Mathematics. — "*The numbers of* STIRLING *expressed by definite integrals*". By Prof. W. KAPTEYN.

(Communicated at the meeting of February 23, 1924).

1. If $x(x+1)\dots(x+n)$ be expanded in a series of ascending powers of x

$$C_{n+1}^0\, x^{n+1} + C_{n+1}^1\, x^n + \dots + C_{n+1}^n\, x$$

the coefficients C_{n+1}^r are called STIRLING's numbers of the first kind.

Equating both forms and dividing by x we get

$$n\,!\,(1+x)\left(1+\frac{x}{2}\right)\cdot\;\left(1+\frac{x}{n}\right) = C_{n+1}^n + C_{n+1}^{n-1}\, x + \dots + C_{n+1}^0\, x^n.$$

Thus we have

$$C_{n+1}^n = n\,!$$

$$C_{n+1}^{n-1} = n\,!\,\overset{n}{\Sigma}\frac{1}{\alpha}$$

$$C_{n+1}^{n-2} = n\,!\,\overset{n}{\Sigma}\frac{1}{\alpha\beta}$$

$$\dots\dots$$

where $\overset{n}{\Sigma}\frac{1}{\alpha}$, $\overset{n}{\Sigma}\frac{1}{\alpha\beta}$, ... represent the sums of all different products of the quantities $\frac{1}{1}, \frac{1}{2}, \dots \frac{1}{n}$ one by one, two by two, etc.

2. The first sum may immediately be written in the form of a definite integral, for

$$\sum_1^n \frac{1}{\alpha} = 1 + \frac{1}{2} + \dots \frac{1}{n} = \int_0^1 \frac{1-x^n}{1-x}\, dx \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The second sum may be reduced to the first, for

$$S_n = \overset{n}{\Sigma}\frac{1}{\alpha\beta} = S_{n-1} + \frac{1}{n}\overset{n-1}{\Sigma}\frac{1}{\alpha}$$

or

$$S_n = S_{n-1} + \frac{1}{n}\int_0^1 \frac{1-y^{n-1}}{1-y}\, dy.$$

Hence

$$S_{n-1} = S_{n-2} + \int_0^1 \frac{1-y^{n-2}}{1-y}\, dy$$

$$S_{n-2} = S_{n-3} + \frac{1}{n-2}\int_0^1 \frac{1-y^{n-3}}{1-y}\, dy$$

. . . .

$$S_3 = S_2 + \frac{1}{3}\int_0^1 \frac{1-y^2}{1-y}\, dy$$

$$S_2 = \qquad \frac{1}{2}\int_0^1 \frac{1-y}{1-y}\, dy$$

and therefore

$$S_n = \int_0^1 \frac{dy}{1-y} \left\{ \frac{1-y}{2} + \frac{1-y^2}{3} + \ldots \frac{1-y^{n-1}}{n} \right\}.$$

Now we know that

$$\frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n} = \int_0^1 \frac{x\,(1-x^{n-1})}{1-x}\, dx = \int_0^1 P dx$$

$$\frac{y}{2} + \frac{y^2}{3} + \ldots \frac{y^{n-1}}{n} = \frac{1}{y}\int_0^y \frac{x\,(1-x^{n-1})}{1-x}\, dx = \frac{1}{y}\int_0^y P dx$$

thus we may write

$$S_n = \int_0^1 \frac{dy}{1-y}\int_0^1 P dx - \int_0^1 \frac{dy}{y(1-y)}\int_0^y P dx$$

or

$$S_n = \int_0^1 \frac{dy}{1-y} \int_y^1 Pdx - \int_0^1 \frac{dy}{y} \int_0^y Pdx.$$

By interchanging the order of the integrations we obtain

$$S_n = \int_0^1 Pdx \int_0^x \frac{dy}{1-y} - \int_0^1 Pdx \int_x^1 \frac{dy}{y}$$

or finally

$$S_n = \int_0^1 Plg \frac{x}{1-x} dx \left(P = \frac{x(1-x^{n-1})}{1-x}\right) \quad . \quad . \quad . \quad . \quad (2)$$

