadth of the

[1]) According to COUETTE's experiments turbulence sets in at $R =$ ca. 1900.

current, we may expect that $-\overline{uv}$ has not a constant value between the boundary layers, but that it is slightly "rounded off". This might be ascribed to slight irregular displacements of the vortices caused by the irregularly distributed velocities which they impart to each other. This "Brownian" movement might give a distribution of the smaller vortices resembling the one determined by the law of BOLTZMANN-MAXWELL for a gas under the influence of gravity, which possibility has been pointed out by VON KÁRMÁN in the lecture mentioned above.

The true distribution of vorticity in the turbulent motion will take some mean position between the two extremes of paragraph 5 (uniform distribution over the whole breadth with C proportional to

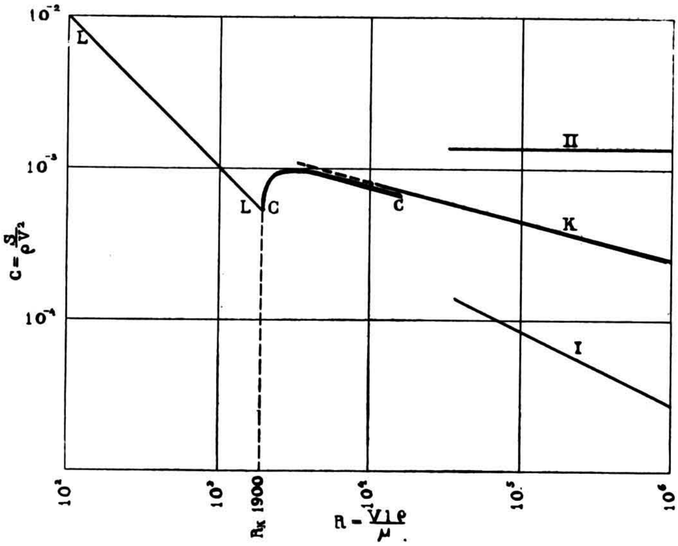

Fig. 4. Logarithmic-scale diagram of the dependence of C on R.

Curve L: laminar region, $C = \dfrac{1}{R}$ (form. 3).

Curve C: results of COUETTE's experiments (the value of R has been calculated using $\mu = 0{,}01096$, comp. COUETTE, l. c. p. 460).

Curve K: $C = 0{,}008\ R^{-1/4}$ (form. 4b), deduced from the investigations by VON KÁRMÁN on the behaviour of $U(y)$.

Curve I: formula (39), deduced from the supposition that all vortices have the same dimensions, and are uniformly distributed over the section.

Curve II: formula (47), deduced from the supposition that the vortices have different dimensions, and are lying against the walls.

$\frac{1}{\sqrt{R}}$, or the best ordered arrangement with all vortices along the walls and C equal to a (high) constant value).

For the sake of comparison the formulae (39), (47) and (46) have been represented together in fig. 4 at a logarithmic scale.

§ 7. *Motion of a fluid between two fixed parallel walls.*

The motion of a fluid between two fixed parallel walls may be treated according to the same scheme as has been used for the motion between a fixed and a moving wall. As the former case has somewhat more resemblance to the types of motion occurring usually in practical cases, the principal features of the calculation will be mentioned here.

The distance of the walls will be taken equal to h; the mean velocity of the current is denoted by V; the pressure gradient $-dp/dx$ will be denoted by J. — REYNOLDS' characteristic number becomes: $R = Vh\varrho/\mu$; the coefficient of the resistance formula is written $C = Jh/\varrho V^2$. Equation (8) of paragraph 2 has to be replaced by the following equation governing the principal motion:

$$\mu \frac{d^2 U}{dy^2} - \frac{d}{dy}(\varrho\, \overline{uv}) = -J \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

A first integration of this formula gives:

$$\mu \frac{dU}{dy} - \varrho\, \overline{uv} = J\left(\frac{h}{2} - y\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

