Mathematics. — "*A Representation of the Line Elements of a Plane on the Points of Space*". By Prof. JAN DE VRIES.

(Communicated at the meeting of January 29, 1924).

1. In the first chapter of his thesis for the doctorate [1]) Dr. G. SCHAAKE has communicated a method to represent the line elements of a plane φ by the points of space. In order to reach this aim by another way I assume a straight line b in φ, a straight line a and a point M outside φ. The point P of the line element $e(P, l)$ defines the straight line $m = PM$, the line l the point $B = lb$. If α is the plane through B and a, I consider the point $S = m\alpha$ as the image of e. Inversely a point S generally defines an element e. For SM cuts φ in P, $\alpha = Sa$ cuts b in B, and $l = PB$.

2. There are three systems of *singular line elements*. For $e = (B, b)$ the point of intersection lb is indefinite, so that also the plane α becomes indefinite; hence any point of the ray BM may be considered as the image S. The *singular* elements (B, b) form a system $(0,1)$, their images are the ranges of points (S) on the rays of the plane pencil (M, β) where $\beta = Mb$.

Let μ be the plane through M and a, c the straight line $\mu\varphi$, C some point of c, $A_0 = ac$, $D = bc$. For the element $e = (C, c)$ $m = CM$ lies in the plane $\alpha = \mu$, so that any point of m may be considered as image S. Also the *singular* elements (C, c) form a $(0,1)$; their images are the point ranges (S) on the rays of the plane pencil (M, μ).

For the element $e = (D, l)$ we have $B = D$, $\alpha = Da = \mu$, $m = DM$; any point of the straight line $d = DM$ may therefore be considered as an image. The *singular* elements (D, l) form a system $(1,0)$.

For the inverse representation M is a *cardinal* point, for each point of φ may be considered as a point P. As $\alpha = \mu$, $B = D$, l always passes through D; M is therefore the image of each element of a null system $N(1,0)$.

Every point A of a is *singular*, for if $S = A$, α is indefinite and

[1]) Afbeeldingen van figuren op de punten eener lineaire ruimte. P. Noordhoff, Groningen, 1922.

$P=(AM,\varphi)$; hence A is the image of each element (C,l). Accordingly the line elements of an $N(0,1)$ correspond to the singular points A; only the points of c are null points.

Also the points $S=B$ are *singular*, for in this case l is indefinite, hence B is the image of each element (B,l).

The image S of any line element of which $l=PA_0$, coincides with the point P. The point A_0 is the image of all the elements (A_0,l), hence a *singular* point S.

3. The straight lines l of a system (1,0) form a plane pencil round a point P_1. The image of this (1,0) is the point range (S) on PM.

The points P of a (0,1) lie on a line l_1. The image of (0,1) is the point range (S) on the intersection of the plane $\alpha=B_1a$ with the plane Ml_1; this line rests accordingly on a.

In a system (1,1) the points P lie on a straight line g and the corresponding lines l pass through a point G. The ranges (P) and (B) are projective, hence the plane pencil (m) is projective with the pencil of planes (α). The image of a (1,1) is therefore a conic σ^2 through M which cuts a. It cuts φ in the point bg and in the point (A_0G,g).

If G lies on c, σ^2 degenerates into the straight line joining M and the point cg and another straight line of the plane Mg. Also if G lies on b, σ^2 degenerates.

In a system (i,k) the locus of the lines l is a curve of the class i and the points P lie on a curve of the order k. Accordingly k points $B=S$ of the image lie on b, and this curve cuts φ besides in the i points P of which the corresponding lines l pass through A_0 (§ 2). The i elements e of which the lines l pass through D, have their images in M, the k elements for which P lies on c, are represented by points A.

The image of a system (i,k) is therefore a curve of the order $(i+k)$ which passes i times through M, and which has the line a as a k-fold line of intersection.

4. In order to determine the image of a *bilinear null system* $N(1,1)$ I consider the elements e that are represented by points of φ. The points P the null rays of which pass through A_0, form a conic α_0^2 through A_0; this "null curve" forms together with the straight line b the intersection of the image Σ^3 with φ.

The points of the null curve δ^2 corresponding to D, define together with D line elements that have their images in M; hence Σ^3 has in M a node of which the cone of tangent lines cuts the plane φ along d^2. This surface is accordingly a *cubic monoid* with vertex M.

The elements e that have their null points on c, are represented by points A; hence Σ^3 contains the line a.

The null point C_0 of c defines the straight line C_0M lying on Σ^3. Analogously the straight line B_0M passes through the null point B_0 of b, and DM is the image of the element e corresponding to D. On Σ^3 there lie three more straight lines m; they are the images of three plane pencils belonging to $N(1,1)$. The null system (1,1) has therefore *three singular null points* [1]. In each plane through two of the lines m there lies another straight line of Σ^3; it is the image of a *singular straight line* of $N(1, 1)$, hence a straight line that has each of its points as a null point.