3. Reducing the third sum to the second we get in the same way

$$S_n = \sum^n \frac{1}{\alpha\beta\gamma} = S_{n-1} + \frac{1}{n} \sum^{n-1} \frac{1}{\alpha\beta}$$

or, introducing the value (2)

$$S_n = S_{n-1} + \frac{1}{n} \int_0^1 \frac{y(1-y^{n-2})}{1-y} lg \frac{y}{1-y} dy$$

which leads to

$$S_n = \int_0^1 \frac{y}{1-y} lg \frac{y}{1-y} dy \left[\frac{1}{3}(1-y) + \frac{1}{4}(1-y^2) + \ldots \frac{1}{n}(1-y^{n-2})\right].$$

Here we have

$$\frac{1}{3} + \frac{1}{4} + \ldots \frac{1}{n} = \int_0^1 \frac{x^2(1-x^{n-2})}{1-x} dx = \int_0^1 Pdx$$

$$\frac{y}{3} + \frac{y^2}{4} + \ldots \frac{y^{n-2}}{n} = \frac{1}{y^2} \int_0^y \frac{x^2(1-x^{n-2})}{1-x} dx = \frac{1}{y^2} \int_0^y Pdx$$

thus

$$S_n = \int_0^1 \frac{y}{1-y} lg \frac{y}{1-y} dy \left[\int_0^1 Pdx - \frac{1}{y^2} \int_0^y Pdx\right]$$

or

$$S_n = \int_0^1 \frac{y}{1-y}\, lg \frac{y}{1-y}\, dy \int_y^1 P dx - \int_0^1 \frac{1+y}{y}\, lg \frac{y}{1-y}\, dy \int_0^y P dx.$$

By interchanging the order of the integrations we obtain

$$S_n = \int_0^1 P dx \left[\int_0^x \frac{y}{1-y}\, lg \frac{y}{1-y}\, dy - \int_x^1 \frac{1+y}{y}\, lg \frac{y}{1-y}\, dy \right]$$

which may be reduced to

$$S_n = \frac{1}{2} \int_0^1 P dx \left[lg^2 \frac{x}{1-x} - \frac{\pi^2}{3} \right] \left(P = \frac{x^2(1-x^{n-2})}{1-x} \right) \quad . \quad . \quad (3)$$

4. In the same way the following results will be found

$$\sum^n \frac{1}{\alpha\beta\gamma\delta} = \frac{1}{3!} \int_0^1 P dx \left[lg^3 \frac{x}{1-x} - \pi^2\, lg \frac{x}{1-x} \right] \left(P = \frac{x^3(1-x^{n-3})}{1-x} \right) \quad (4)$$

$$\sum^n \frac{1}{\alpha\beta\gamma\delta\varepsilon} = \frac{1}{4!} \int_0^1 P dx \left[lg^4 \frac{x}{1-x} - 2\pi^2\, lg^2 \frac{x}{1-x} + \frac{\pi^4}{5} \right] \left(P = \frac{x^4(1-x^{n-4})}{1-x} \right) \quad (5)$$

from which we conclude by induction

$$\sum^n \frac{1}{\alpha\beta\,..\,\lambda} = \frac{1}{p!} \int_0^1 \frac{x^p(1-x^{n-p})}{1-x} \left[lg^p - \frac{p\,.\,p-1}{3!}\, \pi^2\, lg^{p-2} + \right.$$

$$\left. + \frac{p\,.\,p-1\,.\,p-2\,.\,p-3}{5!}\, \pi^4\, lg^{p-4} \ldots \right]_{dx}$$

assuming that the number of the different letters $\alpha\,\beta\,..\,\gamma$ is $p+1$, and omitting the argument $\frac{x}{1-x}$ of the logarithms.

The resulting equation is therefore, if $n > p$

$$C_{n+1}^{n-p-1} = \frac{n!}{p!} \int_0^1 \frac{x^p(1-x^{n-p})}{1-x} \left[lg^p - \frac{p\,.\,p-1}{3!}\, \pi^2\, lg^{p-2} + \,.\,. \right] dx \quad (6)$$

where the last term between the parentheses is

$(-1)^{\frac{p}{2}} \frac{\pi^p}{p+1}$ if p is even, and $(-1)^{\frac{p-1}{2}} \pi^{p-1} lg$ if p is odd.