The constant of the integration is determined by observing that on account of the symmetry of the arrangement both quantities dU/dy and $\overline{uv}$ vanish for $y = h/2$. On integrating a second and a third time, and observing that $U = 0$ at both walls, we get:

$$\mu V h = \frac{1}{12} J h^2 - \int_0^h dy\, \varrho y\, \overline{uv} \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

This equation replaces formula (11). Condition (9) which expresses the dependance of the relative motion on the principal motion, retains its form. Now firstly, using (49), we eliminate dU/dy from (9); then using (50), we eliminate J and we obtain:

$$\frac{\mu V}{h} = \frac{\frac{1}{12}\int_0^h dy \left\{\varrho^2 (\overline{uv})^2 + \mu^2 \overline{\zeta^2}\right\} - \frac{1}{h^2}\left(\int_0^h dy\, \varrho y\, \overline{uv}\right)^2}{\frac{1}{h}\int_0^h dy\, \varrho y\, \overline{uv}} \quad . \quad . \quad (51)$$

After the introduction of undimensioned variables, we make use of the abbreviations:

$$\left.\begin{aligned} \int_0^1 dy\, y\, \overline{uv} &= \sigma \\ \frac{1}{12}\int_0^1 dy\, (\overline{uv})^2 &= (1+\tau)\,\sigma^2 \\ \frac{1}{12}\int_0^1 dy\, \overline{\zeta^2} &= \varkappa\sigma \end{aligned}\right\} \quad \ldots\ldots \quad (52)$$

The equations (50) and (51) now reduce to:

$$\sigma\tau + \frac{\varkappa}{R^2} = \frac{1}{R} \quad \ldots\ldots\ldots \quad (53)$$

$$\frac{1}{12}\frac{Jh}{\varrho V^2} = \frac{C}{12} = \sigma + \frac{1}{R} . \quad \ldots\ldots \quad (54)$$

Distribution of the vortices over the fluid.

As appears from equation (49) the value of $\mu \dfrac{dU}{dy}$ will be small compared to that of $J\left(\dfrac{h}{2} - y\right)$ (as is the case for the real motion) only if $-\varrho\overline{uv}$ becomes approximately equal to $J\left(\dfrac{h}{2} - y\right)$. Or, using the undimensioned variables introduced above, we may say that $-\overline{uv}$ aught to be proportional to $\frac{1}{2} - y$.

Hence the quantity $\overline{uv}$ must take a negative value in the neighbourhood of the wall $y = 0$, and it must take a positive value at the other wall. This can be obtained if we use two groups of vortices whose positions are symmetrical with respect to each other. In the first place a group of elliptic vortices having the same position as those described in paragraphs 4 and 5 (i.e. with the long axis extended from the second to the fourth quadrant) is put in against the wall $y = 0$. The contribution of these vortices to the field of values of $\overline{uv}$ will be denoted by

$$-(\overline{uv})_{\mathrm{I}} = \psi(y).$$

Then a second group is put in, situated symmetrically against the other wall: the contribution of the latter to $\overline{uv}$ will be:

$$-(\overline{uv})_{\mathrm{II}} = -\psi(1-y).$$

The contributions of both groups to the integral $\int dy\, \overline{\zeta^2}$ are of course equal and of equal signs.

If we now take vortices having thicknesses ranging from 1 to a minimum value D_0, and we take their intensities proportional to:

$$B\, d\, D = \left(\frac{1}{D} - \frac{3}{4}\right) dD \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

(this expression has a positive value for all values of D), then we obtain for values of y lying between D_0 and $1 - D_0$ the following expression of $\psi(y)$ (with the omission of a constant factor):

$$\psi(y) = \int\limits_y^1 dD \left(\frac{1}{D} - \frac{3}{4}\right) \varphi\left(\frac{y}{D}\right) =$$

$$= \frac{1}{140} \left\{ \frac{1}{2} - y + 7\, y^5 - 14\, y^6 + 10\, y^7 - \frac{5}{2}\, y^8 \right\}$$

from which follows:

$$\psi(y) - \psi(1 - y) = \frac{1}{140} \left(\frac{1}{2} - y\right) \quad . \quad . \quad . \quad . \quad . \quad (56)$$

Hence between the boundary layers the values of $\overline{uv}$ are correctly distributed.

