Mathematics. — "*Determination of the Bilinear System of ∞^3 Line Elements of Space*". By Dr. G. SCHAAKE. (Communicated by Prof. HENDRIK DE VRIES).

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§ 1. A system S_3 of ∞^3 line elements (P, l) of space each consisting of a straight line l and a point P on it, has three characteristic numbers φ, ψ and χ. φ is the order of the complex of the lines l of S_3, ψ the number of line elements of S_3 for which P lies in a definite point and χ the order of the curve of the points P of the line elements of S_3 the lines l of which lie in a given plane.

For a *bilinear* system S_3 the numbers φ and ψ are both *one*. In this case the lines l of S_3 form a linear complex C. Any plane α contains, therefore, a plane pencil (A, α) of lines l of S_3, which has the point A of α as vertex. Also the straight line l for which P lies in A, belongs to this plane pencil, which contains at the same time all the straight lines of S_3 through A. If l describes the plane pencil (A, α), P describes a curve which has one point outside A in common with each generatrix of (A, α) but which passes at the same time through A and touches there the line l corresponding to A; hence this curve is a conic k^2 through A. The third characteristic number of S_3 is consequently *two*.

On the supposition that a system $S_3\,(1, 1, 2)$ exists, we shall now derive its properties, and then indicate how by the aid of the found properties any such a system may be constructed.

§ 2. If P moves on an arbitrary straight line r, the line l describes a scroll of which r is a single directrix. As the line elements of S_3 in a plane through r have a conic of points P, there lie in this plane two elements of S_3 of which the points P belong to r, and such a plane contains besides r two generatrices of the scroll corresponding to r, which is, therefore, of the third order. This surface ϱ^3 has the straight line r' associated to r relative to C, as a double directrix.

To a straight line of points P there corresponds in S_3 a cubic surface of straight lines l.

The line elements of S_3 the points of which lie in a plane V, have a congruence Φ of lines l. As the elements of S_3 of which the lines l pass through a given point, have a conic of points P, there are two among these line elements that have their points P in V, and the order of Φ is *two*. For the class of Φ the same number is found.

To a plane of points P there corresponds accordingly in S_3 a congruence (2, 2) *of straight lines l.*

The common lines of two congruences Φ_1 and Φ_2 of straight lines l corresponding resp. to the planes V_1 and V_2, form a scroll (Φ_1, Φ_2) of the eighth order. For the lines of Φ_1 and Φ_2 cutting an arbitrary straight line r, form resp. two surfaces of the fourth order φ_1^4 and φ_2^4 for which the lines r and r' are double directrices and which have eight generatrices in common, as r and r' count each four times in the intersection. (Φ_1, Φ_2) consists of the scroll ϱ^3 associated to the straight line (V_1, V_2), and of a scroll of the fifth order ϱ^5 consisting of singular straight lines of S_3, as two different points P, hence an infinite number of points P, correspond to a generatrix l of ϱ^5.

The singular straight lines l of S_3 form a scroll of the fifth order ϱ^5. Each of these straight lines, together with any of its points, gives a line element of S_3.

As an arbitrary plane has a point in common with each singular straight line, all congruences Φ pass through ϱ^5.

To the five points P in which an arbitrary straight line r cuts the surface ϱ^5, there correspond as straight lines l the five generatrices of ϱ^5 through these points. Hence:

Each surface ϱ^3 has five generatrices in common with ϱ^5.

We can also arrive at this conclusion in the following way. An arbitrary scroll of the third order φ^3 consisting of straight lines of C, has six straight lines in common with a congruence Φ. For the straight line r splits off twice and the line r' four times from the intersection of φ^3 with the surface φ^4 consisting of all straight lines of Φ that cut the directrices r and r' of φ^3. The points P associated to an arbitrary scroll φ^3 consisting of straight lines of C, form therefore a curve of the sixth order. Accordingly a surface ϱ^3 associated to a straight line r, must contain five singular lines of S_3.

In the same way the fact that an arbitrary congruence (2, 2) of straight lines of C has six lines in common with a ϱ^3, causes each congruence Φ to pass through ϱ^5.