5. In order to prove exactly the preceding result (6) we want some auxiliary theorems, which we now will examine.

Theorem 1.

$$\int_0^1 \left[lg^p - \frac{p \,.\, p-1}{3!} \pi^2 \, lg^{p-2} + \frac{p \,.\, p-1 \,.\, p-2 \,.\, p-3}{5!} \pi^4 \, lg^{p-4} - \,.\,. \right] dx = 0 \quad (7)$$

Let $\frac{x}{1-x} = e^{-y}$, then

$$a_{2m} = \int_0^1 lg^{2m} \frac{x}{1-x} \, dx = \int_{-\infty}^{\infty} \frac{y^{2m} e^y}{(e^y+1)^2} \, dy = (2^{2m}-2) \, B_m \, \pi^{2m}$$

$$a_{2m+1} = 0.$$

Therefore the theorem is proved if p is odd, and also when p is even, for then it reduces to a known relation betweenBernouillian numbers.

Theorem 2.

$$\int_0^1 x^n \left[lg^p - \frac{p \,.\, p-1}{3!} \pi^2 \, lg^{p-2} + \ldots \right] dx = 0 \quad (n < p) \quad . \quad . \quad (8)$$

Integrating by parts, we get

$$\int_0^1 \left[(n+1) \, x^n - (n+2) \, x^{n+1} \right] \left[lg^{p+1} - \frac{p+1 \,.\, p}{3!} \pi^2 \, lg^{p-1} + \,.\,. \right] dx =$$

$$= \left[\left(x^{n+1} - x^{n+2} \right) \left(lg^{p+1} - \frac{p+1 \,.\, p}{3!} \pi^2 \, lg^{p-1} + \,.\,. \right) \right]_0^1$$

$$- (p+1) \int_0^1 x^n \left[lg^p - \frac{p \,.\, p-1}{3!} \pi^2 \, lg^{p-2} + \,.\,. \right] dx$$

or, because $\left[x \, (1-x) \, lg^r \frac{x}{1-x} \right]_0^1 = 0$

$$\int_0^1 \left[(n+1)x^n - (n+2)x^{n+1}\right]\left[lg^{p+1} - \frac{p+1 \cdot p}{3!}\pi^2 lg^{p-1} + \ldots\right] dx =$$

$$= -(p+1)\int_0^1 x^n \left[lg^p - \frac{p \cdot p-1}{3!}\pi^2 lg^{p-2} + \ldots\right] dx.$$

Assuming in the first place $n = 0$, this formula reduces to

$$\int_0^1 x\left[lg^{p+1} - \frac{p+1 \cdot p}{3!}\pi^2 lg^{p-1} + \ldots\right] dx = 0.$$

Putting secondly $n = 1$, the same formula gives

$$\int_0^1 x^2 \left[lg^{p+1} - \frac{p+1 \cdot p}{3!}\pi^2 lg^{p-1} + \ldots\right] dx = 0.$$

Pursuing in the same way, and substituting finally $n = p-1$, we obtain

$$\int_0^1 x^p \left[lg^{p+1} - \frac{p+1 \cdot p}{3!}\pi^2 lg^{p-1} + \ldots\right] dx = 0.$$

which proves the theorem.

Theorem 3.

$$\int_0^1 x^n\left[lg^p - \frac{p \cdot p-1}{3!}\pi^2 lg^{p-2} + \ldots\right] dx =$$

$$= \frac{p}{n+1}\int_0^1 \frac{1-x^n}{1-x}\left[lg^{p-1} - \frac{p-1 \cdot p-2}{3!}\pi^2 lg^{p-3} + \ldots\right] dx \quad (9).$$

Integrating by parts, we have evidently

$$\int_0^1 \left[nx^{n-1} - (n+1)x^n\right] lg_p\, dx = -p\int_0^1 x^{n-1} lg^{p-1}\, dx.$$

Replacing n successively by $n-1, n-2, \ldots 1$ and adding the results, we obtain

$$\int_0^1 x^n lg^p\, dx = \frac{p}{n+1}\int_0^1 \frac{1-x^n}{1-x} lg^{p-1}\, dx + \frac{1}{n+1}\int_0^1 lg^p\, dx. \qquad (10)$$

Therefore also

$$\int_0^1 x^n \, lg^{p-2} \, dx = \frac{p-2}{n+1}\int_0^1 \frac{1-x^n}{1-x} \, lg^{p-3} \, dx + \frac{1}{n+1}\int_0^1 lg^{p-2} \, dx$$

$$\int_0^1 x^n \, lg^{p-4} \, dx = \frac{p-4}{n+1}\int_0^1 \frac{1-x^n}{1-x} \, lg^{p-5} \, dx + \frac{1}{n+1}\int_0^1 lg^{p-4} \, dx \quad \text{etc.}$$

Multiplying these equations respectively with

$$1, -\frac{p \, . \, p-1}{3!} \pi^2, \frac{p \, . \, p-1 \, . \, p-2 \, . \, p-3}{5!} \pi^4, \text{ etc.}$$

and adding, the theorem is proved.

