Mathematics. — *"On a non-symmetrical affine field theory."* By Prof. J. A. SCHOUTEN. (Communicated by Prof. H. A. LORENTZ.)

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1. *Introduction.* In his last publications [1]) EINSTEIN has given a theory of gravitation which only depends on a symmetrical linear pseudo-parallel displacement ("affine Uebertragung") and a principle of variation. From the equations, that result in this case, we see that the electromagnetic field only depends on the curl of the electric current vector, so that the difficulty arises that the electromagnetic field cannot exist in a place with vanishing current density.

In the following pages will be shown that this difficulty disappears when the more general supposition is made that the original displacement is not necessarily symmetrical.

The equations which define such a displacement are

$$\nabla_\mu v^\nu = \frac{\partial v^\nu}{\partial x^\mu} + \Gamma'^{\nu}_{\lambda\mu} v^\lambda$$

$$\nabla_\mu w_\lambda = \frac{\partial w_\lambda}{\partial x^\mu} - \Gamma'^{\nu}_{\lambda\mu} w_\nu ,$$

in which the parameters $\Gamma'^{\nu}_{\lambda\mu}$ (with an accent to distinguish them from the $\Gamma^{\nu}_{\lambda\mu}$ of a symmetrical displacement) are not symmetrical in λ, μ.

EINSTEIN [2]) has defended the use of symmetrical parameters with the remark that in the non symmetrical case not only

$$\frac{\partial w_\lambda}{\partial x^\mu} - \Gamma'^{\nu}_{\lambda\mu} w_\nu$$

but also

$$\frac{\partial w^\lambda}{\partial x^\mu} - \Gamma'^{\nu}_{\mu\lambda} w_\nu$$

can be regarded as the covariant differential quotient (Erweiterung)

1) Berliner Sitzungsberichte 1923 p. 32—38, 76—77, 137—140.

2) L.c. p. 33.

of a covariant vector, and thus the unambiguous character of this quotient would vanish. But when the second expression is used the transvection $v^\lambda w_\lambda$ of two vectors v^ν and w_λ is no more an invariant with a pseudo-parallel displacement, so that the differential quotient of the first formula occupies a well defined preferred position.

We will not consider the most general case, but the *semi-symmetrical* case in which the alternating part of the parameters has the form

$$^1/_2(\Gamma'^{\nu}_{\lambda\mu} - \Gamma'^{\nu}_{\mu\lambda}) = {}^1/_2(S_\lambda A^\nu_\mu - S_\mu A^\nu_\lambda) \quad ; \quad A^\nu_\lambda = \begin{cases} 1, & \nu = \lambda \\ 0, & \nu \neq \lambda, \end{cases}$$

in which S_λ is a general covariant vector [1]). It will be shown that already with this simplified supposition the above mentioned difficulty can be made to disappear.

About the *special form* of the world function $\mathfrak{H}$, nothing will be supposed, so that the resulting expressions are quite general.

2. *Deduction of the field equations.* The $\Gamma'^{\nu}_{\lambda\mu}$ of a semi-symmetrical displacement can always be divided into a symmetrical and an antisymmetrical part:

$$(1) \qquad \Gamma'^{\nu}_{\lambda\mu} = \Lambda^\nu_{\lambda\mu} + S_{[\lambda} A^\nu_{\mu]} \quad ; \quad \Lambda^\nu_{\lambda\mu} = \Lambda^\nu_{\mu\lambda}\ ^{2)}.$$

Be $R'^{\cdot\cdot\cdot\nu}_{\omega\lambda\mu}$ the curvature quantity belonging to $\Gamma'^{\nu}_{\lambda\mu}$:

$$(2) \qquad R'^{\cdot\cdot\cdot\nu}_{\omega\mu\lambda} = \frac{\partial}{\partial x^\mu} \Gamma'^{\nu}_{\lambda\omega} - \frac{\partial}{\partial x^\omega} \Gamma'^{\nu}_{\lambda\mu} - \Gamma'^{\nu}_{\varkappa\omega} \Gamma'^{\varkappa}_{\lambda\mu} + \Gamma'^{\nu}_{\varkappa\mu} \Gamma'^{\varkappa}_{\lambda\omega},$$