§ 3. The rays l of C which cut two arbitrary lines r_1 and r_2, form a scroll λ^3. To this scroll there corresponds a curve of points P which cuts each generatrix of λ^3 once, namely in the point associated to it. The three lines l of the surface $\varrho_1{}^3$ corresponding to r_1 which cut r_2, are the generatrices of λ^3 the points P of which lie on r_1. The curve associated to λ^3 has, therefore, four points in common with an arbitrary plane through r_1.

To a scroll of straight lines l of C there corresponds in S_3 a rational curve of the fourth order k^4 of points P.

To the straight lines l which cut an arbitrary line r and which form accordingly a bilinear congruence with directrices r and r', the points P of a surface are associated. This surface passes through r, because each point of r is the point P of a line l, and also through r', because the line l corresponding to a point of r', always cuts r. Besides this surface cuts each line l resting on r, hence also on r', outside r and r' in the points P associated to l, so that it is of the third order.

To a bilinear congruence of C there corresponds accordingly a cubic surface Ω^3.

To the scroll which two bilinear congruences of C have in common, a k^4 is associated lying on both the surfaces Ω^3 corresponding to the congruences mentioned. These surfaces have one more curve k^5 in common, consisting of points that are singular for S_3. The lines l corresponding to a point of k^5, form the plane pencil of straight lines of C passing through this point.

There is a quintic k^5 of points that are singular for S_3. To each of the points of k^5 corresponds a plane pencil of straight lines l. The lines l associated to these singular points, form a congruence K **(5,5)**.

As a straight line of any bilinear congruence of C passes through each point of k^5, k^5 lies on all surfaces Ω^3.

A singular line l, i. e. a generatrix of ϱ^5, cannot intersect a surface Ω^3 in a point that is not singular for S_3, as the line in S_3 associated to this point, i. e. l, does not cut the line r corresponding to Ω^3. Consequently each singular straight line has three points in common with k^5. This ensues also from the fact that according to § 2 the straight lines l associated to points P of a singular straight line, form a cubic scroll ϱ^3 which must consist of three plane pencils, so that each singular line contains three singular points.

Inversely any straight line t cutting k^5 three times, must be a singular line l for S_3. For the surface ϱ^3 corresponding to this line, is formed by the three plane pencils that correspond to the points of intersection with k^5, so that to the other points of t a constant ray is associated which must coincide with t.

The scroll ϱ^5 of the singular straight lines consists accordingly of the trisecants of the curve k^5.

The trisecants of k^5 passing through an arbitrary point A of this curve, lie in the plane pencil (A, α) of the lines l of C through this point. When A is chosen arbitrarily, the generatrices of (A, α) have a conic of points P; in this case however the point A is associated to any generatrix of (A, α), so that the generatrices of (A, α) contain two straight lines that are singular for S_3, and belong to the generatrices of ϱ^5. Through any point of k^5 there pass therefore always two of its trisecants.

The curve k^5 is a double curve of the surface of its trisecants.

Two trisecants of k^5 cannot intersect each other outside k^5, as in this case the plane through these two lines would contain six points of k^5. A plane section of ϱ^5 has consequently five double points.

The surface ϱ^5 is therefore of the genus one.

The straight lines l associated to the points P of a chord k of k^5, form a plane pencil w_k as the two plane pencils of straight lines l corresponding to the points of intersection of k and k^5, split off from the surface ϱ^3 corresponding to an arbitrary straight line. As outside this curve k cuts one trisecant of k^5, w_k contains one trisecant of k^5.

Inversely to a plane pencil of lines l containing one trisecant of k^5, there corresponds a straight line of points P cutting k^5 twice. For in this case a straight line which cuts k^5 three times, splits off from the conic associated to an arbitrary plane pencil of C which intersects k^5 five times. Hence the number of bisecants of k^5 through a point P is equal to the number of plane pencils through a line l which contain at the same time a generatrix of ϱ^5, that is five.

The number of apparent double points of k^5 is five and the genus of this curve is consequently one.

The curve k^5 cuts resp. five and ten generatrices of a plane pencil and of a scroll of lines l. Hence:

The conic k^2 associated to a plane pencil of C, and the curve k^4 corresponding to a scroll of straight lines l, have resp. five and ten points in common with k^5.