6. Returning to the equation (6) we may now give an exact proof in the following way.

According to (6) we have

$$C_{n+1}^{n-p-1} - n \, C_n^{n-p-2} = \frac{n!}{p!}\int_0^1 x^{n-1} \left[lg^p - \frac{p \, . \, p-1}{3!} \pi^2 \, lg^{p-2} + \, .. \right] dx$$

where the second member by means of Theorem 3 may be written

$$\frac{n-1!}{p-1!}\int_0^1 \frac{1-x^{n-1}}{1-x} \left[lg^{p-1} - \frac{p-1 \, . \, p-2}{3!} \pi^2 \, lg^{p-3} + \, .. \right] dx$$

or, if we subtract from it, as we know from Theorem 2

$$\frac{n-1!}{p-1!}\int_0^1 \frac{1-x^{p-1}}{1-x} \left[lg^{p-1} - \frac{p-1 \, . \, p-2}{3!} \pi^2 \, lg^{p-3} + \, .. \right] dx = 0$$

$$C_{n+1}^{n-p-1} - n \, C_n^{n-p-2} =$$

$$= \frac{n-1!}{p-1!}\int_0^1 \frac{x^{p-1}(1-x^{n-p})}{1-x} \left[lg^{p-1} - \frac{p-1 \, . \, p-2}{3!} \pi^2 \, lg^{p-3} + \, .. \right] dx.$$

Here the second member, as follows from (6) is

$$C_n^{n-p-1}.$$

In this way the equation (6) leads to a known relation between the numbers of STIRLING of the first kind, which proves that this equation is valid generally.

If to (6) we add

$$0 = \frac{n!}{p!} \int_0^1 \frac{1-x^p}{1-x} \left[lg_p - \frac{p \cdot p-1}{3!} \pi^2 \, lg^{p-2} + \ldots \right] dx$$

this equation may also be written

$$C_{n+1}^{n-p-1} = \frac{n!}{p!} \int_0^1 \frac{1-x^n}{1-x} \left[lg_p - \frac{p \cdot p-1}{3!} \pi^2 \, lg^{p-2} + \ldots \right] dx \quad . \quad (11)$$

7. Before considering the STIRLING numbers of the second kind, we wish to show how the integrals we met with in the preceding part, may be determined directly.

First case $n \leqq p$.

Putting successively $n = 1, 2, 3 \ldots$ in the equation

$$(n+1) \int_0^1 x^n \, lg^p \, dx = p \int_0^1 \frac{1-x^n}{1-x} \, lg^{p-1} \, dx + a_p \quad . \quad . \quad . \quad (10)$$

we obtain

$$2! \int_0^1 x \, lg^p \, dx = p a_{p-1} + a_p$$

$$3! \int x^2 \, lg^p \, dx = p \cdot p-1 \; a_{p-2} + 3 \, p a_{p-1} + 2 \, a_p$$

$$4! \int x^3 \, lg^p \, dx = p \cdot p-1 \cdot p-2 \; a_{p-3} + 6 \, p \cdot p-1 \; a_{p-2} + 11 \, p a_{p-1} + 6 \, a_p$$

$$5! \int x^4 \, lg^p \, dx = p \cdot p-1 \cdot p-2 \cdot p-3 \; a_{p-4} + 10 \, p \cdot p-1 \cdot p-2 \; a_{p-3} +$$

$$+ 35 \, p \cdot p-1 \cdot a_{p-2} + 50 \, p \cdot a_{p-1} + 24 \, a_p$$

. . . .

The coefficients in the second members of these equations are easily recognised as the STIRLING numbers

$$C_2^0 \; C_2^1$$
$$C_3^0 \; C_3^1 \; C_3^2$$
$$C_4^0 \; C_4^1 \; C_4^2 \; C_4^3$$
$$C_5^0 \; C_5^1 \; C_5^2 \; C_5^3 \; C_5^4$$

. . . .