$\overset{*}{R}{}^{\cdot\cdot\cdot\nu}_{\omega\mu\lambda}$ the curvature quantity formed in the same way with the parameters $\Lambda^\nu_{\lambda\mu}$, $R'_{\mu\lambda}$ the quantity obtained from $R'^{\cdot\cdot\cdot\nu}_{\omega\mu\lambda}$ by contracting, $\omega = \nu$:

$$(3) \qquad R'_{\mu\lambda} = \frac{\partial}{\partial x^\mu} \Gamma'^{\alpha}_{\lambda\alpha} - \frac{\partial}{\partial x^\alpha} \Gamma'^{\alpha}_{\lambda\mu} - \Gamma'^{\alpha}_{\varkappa\alpha} \Gamma'^{\varkappa}_{\lambda\mu} + \Gamma'^{\alpha}_{\varkappa\mu} \Gamma'^{\varkappa}_{\lambda\alpha}$$

and $\overset{*}{R}_{\mu\lambda}$ the quantity obtained in the same way from $\overset{*}{R}{}^{\cdot\cdot\cdot\nu}_{\omega\mu\lambda}$, then we can easily deduce the relation

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1) That the differences $\Gamma'^{\nu}_{\lambda\mu} - \Gamma'^{\nu}_{\mu\lambda}$ always are the components of a quantity of the third rank may be supposed as known. Cf. the author's paper in Math. Zeitschrift 13 (1922), p. 56—81, Nachtrag 15 (1922) p. 168.

2) In this paper the symbol $v_{[\lambda} w_{\mu]}$ means $^1/_2(v_\lambda w_\mu - v_\mu w_\lambda)$.

$$(4)\quad R'_{\mu\lambda} = \overset{*}{R}_{\mu\lambda} - {}^1/_2\left(\frac{\partial S_\lambda}{\partial x^\mu} - \frac{\partial S_\mu}{\partial x^\lambda}\right) + {}^1/_2(n-1)\left(\frac{\partial S_\lambda}{\partial x^\mu} - \Lambda^\nu_{\lambda\mu} S_\nu\right) - {}^1/_4(n-1) S_\lambda S_\mu =$$

$$= \overset{*}{R}_{\mu\lambda} - \overset{*}{\nabla}_{[\mu} S_{\lambda]} + {}^1/_2 (n-1)\, \overset{*}{\nabla}_\mu S_\lambda - {}^1/_4 (n-1)\, S_\lambda S_\mu,$$

in which ∇^* is the covariant differential operator belonging to $\Lambda^\nu_{\lambda\mu}$. We suppose that the determinant $R' = |R'_{\lambda\mu}|$ does not vanish. Hence there exists an inverse quantity $r'^{\lambda\mu}$:

$$(5)\qquad R'\, r'^{\mu\lambda} = \frac{\partial R'}{\partial R'_{\lambda\mu}} \quad ; \quad r'^{\nu\mu} R'_{\mu\lambda} = r'^{\mu\nu} R'_{\lambda\mu} = A^\nu_\lambda\,.$$

When $F'_{\mu\lambda}$ and $G'_{\mu\lambda}$ are the antisymmetrical and the symmetrical part of $R'_{\mu\lambda}$:

$$(6)\qquad F'_{\mu\lambda} = R'_{[\mu\lambda]} \quad ; \quad G'_{\mu\lambda} = R'_{(\mu\lambda)}\ {}^{1)}$$

and when the word function $\mathfrak{H} = H\sqrt{-R}$ (scalar density) is a still unknown function of $G'_{\mu\lambda}$ and $F'_{\mu\lambda}$, we then have the variation equation:

$$(7)\qquad \bar{d}\int \mathfrak{H}\, d\tau = \int \mathfrak{v}'^{\lambda\mu}\, \bar{d}R'_{\mu\lambda}\, d\tau = 0\ {}^{2)}\,,$$

in which

$$(8a)\qquad \mathfrak{v}'^{\lambda\mu} = v'^{\lambda\mu}\sqrt{-R'} = (g'^{\lambda\mu} + f'^{\lambda\mu})\sqrt{-R'}$$

$$(8b)\qquad g'^{\lambda\mu}\sqrt{-R'} = \frac{\partial \mathfrak{H}}{\partial G'_{\mu\lambda}} \quad ; \quad f'^{\lambda\mu}\sqrt{-R'} = \frac{\partial \mathfrak{H}}{\partial F'_{\mu\lambda}}.$$