We remark also that the point P associated to a line l, may be determined by constructing in a plane α through l the conic k^2 which cuts k^5 five times. Besides in the vertex A of the plane pencil of C in α, this conic must cut l in the point P corresponding to l. Hence:

The conics k^2 cutting k^5 five times and intersecting a straight line of C twice, all pass through the point P associated to this line.

§ 4. Starting from a twisted curve of the fifth order and the genus one, k^5, we shall now construct a system S_3 which has the properties of the system that we until now supposed to exist and of which k^5 is the locus of the singular points.

In the same way as every twisted quintic, k^5 lies on a cubic surface Ω_1^3. We shall make use of the simplest representation of Ω_1^3 on a plane V, which has in V six singular points $F_1, \ldots, F_6$, to which resp. six crossing straight lines $f_1, \ldots, f_6$ of Ω_1^3 are associated. If e.g. we assume in V a curve k'^5 of the fifth order that has double points in $F_1, \ldots, F_6$, there corresponds to it on Ω_1^3 a curve of the fifth order and the genus one. For the curve assumed in V has five points that are not singular for the representation, in common with the image of a plane section of Ω_1^3, i.e. a cubic through $F_1, \ldots, F_6$.

The image in V of the intersection k^9 of an arbitrary cubic surface Ω_2^3 with Ω_1^3 is a curve k'^9 of the ninth order which has triple points in $F_1, \ldots, F_6$. The curve k'^5 is therefore completed into a k'^9 by a rational quartic k'^4 that has a triple point in F_6 and single points in $F_1, \ldots, F_5$. As consequently a given curve k'^5 together with any individual of a linear system of ∞^3 curves k'^4, is the image of the base curve of a pencil of surfaces Ω^3 all passing through k^5 which contains Ω_1^3, the surfaces of the third order through k^5 form a linear system Σ_4 of ∞^4 individuals.

A curve k'^4 has in common with k'^5 ten points that are not singular for the representation of Ω_1^3 on V. Two surfaces Ω^3 of Σ_4 have therefore besides k^5 another rational curve of intersection of the fourth order k^4, resting on k^5 in ten points.

k^5 is a double curve of the surface of its trisecants. For the projection of k^5 out of one of its points on an arbitrary plane, a curve of the order four and the genus one, has two double points and through such a point there pass accordingly two trisecants of k^5. Further this surface has in common with Ω_1^3 five straight lines that are represented on the five straight lines of V which join F_6 and the other five points F; hence the intersection of the surfaces is of the order fifteen, so that the surface of the trisecants is a surface of the fifth order ϱ^5.

Σ_4 contains one surface Ω^3 to which belongs an arbitrary given straight line r. This surface is the locus of the ∞^1 individuals of the ∞^3 conics k^2 intersecting k^5 five times and cutting r twice. For seven points of intersection of such a conic k^2 and Ω^3 may at once be indicated, so that any conic k^2 of which the plane passes through r, lies on the surface Ω^3 which contains r.

The conics k^2 which cut r twice, define therefore on this straight line an involution I, so that there are two conics k^2 touching r (in the double points of I).

Σ_4 further contains one monoid that has its vertex in an arbitrary given point P. This surface Ω^4_P is the locus of the conics k^2 through P. It contains the five bisecants of k^6 through P, as each of these, together with one trisecant of k^6, forms a conic k^2 through P. Besides these five straight lines there lies on Ω^4_P one more straight line l through P which does not cut k^6. For the quadratic cone of the tangents of Ω^4_P in P has in common with this surface six straight lines through P and the ten points of intersection of the cone with k^6 lie on the five bisecants.

The planes of the conics k^2 through P have in common with Ω^4_P one more straight line through P which does not intersect k^6, and pass therefore through l. Inversely each conic k^2 that intersects l twice, must lie on Ω^4_P and passes therefore through P. For a straight line l corresponding to a point P the involution I is accordingly parabolic. The two conics k^2 touching l, coincide in a conic through P.