Therefore we may expect generally

$$(n+1)!\int_0^1 x^n\, lg^p\, dx = C^0_{n+1}\, p\,.\,p-1\,..\,p-n+1\; a_{p-n}$$
$$+\; C^1_{n+1}\, p\,.\,p-1\,..\,p-n+2\; a_{p-n+1}$$
$$+\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\qquad (12)$$
$$+\; C^{n-1}_{n+1}\, p a_{p-1}$$
$$+\; C^n_{n+1} a_p$$

Assuming the validity of this formula, we may prove that it also holds if n is replaced by $n+1$. For then

$$\int_0^1 \frac{1-x^{n+1}}{1-x}\, lg^{p-1}\, dx = \int_0^1 (1+x+x^2+\ldots x^n)\, lg^{p-1}\, dx =$$

$$= a_{p-1}\left\{1+\frac{C^1_2}{2!}+\frac{C^2_3}{3!}+\ldots\frac{C^n_{n+1}}{n+1!}\right\}$$

$$+\; p-1\,.\,a_{p-2}\left\{\frac{C^0_2}{2!}+\frac{C^1_3}{3!}+\ldots\frac{C^{n-1}_{n+1}}{n+1!}\right\}$$

$$+\; p-1\,.\,p-2\,.\,a_{p-3}\left\{\frac{C^0_3}{3!}+\frac{C^1_4}{4!}+\ldots\frac{C^{n-2}_{n+1}}{n+1!}\right\}$$

$$+\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.$$

$$+\; p-1\,.\,p-2\,..\,p-n+1\; a_{p-n}\left\{\frac{C^0_n}{n!}+\frac{C^1_{n+1}}{n+1!}\right\}$$

$$+\; p-1\,.\,p-2\,..\,p-n\,.\,a_{p-n-1}\,\frac{C^0_{n+1}}{n+1!}$$

$$=\frac{1}{n+1!}\Big[C^n_{n+2}\, a_{p-1} + C^{n-1}_{n+2}\, p-1\,.\,a_{p-2}+\ldots$$

$$+\; C^1_{n+2}\, p-1\,..\,p-n+1\; a_{p-n} + C^0_{n+2}\, p-1\,..\,p-n\,.\,a_{p-n-1}\Big]$$

Substituting this result in

$$(n+2)\int_0^1 x^{n+1}\, lg^p\, dx = p\int_0^1 \frac{1-x^{n+1}}{1-x}\, lg^{p+1}\, dx + a_p \quad .\;.\quad (10)$$

we find

$$(n+2)\int_0^1 x^{n+1}\, lg^p\, dx = C^0_{n+2}\, p \,.\, p-1 \,..\, p-n \,.\, a_{p-n-1}$$

$$+ C^1_{n+2}\, p \,.\, p-1 \,..\, p-n+1 a_{p-n}$$

$$+ \;.\;.\;.\;.\;.\;.\;.\;.\;.$$

$$+ C^n_{n+2}\, p \,.\, a_{p-1}$$

$$+ C^{n+1}_{n+2}\, a_p.$$

8. *Second case* $n \geqq p$.

Putting successively $p = 1, 2, 3 \ldots$ in the general formula

$$\int_0^1 x^n\, lg^p \frac{x}{1-x}\, dx = \frac{p}{n+1}\int_0^1 \frac{1-x^n}{1-x}\, lg^{p-1} \frac{x}{1-x}\, dx + \frac{1}{n+1}\, a_p \quad . \quad (10)$$

we find

$$\int_0^1 x^n\, lg\, dx = \frac{1}{n+1}\int_0^1 (1+x+\ldots x^{n-1})\, dx + \frac{1}{n+1}\, a_1$$

or, as $a_1 = 0$

$$\int_0^1 x^n\, lg\, dx = \frac{1}{n+1}\left(1 + \frac{1}{2} + \ldots \frac{1}{n}\right) = \frac{1}{n+1!}\, C^{n-1}_{n+1}.$$