Within each boundary layer $|\overline{uv}|$ decreases from 1/280 to zero. The full expression of the value of $\overline{uv}$ having been worked out, we obtain the integrals:

$$\int\limits_0^1 dy\, y\, \overline{uv} = \frac{1}{1680} \left(1 - 2{,}667\, D_0 + \ldots\right)$$

$$\frac{1}{12} \int\limits_0^1 dy\, (\overline{uv})^2 = \left(\frac{1}{1680}\right)^2 (1 - 3{,}204\, D_0 + \ldots)$$

from which:

$$\tau = 2{,}129\, D_0 - \text{terms of the order } D_0^2 \ldots \quad . \quad . \quad . \quad (57)$$

The value of the integral $\int\limits_0^1 dy\, \overline{\zeta^2}$ becomes:

$$2 \int\limits_{D_0}^1 dD\, \frac{294}{630}\, \frac{1}{D} \left(\frac{1}{D} - \frac{3}{4}\right) = \frac{294}{315} \left(\frac{1}{D_0} - \frac{3}{4}\, lg\, \frac{1}{D_0} - \ldots\right).$$

This gives:

$$\varkappa = \frac{131}{D_0}\left(1 - \frac{3D_0}{4}\, lg \frac{1}{D_0} + \text{terms of the order } D_0 \ldots\right) \quad . \quad . \quad (58)$$

The results of (57) and (58) are substituted into equation (53); and the maximum value of σ is determined. This maximum occurs if:

$$D_0 = \frac{262}{R}\left(1 - \frac{98}{R}\, lg \frac{R}{262} \ldots\right)$$

Finally equation (54) gives:

$$C = 0{,}0108 + \frac{2{,}11}{R}\, lg\, R + \text{terms of the order } \frac{1}{R}\,^{1)} \quad . \quad . \quad . \quad . \quad (59)$$

Discussion.

In this case too the *quadratic* law of resistance is asymptotically arrived at (for values of R surpassing 100000 the logarithmic term is little more than 2°/₀ of the constant term). Just like what occurred in the more simple case the value of the coefficient C is *too high*. For channels with smooth walls VON MISES gives that C ranges from 0,006 to 0,0024 if R ranges from 10000 to the greatest values obtained; the formula derived by VON KÁRMÁN's theory gives:

$$C = \text{ca. } 0{,}07 \ R^{-1/4}$$

For channels with rough walls the dependance of the coefficient C on the value of R is usually very small, so that a quadratic resistance formula can be used, the value of C depending, however, on the dimensions of the irregularities of the walls as compared to the diameter of the channel. The value of C is much higher than in the case of smooth walls; it may even surpass that given by (59). So GIBSON mentions values ranging to 0,015 for old cast iron tubes or channels, lightly tuberculated [2]).

Laboratorium voor Aero- en Hydrodynamica der T. H.
Delft, May 1923.

[1]) The constant term of C in this formula has a value of 8 times that of formula (47). An elementary but superficial comparison of the magnitude of the frictional forces exerted on the walls in both cases leads to the same result.

[2]) R. VON MISES, l.c. p. 63, in connection with the definition of r, given at p. 83/84. In the case of a channel of infinite depth as the one treated here, r is equal to h.

A. H. GIBSON, Hydraulics and its applications (1919), p. 209 (in the formula mentioned at p. 206 is m is ½ time the quantity r introduced by VON MISES; comp. GIBSON, l.c. p. 194).

Comp. also L. SCHILLER, ZS. für angew. Math. u. Mechanik, **3**, p. 2, 1923. and others.