When we substitute into (7) the value of (4), we get for $n = 4$

$$(9)\quad 0 = \int \mathfrak{v}'^{\lambda\mu}\, d\tau \left\{ \bar{d}\overset{*}{R}_{\mu\lambda} - {}^1/_2\bar{d}\left(\frac{\partial S_\lambda}{\partial x^\mu} - \frac{\partial S_\mu}{\partial x^\lambda}\right) + 2\bar{d}\left(\frac{\partial S_\lambda}{\partial x^\mu} - \Gamma'^\nu_{\lambda\mu} S\right) - {}^3/_4\, \bar{d}(S_\lambda S_\mu) \right\},$$

an equation that, $\overset{*}{R}_{\mu\lambda}$ being independent of S_λ, is equivalent with the two equations

$$(10)'\qquad d\Lambda^\alpha_{\varkappa\mu}\{-A^\mu_\alpha(\overset{*}{\nabla}_\beta v'^{\lambda\beta} - P_\beta v'^{\lambda\beta}) + \nabla_\alpha v'^{\lambda\mu} - P_\alpha v'^{\lambda\mu} - {}^3/_2 S_\alpha v'^{\lambda\mu}\} = 0$$

$$(11)\quad d S_\lambda \{\overset{*}{\nabla}_\mu f'^{\lambda\mu} - P_\mu f'^{\lambda\mu} - {}^3/_2(\overset{*}{\nabla}_\mu v'^{\lambda\mu} - P_\mu v'^{\lambda\mu}) - {}^3/_2 S_\mu g'^{\lambda\mu}\} = 0,$$

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[1]) In this paper $v_{(\lambda} w_{\mu)}$ means ${}^1/_2 (v_\lambda w_\mu + v_\mu w_\lambda)$.

[2]) We use the variation symbol $\bar{d}$ in place of δ to prevent confusion with the symbol δ of the covariant differentiation.

in which P_λ is a vector depending on $R'_{\mu\lambda}$ and $r'^{\lambda\mu}$ in the following way:

(12) $$P_\mu = {}^1/_2\, R'_{\lambda\nu} \overset{*}{\nabla}_\mu r'^{\nu\lambda} = -\frac{\partial \log \sqrt{-R'}}{\partial x^\mu} + \Lambda^\alpha_{\alpha\mu}.$$

Since $\Lambda^\nu_{\lambda\mu}$ is symmetrical in $\lambda\mu$, we get from (10):

(*I*) $$\boxed{\begin{aligned} &- A^{(\mu}_\alpha \overset{*}{\nabla}_\beta g'^{\lambda)\beta} + A^{(\mu}_\alpha P_\beta g'^{\lambda)\beta} - A^{(\mu}_\alpha \overset{*}{\nabla}_\beta f'^{\lambda)\beta} + A^{(\mu}_\alpha P_\beta f'^{\lambda)\beta} + \\ &+ \overset{*}{\nabla}_\alpha g'^{\lambda\mu} - P_\alpha g'^{\lambda\mu} - {}^2/_3\, S_\alpha g'^{\lambda\mu} = 0 \end{aligned}}$$

and from (11):

(*II*) $$\boxed{\overset{*}{\nabla}_\mu f'^{\lambda\mu} - P_\mu f'^{\lambda\mu} - {}^3/_2 (\overset{*}{\nabla}_\mu r'^{\lambda\mu} - P_\mu r'^{\lambda\mu}) - {}^2/_3\, S_\mu g'^{\lambda\mu} = 0.}$$

For $\overset{*}{\nabla}_\mu f'^{\lambda\mu} - P_\mu f'^{\lambda\mu}$ we introduce the notation i'^λ. It is easily shown that