For $p = 2$ we obtain

$$\int_0^1 x^n\, lg^2\, dx = \frac{2}{n+1}\int_0^1 (1+x+\ldots x^{n-1})\, lg\, dx + \frac{1}{n+1}\, a_2$$

$$= \frac{2}{n+1}\left[\frac{1}{2!}\, C^0_2 + \frac{1}{3!}\, C^0_3 + \ldots \frac{1}{n!}\, C^{n-2}_n\right] + \frac{1}{n+1}\, a_2$$

$$= \frac{2}{n+1!}\, C^{n-2}_{n+1} + \frac{C^n_{n+1}}{n+1!}\, a_2$$

If $p = 3$, we have

$$\int_0^1 x^n\, lg^3\, dx = \frac{3}{n+1}\int_0^1 (1+x)\, lg^3\, dx + \frac{3}{n+1}\int_0^1 (x^2+x^3+\ldots x^{n-1})\, lg^3\, dx + + \frac{1}{n+1} a_3$$

where according to (12)

$$\int_0^1 (1+x)\, lg^3\, dx = C_1{}^0 ap + \frac{1}{2!}(C_2{}^0\, pa_{p-1} + C_2{}^1 a_p)$$

and

$$\int_0^1 (x^2+x^3+\ldots x^{n-1})\, lg^3\, dx = 2\left(\frac{C_3{}^0}{3!} + \frac{C_4{}^1}{4!} + \ldots \frac{C_n^{n-3}}{n!}\right) + \left(\frac{C_3{}^2}{3!} + \frac{C_4{}^3}{4!} + \ldots \frac{C_n^{n-1}}{n!}\right) a_2$$

thus

$$\int_0^1 x^n\, lg^3\, dx = \frac{3!}{n+1!}\, C^{n-3}_{n+1} + \frac{3}{n+1!}\, C^{n-1}_{n+1}\, a_2$$

omitting $a_3 = 0$.

In the same way we find

$$\int_0^1 x^n\, lg^4\, dx = \frac{4!}{n+1!}\, C^{n-4}_{n+1} + \frac{4!}{2!n+1!}\, C^{n-2}_{n+1}\, a_2 + \frac{4!}{4!n+1!}\, C^{n}_{n+1}\, a_4$$

$$\int_0^1 x^n\, lg^5\, dx = \frac{5!}{n+1!}\, C^{n-5}_{n+1} + \frac{5!}{2!n+1!}\, C^{n-3}_{n+1}\, a_2 + \frac{5!}{4!n+1!}\, C^{n-1}_{n+1}\, a_4$$

and by induction

$$\frac{n+1!}{p!}\int_0^1 x^n\, lg^p\, dx = C^{n-p}_{n-1} + \frac{1}{2!}\, C^{n-p+2}_{n+1}\, a_2 + \frac{1}{4!}\, C^{n-p+4}_{n+1}\, a_4 + \ldots \quad (13)$$

where the last term is $\frac{1}{p!}\, C^n_{n+1}\, a_p$ if p is even, and $\frac{1}{p-1!}\, C^{n-1}_{n+1}\, a_{p-1}$ when p is odd.

If we substitute this result in (10) remarking that

$$C^r_{n+1} + (n+1)\, C^{r-1}_{n+1} = C^r_{n+2}$$

we easily find that this equation is satisfied, which proves that (13) is true generally.

9. The numbers of STIRLING of the second kind $\mathfrak{S}^p_{n+1}$, which are determined by the expansion

$$\frac{1}{x(x+1)\ldots(x+n)} = \frac{\mathfrak{S}^0_{n+1}}{x^{n+1}} - \frac{\mathfrak{S}^1_{n+1}}{x^{n+2}} + \frac{\mathfrak{S}^2_{n+1}}{x^{n+3}} - \ldots$$

may also be written in the form of definite integrals.

Starting from the higher difference of a power of the variable x

$$\Delta^n x^p = \sum_{s=0}^{n} (-1)^s \binom{n}{s} (x+n-s)^p$$

and integrating both members between the limits 0 and 1, we get

$$\int_0^1 \Delta^n x^p \, dx = \frac{1}{p+1} \sum_{s=0}^{n} (-1)^s \binom{n}{s} \left[(n+1-s)^{p+1} - (n-s)^{p+1} \right]$$

$$= \frac{1}{p+1} \sum_{s=0}^{n+1} (-1)^s \binom{n+1}{s} (n+1-s)^{p+1}.$$

Comparing this result with the known formula

$$\mathfrak{S}^{p-n}_{n+2} = \frac{1}{n+1!} \sum_{s=0}^{n+1} (-1)^s \binom{n+1}{s} (n+1-s)^{p+1}$$

we obtain

$$\mathfrak{S}^{p-n}_{n+2} = \frac{p+1}{n+1!} \int_0^1 \Delta^n x^p \, dx.$$

Now $\Delta^n x^p$ has been expressed in the form of a definite integral by LAPLACE (Oeuvres 7 p. 518) in the following way.