Besides the complex of the lines l there is also a linecomplex of the fifth order for which the involution I is parabolic. Let us consider a straight line a which cuts k^6 once. A conic k^2 cutting a twice, must pass through the point of intersection of a and k^6, because else the plane of k^2 would have six points of intersection with k^6. Through each point P of a there passes one such a conic k^2, which is the intersection of Ω^4_P and the plane that passes through a and the straight line l corresponding to P. Also for a line a we have, therefore, only one point where it is touched by a conic k^2.

With a view to determining the order of the complex of the lines l, we take a plane pencil (P, φ) of lines r and investigate the locus of the points where conics k^2 touch these straight lines r. This is a curve which cuts each straight line of (P, φ) twice besides in P and which has a double point in P. The tangents at this double point are at the same time the tangents of Ω^4_P in P which lie in φ. To this curve, which is accordingly of the fourth order, out of its double point P six tangents can be drawn and these are the straight lines for which the involution I is parabolic. As the plane pencil (P, φ) contains five lines a, one line l belongs to (P, φ), so that the complex C of the lines l is linear.

C contains the surface ϱ^6 of the trisecants t of k^6. For if we choose P on a straight line t, Ω^4_P becomes the surface of the bisecants of k^6 which cut t and which, together with t, form therefore conics k^2 through P. For the surface of the bisecants of k^5 intersecting an arbitrary straight line, is of the fifteenth order, as it has the directrix as a five-fold line, and has ten generatrices in a plane through the directrix. If we take a trisecant t of k^5 as directrix, three cones of the fourth order through t are split off from this surface, so that there remains a cubical surface with t as a double line. The planes of the conics k^2 containing P all pass through the line t, which is therefore associated to P as a line l.

Consequently if k^5 is not degenerate, C is a general linear complex. For if C were special, the axis of C would be a directrix of ϱ^5 and even a multiple directrix, as ϱ^5 is not rational. But outside k^5 two trisecants of this curve cannot cut each other.

We remark also that a trisecant t corresponds to each of its points P as a line l.

To a point P of k^5 an infinite number of straight lines is associated. These form the plane pencil of C that has P for vertex and that is defined by the two trisecants of k^5 through P. For any of the lines of this plane pencil the associated point P must lie in the point of intersection with k^5. If we choose P outside k^5 and if this point approaches k^5, Ω^3_P is transformed into the surface formed by the conics k^2 passing through a given point of k^5 and touching at this point a plane through the tangent to k^5. Hence there correspond indeed to a point P of k^5 ∞^1 monoids Ω^3_P that have their vertices in P, and the straight lines l of these monoids form the plane pencil of the straight lines of C through P.

The line elements (P, l) of this § form indeed a bilinear system of ∞^3 individuals for which k^5 is the locus of the singular points P and ϱ^5 the scroll of the singular lines l.

A bilinear system of ∞^3 line elements (P, l) may always be derived from a twisted curve k^5 of the genus one by associating to each point P the line l through P which does not cut k^5, of the monoid of the third order that passes through k^5 and has its vertex in P, or, what amounts to the same, by associating the centre of the parabolic involution that is defined on lines l which do not cut k^5, by the conics intersecting k^5 five times, to these lines l. Inversely in the way indicated a bilinear system of ∞^3 line elements may be derived from any curve k^5 of the genus one.

From the representations of a cubic surface on a plane used in the beginning of this §, there ensues that ∞^5 twisted quintics of the genus one lie on any given cubic surface. As there lie ∞^{19} cubic surfaces in space, and through any k^5 of the genus one there pass ∞^4 cubic surfaces, there lie in space ∞^{24} curves k^5 of the genus one.

There are, accordingly, ∞^{20} *bilinear systems of* ∞^3 *line elements.*

§ 5. There are ∞^{15} bilinear systems S_3 of ∞^3 line elements for which the complex of the lines l coincides with a given linear complex C. This may be proved by the aid of the representation of NÖTHER [1]) of the rays l of C on the points Q of space. For this representation there is one cardinal ray l_1 in C to which all the points Q of a plane V are associated and there is one conic k'^2 of singular points Q in V, to each of which a plane pencil of C containing l_1 corresponds.

To a scroll in C of the order ν which has a v-fold line in l_1, a curve corresponds of the order $\nu—v$ which cuts k'^2 in $\nu—2v$ points. Inversely a curve of the n^{th} order of points Q, intersecting k'^2 in s points, is associated to a scroll in C of the order $2n—s$ which has in l_1 an $(n—s)$-fold line.