The remaining 10 straight lines are the images of elements e of the monoid that have their null points on a straight line of φ. The null rays of an arbitrary straight line g envelop a conic touching g. The image of the system (2,1) defined in this way is a nodal cubic with double point M which cuts φ in $B = bg$ and in two other points of g.

For the 10 straight lines mentioned the image degenerates into three straight lines; the line r which Σ^3 also has in common with the plane B_0C_0M, forms, together with B_0M and C_0M, the image of a (2,1) the null points of which are projected out of M on r. To the null curves of $N(1,1)$ there correspond twisted cubics of Σ^3 which pass through M and have a as a chord.

5. The image of a null system $N(1, k)$ is a *monoid* Σ^{k+2} with a $(k+1)$-fold point M. On this monoid lie the straight lines a, b and d besides k straight lines B_0M and k lines C_0M. The remaining $(k+1)(k+2)-2(k+1)$ straight lines m are images of plane pencils; hence the null system has (k^2+k+1) *singular points* [2]).

As the plane through two of these lines m generally cuts the monoid along a curve of the order k, as a rule an $N(1, k)$ has *no* singular straight lines. As for $n > 3$ a monoid Σ^n generally does not contain any straight lines that do not pass through the vertex, Σ^{k+2} is not the most general monoid of the order $(k+2)$.

The image of a null system $N(i, k)$ where a point P is the null point of i rays l and a straight line l is the null ray of k points,

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[1]) See my communication on plane linear null systems. These Proceedings, Vol. XV, p. 1165.

[2]) l. c.

is a surface Σ of the order $(2i+k)$ with an $(i+k)$-fold point M of which the cone of tangent lines has an i-fold generatrix d.

Σ further contains the i-fold lines a, b and d.

The intersection with φ consists of the i-fold straight line b, and the null curve a_0^{i+k} corresponding to A_0. Each of the null points of c defines a straight line on Σ; hence the intersection with the plane μ consists of k lines CM and the i-fold lines a and d. Analogously Σ has the i-fold lines b and d and k lines BM in common with the plane β.

Especially the image of an $N(1,0)$ of which the ∞^2 elements e lie on the rays l of a plane pencil, is a quadratic scroll through the lines a, b, and d. The regulus containing d consists of the images of the elements on the singular null rays.

The null system $N(0,1)$ where any point of a fixed straight line g is the null point of a plane pencil, has apparently for image the points of the plane Mg.

6. If the point S describes a straight line r, the pencil of planes round a becomes projective with the range of points on b and with the range of points on the straight line $g=(Mr,\varphi)$. Hence the straight line l envelops a conic λ^2 touching b and g.

The point $S=r\mu$ is the image of an e formed by $C=cg$ and the line c; accordingly λ^2 is inscribed in the triangle bcg. The other tangent line out of A_0 cuts g in the point $P=gr$.

Together with b the point $B=bg$ defines an element e that is represented by BM. Analogously $C=cg$ defines an element that has CM for image. The complete image of the system (2,1) defined by g and λ^2 consequently consists of the three straight lines r, BM and CM.

If S describes a twisted curve σ^n resting on a in k points A and passing i times through M, the locus of P is a curve of the order $(n-i)$. As a plane α contains $(n-k)$ points S, a point B is associated to $(n-k)$ points P. By the correspondence between the points B and P a correspondence $(n-i,\ n-k)$ is established between the rays of a plane pencil chosen arbitrarily in φ. Consequently the lines l envelop a curve (l) of the class $(2n-i-k)$.

The plane μ contains $(n-i-k)$ points S each of which is the image of an element (C,c); hence c is an $(n-i-k)$-fold tangent of (l). Analogously (l) has the $(n-i)$-fold tangent b. The complete image of the system $(2n-i-k,\ n-i)$ consists apparently of the curve σ^n, $(n-i)$ straight lines BM, and $(n-i-k)$ straight lines CM.

7. A surface Σ^n with an i-fold point in M and a k-fold straight line in a is the image of a null system in φ.

A straight line $m = MP$ contains $(n—i)$ points S; hence P is the null point of $(n—i)$ null rays. The element e of a straight line l are represented by a straight line cutting a; l has accordingly $(n—k)$ null points. Consequently a null system $N(n—i, n—k)$ corresponds to Σ^n.

Evidently b and c are singular null rays; any point B or C may be considered $(n—i)$ times as the null point of b or c. If Σ^n contains a straight line m (through M) this line is the image of a singular plane pencil.

An arbitrary plane Σ is in particular the image of an $N(1,1)$ for which b and c are singular null rays; the third singular null ray passes through the points $b\Sigma$ and A_0. The complete image of this null system consists of the planes Σ, μ and β.

The plane φ is the image of the two null systems $N(1,0)$ and $N(0,1)$, which have resp. A_0 as a singular point and b as a singular line.