By partial integration

$$\int_{-\infty}^{\infty} \frac{e^{-ixs}\, ds}{(1-is)^{p+1}} = \frac{1}{ip} \left[\left(\frac{e^{-ixs}}{(1-is)^p} \right)_{-\infty}^{\infty} + ix \int_{-\infty}^{\infty} \frac{e^{-ixs}}{(1-is)^p}\, ds \right]$$

or, because

$$mod. \left| \frac{e^{-ixs}}{(1-is)^p} \right| = \frac{1}{(1+s^2)^{\frac{p}{2}}}$$

$$\int_{-\infty}^{\infty} \frac{e^{-ixs}\, ds}{(1-is)^{p+1}} = \frac{x}{p} \int_{-\infty}^{\infty} \frac{e^{-ixs}\, ds}{(1-is)^p},$$

Thus also

$$\int_{-\infty}^{\infty} \frac{e^{-ixs}\, ds}{(1-is)^p} = \frac{x}{p-1} \int_{-\infty}^{\infty} \frac{e^{-ixs}\, ds}{(1-is)^{p-1}}$$

. . . .

$$\int_{-\infty}^{\infty} \frac{e^{-ixs}\, ds}{(1-is)^2} = \frac{x}{1} \int_{-\infty}^{\infty} \frac{e^{-ixs}}{1-is}\, ds$$

and by multiplying all these equations

$$\int_{-\infty}^{\infty} \frac{e^{-ixs}\, ds}{(1-is)^{p+1}} = \frac{x^p}{p!} \int_{-\infty}^{\infty} \frac{e^{-ixs}\, ds}{1-is}.$$

Now the last integral has the value

$$\int_{-\infty}^{\infty} \frac{(\cos xs - i \sin xs)(1+is)}{1+s^2}\, ds = 2 \int_{-\infty}^{\infty} \frac{\cos xs + s \sin xs}{1+s^2}\, ds = 2\, \pi e^{-x}$$

therefore

$$\int_{-\infty}^{\infty} \frac{e^{x(1-is)}}{(1-is)^{p+1}}\, ds = 2\pi \frac{x^p}{p!}$$

and

$$x^p = \frac{p!}{2\pi} \int_{-\infty}^{\infty} \frac{e^{x(1-is)}\, ds}{(1-is)^{p+1}}.$$

Hence

$$\Delta^n x^p\, dx = \frac{p!}{2\pi} \int_{-\infty}^{\infty} \frac{\Delta^n e^{x(1-is)}}{(1-is)^{p+1}}\, ds = \frac{p!}{2\pi} \int_{-\infty}^{\infty} \frac{e^{x(1-is)}\, [e^{1-is} - 1]^n\, ds}{(1-is)^{p+1}}$$

and, integrating between 0 and 1

$$\int_0^1 \Delta^n x^p \, dx = \frac{p!}{2\pi} \int_{-\infty}^{\infty} \frac{(e^{1-is} - 1)^{n+1}}{(1-is)^{p+2}} \, ds$$

thus

$$\mathfrak{C}^{p-n}_{n+2} = \frac{p+1!}{2\pi \, (n+1)!} \int_{-\infty}^{\infty} \frac{(e^{1-is} - 1)^{n+1} \, ds}{(1-is)^{p+2}}$$

or

$$\mathfrak{C}^{p}_{n+1} = \frac{p+n!}{2\pi \, . \, n!} \int_{-\infty}^{\infty} \frac{(e^{1-is} - 1)^{n} \, ds}{(1-is)^{p+n+1}}.$$

Finally we remark that the function

$$\mathfrak{C}^{p}_{n+1}(\alpha) = \frac{1}{n!} \sum_{s=0}^{n} (-1)^s \binom{n}{s} (\alpha + n - s)^{n+p}$$

may also be expressed by the definite integral

$$\frac{p+n!}{2\pi \, . \, n!} \int_{-\infty}^{\infty} \frac{(e^{1-is} - 1)^{n} \, e^{(1-is)\alpha}}{(1-is)^{p+n+1}} \, ds.$$