A congruence (μ, μ) with a ϱ-fold line in l_1 is represented on a surface of the order $2\mu—\varrho$ of which k'^2 is a $(\mu—\varrho)$-fold conic, and to a surface of the m^{th} order of points Q containing k'^2 as an m_1-fold conic, a congruence of rays $(m—m_1, m—m_1)$ is associated that has an $(m—2m_1)$-fold line in l_1,

Now let us assume a curve k'^6 of the genus one, formed by points Q, which cuts k'^2 five times. This curve is the image of a scroll ϱ^5 of the order five and the genus one the generatrices of which belong to C.

Let us now consider the surface formed by the bisecants of k'^6 which intersect k'^2. This surface has k'^2 as a five-fold and k'^6 as a three-fold curve and is a surface of the tenth order ϱ'^{10}. For k'^2 cuts ten times outside k'^6 the surface of the fifteenth order of the bisecants of k'^6 that cut a given straight line, which surface has k'^6 as a quadruple curve.

To ϱ'^{10} there corresponds a congruence K (5, 5) formed by the plane pencils of C that contain two lines of ϱ^5. The vertices of these plane pencils form accordingly the double curve of ϱ^5, which is of the fifth order; for in a plane there lie five generatrices of the congruence corresponding to ϱ'^{10}, hence also five vertices of plane pencils of this congruence. As a point of k'^6 carries three generatrices of ϱ'^{10}, the straight lines of ϱ^5 are trisecants of k^5. Inversely each trisecant t of k^5 lies on ϱ^5, because six points of intersection of

[1]) "*Zur Theorie algebraischer Functionen*", Gött. Nachrichten 1869.

t and ϱ^5 may be indicated, and ϱ^5 is consequently the surface of the trisecants of k^5. As a point of k^5 carries two trisecants, this curve is of the genus one. As a rule it is not degenerate. For if k^5 consisted of a biquadratic curve of the first kind and a line of intersection of this curve, C would be a special linear complex, and for any other degeneration of k^5 ϱ^5, and accordingly k'^5, would be degenerate.

Of the bilinear system S_3 of ∞^3 line elements which according to § 5 may be derived from k^5, C is the complex of the lines l. Else the surface ϱ^5 would be common to two linear complexes, and as in this case it would belong to a bilinear congruence, it would have two straight directrices, which cannot be the case, even if two straight lines belonged to k^5. For if e. g. k^5 degenerated into a twisted cubic with an intersecting line and a bisecant, also the bisecants of the cubic which meet the intersecting line, would belong to ϱ^5.

§ 6. If we associate to each point P corresponding in S_3 to a line l, the point Q which is conjugated to the same straight line by a representation of NÖTHER, we get ∞^3 pairs of points (P, Q) which define a birational transformation in space. The point P of the line l_1, which we shall call P_1, is a cardinal point for this transformation. The corresponding points Q form the plane V. Further k^5 is a curve of singular points P. To each point of k^5 there corresponds a straight line of points Q which cuts k'^2. The straight lines associated to the points of k^5, form the surface ϱ'^{10}.

There are two curves of singular points Q, namely k'^2 and k'^5. To a point of k'^2 the points P of a plane pencil of C containing l_1 are associated which form a conic k^2 through P_1. The conics k^2 corresponding to the points Q of k'^2, form the monoid $\Omega^3{}_{P_1}$ that has its vertex in P_1. To the points Q of k'^5 are associated straight lines of points P that form the surface ϱ^5.

If P moves on a straight line, l describes a cubic scroll which contains five generatrices of ϱ^5 and Q accordingly describes a cubic which cuts k'^2 three times and k'^5 five times. To a plane of points P there corresponds a congruence (2, 2) of lines l containing ϱ^5, hence a biquadratic surface of points Q of which k'^2 is a double curve and which contains k'^5.

If Q moves on a straight line, l describes a scroll containing l_1 and P therefore a rational quartic that passes through P_1 and intersects k^5 in ten points. To a plane of points Q a bilinear congruence of lines l is associated containing l_1, hence a cubic surface of points P through P_1 containing k^5.