(13) $$i'^\lambda = \overset{*}{\nabla}_\mu f'^{\lambda\mu} - P_\mu f'^{\lambda\mu} = \frac{1}{\sqrt{-R'}} \frac{\partial f'^{\lambda\mu} \sqrt{-R'}}{\partial x^\mu}.$$

From (*I*) follows by contracting, $\alpha = \mu$:

(14) $$\overset{*}{\nabla}_\mu g'^{\lambda\mu} - P_\mu g'^{\lambda\mu} = - i'^\lambda - S_\mu g'^{\lambda\mu}.$$

When this value is substituted into (*I*), we get

(15) $$\nabla^*_\alpha g'^{\lambda\mu} - P_\alpha g'^{\lambda\mu} = - {}^1/_2\, A^{(\mu}_\alpha i'^{\lambda)} - A^{(\mu}_\alpha g'^{\lambda)\beta} S_\beta + {}^2/_3\, S_\alpha g'^{\lambda\mu}.$$

In the supposition that also the determinant $|g'^{\lambda\mu}|$ does not vanish this equation can be simplified by the introduction of the tensor

(16) $$g^{\lambda\mu} = \frac{\sqrt{-R'}}{\sqrt{-g}} g'^{\lambda\mu} \quad ; \quad g = |g^{\lambda\mu}|^{-1}$$

as *fundamental tensor* and the vector

(17) $$i_\nu = \frac{\sqrt{-R'}}{\sqrt{-g}} i'^\nu.$$

Then, because of

(18) $$P_\mu - {}^1/_2\, g_{\lambda\nu} \overset{*}{\nabla}_\mu g^{\lambda\nu} = - \frac{\partial}{\partial x^\mu} \log \frac{\sqrt{-R'}}{\sqrt{-g}},$$

the equation (15) passes into:

$$(19)\quad \nabla^*_\alpha g^{\lambda\mu} - {}^1/_2\, g^{\lambda\mu} g_{\beta\gamma} \nabla_\alpha g^{\beta\gamma} = - {}^3/_2\, A^{(\mu}_\alpha i^{\lambda)} - A^{(\mu}_\alpha S^{\lambda)} + {}^3/_2\, S_\alpha g^{\lambda\mu}.$$

Transvection of this equation with $g_{\lambda\mu}$ gives:

$$(20)\qquad - g_{\beta\gamma} \nabla_\alpha g^{\beta\gamma} = - {}^3/_2\, i_\alpha + 5\, S_\alpha,$$

so that we get the resulting equation:

$$(21)\qquad \nabla^*_\alpha g^{\lambda\mu} = - {}^3/_2\, A^{(\mu}_\alpha i^{\lambda)} + {}^1/_2\, i_\alpha g^{\lambda\mu} - A^{(\mu}_\alpha S^{\lambda)} - S_\alpha g^{\lambda\mu}$$

and

$$(III)\qquad \boxed{\nabla'_\alpha g^{\lambda\mu} = - {}^3/_2\, A^{(\mu}_\alpha i^{\lambda)} + {}^1/_2\, i_\alpha g^{\lambda\mu} - 2\, S_\alpha g^{\lambda\mu},}$$

in which ∇' is the differential operator belonging to $\Gamma'^{\nu}_{\lambda\mu}$.

From (21) we deduce:

$$(22)\quad A^{\nu}_{\lambda\mu} = \left\{ {\lambda\mu \atop \nu} \right\} - {}^1/_2\, g_{\lambda\mu} i^{\nu} + {}^1/_6\, A^{\nu}_{\lambda} i_\mu + {}^1/_6\, A^{\nu}_{\mu} i_\lambda - {}^1/_2\, A^{\nu}_{\lambda} S_\mu - {}^1/_2\, A^{\nu}_{\mu} S_\lambda .$$

so that, with regard to (1):

$$(23)\qquad \Gamma'^{\nu}_{\lambda\mu} = \left\{ {\lambda\mu \atop \nu} \right\} - {}^1/_2\, g_{\lambda\mu} i^{\nu} + {}^1/_6\, A^{\nu}_{\lambda} i_\mu + {}^1/_6\, A^{\nu}_{\mu} i_\lambda - A^{\nu}_{\lambda} S_\mu .$$