The pairs of points (P, Q) *accordingly define a birational transformation* (3, 4)[1].

§ 7. A curve of the n^{th} order which cuts k^6 m times, intersects a surface Ω^3 in $3n-m$ points that are not singular for S_3 and meets $5n-2m$ generatrices of ϱ^6 outside k^6. Hence:

The lines l associated in S_3 to the points P of a curve of the n^{th} order that cuts k^6 m times, form a scroll of the order $3n-m$ which has $5n-2m$ generatrices in common with ϱ^6.

If inversely we consider a scroll of the order ν that has μ generatrices in common with ϱ^6, we get by making ν and μ resp. equal to $3n-m$ and $5n-2m$ and by solving n and m out of these equations:

The points P corresponding in S_3 to the lines l forming a surface of the order ν which has μ generatrices in common with ϱ^6, form a curve of the order $2\nu-\mu$ which cuts k^6 in $5\nu-3\mu$ points.

A surface of the order p containing k^6 as a q-fold curve, is cut by a conic k^2 and a generatrix of ϱ^6 resp. in $2p-5q$ and $p-3q$ points that are not singular for S_3.

To the points P of a surface of the order p with k^6 as a q-fold curve, there correspond accordingly in S_3 the lines l of a congruence $(2p-5q,\ 2p-5q)$, of which the generatrices of ϱ^6 are $(p-3q)$-fold lines.

Inversely it is easily seen that

To a congruence (π, π) of lines l containing the generatrices of ϱ^6 as $\varkappa$-fold lines, a surface of points P is associated which is of the order $3\pi-5\varkappa$ and has k^6 as a $(\pi-2\varkappa)$-fold curve.

Several applications can be made of the representation defined by S_3 of the rays of C on the points of space. Let us for instance try to find the number of the conics which cut k^6 five times and which meet besides three given straight lines r. These conics are the representations of the plane pencils of C which contain one straight line of each of the three surfaces ϱ^3 corresponding to the lines r and which have accordingly their vertices in the 27 points of intersection of these three surfaces.

There are 27 conics intersecting five times a twisted quintic of the genus one and cutting besides three given straight lines.

§ 8. Finally we determine the scrolls belonging to C that are associated to the straight lines of a cubic surface Ω^3 which is the locus of the points P of the lines l intersecting an arbitrary straight line r, hence also the line r' associated to r relative to C.

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[1] STURM: "*Geometrische Verwandtschaften*", IV p. 371.

The straight lines r and r', which both lie on Ω^3, are the images of the surfaces ϱ^3 and ϱ'^3 corresponding resp. to these lines.

Further the five lines t of ϱ^6 that are singular for S_3 which cut r, belong to Ω^3 as to each of these lines all its points are associated as points P. Besides r these lines also cut r', and they are trisecants of k^6.

As the line t belonging to the plane pencil of C which has the point of intersection of r with a line t as vertex and of which the plane passes therefore through r', splits off from the associated conic, this plane pencil contains a straight line of points P cutting k^6 twice and cutting r'. Accordingly five bisecants of k^5 intersecting r', lie on Ω^3. In the same way we find on Ω^3 five bisecants of k^6 which cut r and which are associated to the plane pencils of C that have the points of intersection of r' and ϱ^6 as vertices.

Finally for a scroll with r and r' as directrices containing three generatrices of ϱ^5, and belonging therefore to C, three trisecants of k^6 split off from the associated quartic. Such a scroll is represented on a straight line which cuts k^5 once, but has no point in common with r and r'. On Ω^3 there lie ten lines of this kind.

In this way the images of the 27 straight lines of Ω^3 are found.

If the straight line r belongs to C and is accordingly a line l, we have to do with a monoid $\Omega^3{}_P$ that has the point P of l as vertex. In this case r and r' coincide with l. Also the straight lines that were associated to the ten plane pencils of C which had their vertices in the points of intersection of r and r' with ϱ^5, coincide in pairs in five lines through P, as all these plane pencils contain l. These five lines are the bisecants of k^5 through P. Further there lie on $\Omega^3{}_P$ the five trisecants of k^5 cutting l, and the ten straight lines belonging to scrolls of C which cut k^6 once and which have no point in common with l.