Substituting (22) into (3), we obtain:

$$(24)\qquad R^*_{\mu\lambda} = K_{\mu\lambda} + {}^1/_6\, (\nabla^*_\mu i_\lambda - \nabla^*_\lambda i_\mu) + {}^1/_6\, i_\mu i_\lambda - {}^1/_2 (\nabla^*_\mu S_\lambda - \nabla^*_\lambda S_\mu) - $$
$$- {}^3/_2\, \nabla_\mu S_\lambda + {}^9/_4\, S_\mu S_\lambda,$$

in which $K_{\lambda\mu}$ is the contracted curvature quantity $K_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ belonging to the fundamental tensor $g_{\lambda\mu}$. By substituting (24) into (4) we obtain the field equations:

$$(IV)\qquad \boxed{\begin{aligned} R'_{\mu\lambda} &= K_{\mu\lambda} + {}^1/_6\, (\nabla^*_\mu i_\lambda - \nabla^*_\lambda i_\mu) + {}^1/_6\, i_\mu i_\lambda - (\nabla^*_\mu S_\lambda - \nabla^*_\lambda S_\mu) \\ &= K_{\mu\lambda} + {}^1/_6 \left(\frac{\partial i_\lambda}{\partial x^\mu} - \frac{\partial i_\mu}{\partial x^\lambda} \right) + {}^1/_6\, i_\mu i_\lambda - \left(\frac{\partial S_\lambda}{\partial x^\mu} - \frac{\partial S_\mu}{\partial x^\lambda} \right) \end{aligned}}$$

From (IV) follows for the bivector $F'_{\mu\lambda}$ of the electromagnetic field:

$$(25)\qquad F'_{\mu\lambda} = R'_{[\mu\lambda]} = {}^1/_6\left(\frac{\partial i_\lambda}{\partial x^\mu} - \frac{\partial i_\mu}{\partial x^\lambda}\right) - \left(\frac{\partial S_\lambda}{\partial x^\mu} - \frac{\partial S_\mu}{\partial x^\lambda}\right).$$

We now return to the equation (II) obtained from the variation principle. With regard to (13), (14) and (17) this equation leads to

$$(26)\qquad i^\nu = 0.$$

Since i^ν has the character of a current vector, it is not allowed to consider variations of the *alternating* part of $\Gamma'^{\nu}_{\lambda\mu}$, when we wish to keep the current vector in the equations. In regions where only an electromagnetic field exists and no current, the variation principle remains valid without any restriction.

The expressions (IV) and (25) only differ from those of EINSTEIN by the terms in S_λ, hence an electromagnetic field is also possible in places with vanishing current vector i^ν. There the vector S_λ behaves as a potential vector.

We can further make the following important remarks:

1. In the field equations (IV) S_λ does not contribute to the *symmetrical* part of $R'_{\mu\lambda}$.

2. When there is no current the displacement is by (III) *conformal*, the fundamental tensor being diminished with $2\,dx^\alpha S_\alpha g^{\lambda\mu}$ when the pseudoparallel displacement is dx^ν.

3. When there is no current and no potential (23) passes into the ordinary equation of the gravitational field, in the same way as EINSTEIN's equation.

3. *The potentialvector* S_λ. It is remarkable that here the potential vector S_λ occurs as unambiguously determinated, not as a vector to which an arbitrary gradient vector may be added. This difficulty disappears when we make the supposition that the parameters which define the displacement are not the same for covariant and for contravariant vectors[1]) and thus no longer adopt the invariance of transvection. It is namely possible to alter covariant parameters independently of the transformation of the original variables by changing the *measure*[2]) of the *covariant* vectors. This change of measure

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[1]) For these displacements cf. the above mentioned paper in Math. Zeitschrift **13**.

[2]) This change of measure has nothing to do with an introduction of a *ds*.

(27) $$\tau\, 'w_\lambda = w_\lambda$$

in which τ is an arbitrary function, leaves the parameters of the contravariant displacement unaltered, while the covariant parameters, which we will also further denote with $\Gamma'^{\nu}_{\lambda\mu}$, will be transformed in the following way:

(28) $$'\Gamma'^{\nu}_{\lambda\mu} = \Gamma'^{\nu}_{\lambda\mu} - \frac{\partial\, lg\, \tau}{\partial x^\mu} A^{\nu}_{\lambda}\,.$$

Such a change of measure cannot be applied in the same easy way to contravariant vectors, the new components $\tau^{-1}\, dx^\nu$ being in general no more exact differentials. In this case we would be obliged to consider space-time as a system of non-exact differentials, and it would no more be possible to represent a point by four finite coordinates. This case has doubtlessly but little attraction so long as there are other possibilities.

When we wish to "loose" the vector S_ν in the above mentioned sense, we have only to consider the $\Gamma'^{\nu}_{\lambda\mu}$ as the parameters of the *covariant* displacement and to define the $\underset{\cdot}{\Gamma}{}^{\nu}_{\lambda\mu}$, the parameters of the contravariant displacement, in the following way:

(29) $$\underset{\cdot}{\Gamma}{}^{\nu}_{\lambda\mu} = \Gamma'^{\nu}_{\lambda\mu} + S_\mu A^{\nu}_{\lambda} = \left\{ {\lambda\mu \atop \nu} \right\} - {}^1/_2\, g_{\lambda\mu}\, i^\nu + {}^1/_6\, A^{\nu}_{\lambda}\, i_\mu + {}^1/_6\, A^{\nu}_{\mu}\, i_\lambda,$$

We then have obtained that $\Gamma^{\nu}_{\lambda\mu}$ is independent of S_λ and that, when covariant measure is changed, S_λ is transformed in the following way:

(30) $$'S_\mu = S_\mu + \frac{\partial\, lg\, \tau}{\partial x_\mu}\,.$$

It is very remarkable that by (23) $\Gamma'^{\nu}_{\lambda\mu}$ has just a form that leads to the desired transformation of the potential vector. If f.i. $\Gamma'^{\nu}_{\lambda\mu}$ contained a term with $S_\lambda A^{\nu}_{\mu}$, it would not be possible to obtain an equation of the form (30).

Representing the covariant differential operator determined by $\underset{\cdot}{\Gamma}{}^{\nu}_{\lambda\mu}$ and $\Gamma'^{\nu}_{\lambda\mu}$ by ∇, (III) is changed into:

(*III'*)
$$\nabla_\alpha\, g^{\lambda\mu} = -\, {}^2/_3\, A^{(\mu}_{\alpha}\, i^{\lambda)} + {}^1/_3\, i_\alpha\, g^{\lambda\mu}$$
$$\nabla_\alpha\, g_{\lambda\mu} = -\, {}^1/_3\, g_{\alpha\mu}\, i_\lambda - {}^1/_3\, g_{\alpha\lambda}\, i_\mu + {}^1/_3\, i_\alpha\, g_{\lambda\mu} - 2\, S_\alpha\, g_{\lambda\mu}\,.$$

The tensor $g_{\lambda\mu}$ is a quantity variable with transformation of covariant measure, for its components do *not* change, while the components of a genuine quantity of second order obtain the factor τ^{-2}. When the current vanishes, this quantity has the same character as the variable fundamental tensor of WEYL's theory, and $-2\,S_\alpha$ behaves as the vector which WEYL calls φ_α.

4. *On the law of conservation of energy and momentum.* The law of conservation of energy and momentum in gravitation theory is a consequence of the identity of BIANCHI. The form of this identity is known for non-symmetrical displacements and for displacements with non-invariant transvection [1]). Hence it must be possible to deduce, starting with this identity, an equation that can be considered as an analogon of the equation that expresses the law of energy and momentum. This possibility exists already before any supposition is made relating to the special form of Hamilton's function.

[1]) Cf. Math. Zeitschrift 1923, 17, p. 111—115; R. WEITZENBÖCK, Invariantentheorie (NOORDHOFF, Groningen 1923), p. 357.