Geophysics. — *"On the flattening and the constitution of the earth".* By Prof. W. DE SITTER.

(Communicated at the meeting of March 29, 1924).

1. The outer potential of any body, which is symmetrical with reference to an equatorial plane and an axis perpendicular to this plane, may be developed in a series, which, to the order of accuracy here required is

$$V = \frac{fM_1}{r}\left[1 - \frac{2}{3}\frac{Jb^2}{r^2}P_2(\sin\delta) + \frac{4}{15}\frac{Kb^4}{r^4}P_4(\sin\delta)\right], \quad . \quad . \quad (1)$$

M_1 being the total mass, f the constant of gravitation, and b the equatorial radius. P_2 and P_4 are spherical harmonics of the declination δ above the equatorial plane and J and K are constants characteristic of the body. We have

$$J = \frac{3}{2}\frac{C-A}{M_1 b^2},$$

C en A being the moments of inertia with respect to the polar and an equatorial diameter respectively. If we introduce the ratio

$$H = \frac{C-A}{C},$$

we have

$$J = q\,H \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

with

$$q = \frac{3}{2}\frac{C}{M_1 b^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the surface of the body is an equipotential surface, its equation may be written as that of a spheroid

$$r = b\left[1 - \varepsilon\sin^2\varphi' - \left(\frac{3}{8}\varepsilon^2 + \varkappa\right)\sin^2 2\varphi'\right], \quad . \quad . \quad . \quad (4)$$

φ' being the geocentric latitude and ε the flattening.

This spheroid deviates from an ellipsoid of rotation by a depression $-b\varkappa\sin^2 2\varphi'$ reaching its maximum at the latitude 45°.

If the body rotates with a constant angular velocity ω, so that at its surface the potential is (V_1 being the value of V for the surface, $\varphi' = \delta$):

$$V_1 + \frac{1}{2} r^2 \omega^2 \cos^2 \varphi',$$

then the conditions that the surface shall be an equipotential surface are

$$\varepsilon - J = \frac{1}{2} \varrho_1 + \frac{1}{2} \varepsilon^2 - \frac{1}{7} \varepsilon \varrho - \frac{4}{7} \varkappa = \cdot 0017287. \quad . \quad . \quad (5)$$

$$K = \frac{24}{7} \varkappa + 3 \varepsilon^2 - \frac{15}{7} \varepsilon \varrho = \cdot 0000109,$$

where

$$\varrho_1 = \frac{\omega^2 r_1^3}{f M_1}$$

is determined by

$$\varrho_1 + \frac{2}{3} \varrho_1^2 = \frac{\omega^2 r_1}{g_1},$$

r_1 being the mean radius, and g_1 the acceleration of gravity at the latitude of this mean radius. This gives

$$\varrho_1 = \cdot 00344992 \pm \cdot 00000002.$$

In the small terms in (5) and (6) I have omitted the index of ϱ_1, and I have adopted

$$\varepsilon = {}^1/296{\cdot}5 = \cdot 003373$$
$$\varkappa = \cdot 00000050.$$

The value of $\varkappa$ is, of course, entirely unknown. If the earth were homogeneous, it would be zero, on the other hand it cannot exceed $\frac{5}{16} \varepsilon \varrho - \frac{1}{4} \varepsilon^2 = \cdot 00000080$, so that the adopted value appears plausible. It corresponds to 3.2 meters, and is thus entirely irrelevant.

The equation (5) is independent of any assumption regarding the distribution of mass inside the earth. For q on the other hand we have no rigorous equation of this kind, but the theory of CLAIRAUT on the constitution of the earth enables us to derive a very approximate value of q.

If we wish to go beyond an accuracy of the order of a unit in the denominator of ε, corresponding to the fifth decimal place in ε itself, it is necessary to include the second order. On the other hand, if the third order is neglected, all figures beyond the seventh decimal are meaningless. The theory of CLAIRAUT has been developed to the second order by DARWIN[1]), and others. By taking as independent variable the *mean*, instead of the equatorial, radius, the formulas become somewhat simpler, and at the same time the range of uncertainty of q is considerably narrowed. As this change of independent variable does not affect the essential parts of the theory, which are well known, I will only state the principal steps and formulas very succinctly, without going into the details of their derivation. As we require only one term beyond the one of the lowest order in any equation, we can choose at random any one of the several definitions of the mean radius, which are equivalent to the first order. We may suppose it to be the radius of the sphere of equal volume.

2. On the theory of CLAIRAUT the surfaces of equal density are equipotential surfaces. Let β be the mean radius of any such surface, expressed in that of the outer surface as unit, then the equation of this surface becomes

$$r = \beta\left[1 - \frac{2}{3}\left(\varepsilon' + \frac{2}{3}\varepsilon^2\right)P_2(\sin\varphi') + \ldots\right], \quad . \quad . \quad (7)^2)$$

P_2 being again the spherical harmonic. The harmonic of the fourth order is not needed for our immediate purpose, and has been omitted from the formulas. We have put

$$\varepsilon' = \varepsilon - \frac{5}{42}\varepsilon^2 + \frac{4}{7}\varkappa \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

In the terms of the second order it is not necessary to distinguish between ε and ε', and the accent is dropped.

The potential V at any point within the earth, of which the coordinates are r and φ', is given by

$$\frac{V}{fW}\cdot r = D\left[1 + \frac{1}{2}\varrho\frac{r^3}{\beta^3}\cos^2\varphi'\right] - \frac{2}{5}\left[S\frac{\beta^2}{r^2} + T\frac{r^3}{\beta^3}\right]P_2\sin\varphi' + \ldots \quad (9)$$

where

$$\varrho = \frac{\omega^2\beta^3}{fM} = \frac{\varrho_1}{D},$$

W being the volume, and M the mass within the surface of which the mean radius in β, and

[1]) Scientific papers III, p. 78 = M.N. 60, p. 82 (1900).

[2]) The numbering of the formulas is the same as in the somewhat more extended publication in B. A. N. 55, which explains why some numbers are missing here.

$$D = \frac{3}{\beta^3} \int_0^\beta \delta \beta^2 d\beta,$$

$$S = \frac{1}{\beta^5} \int_0^\beta \delta \frac{d}{d\beta} \left[\beta^5 \left(\varepsilon' + \frac{2}{7} \varepsilon^2 \right) \right] d\beta,$$

$$T = \int_\beta^1 \delta \frac{d}{d\beta} \left[\varepsilon' + \frac{16}{21} \varepsilon^2 \right] d\beta.$$

In these formulas δ is the density, expressed in the mean density as a unit, and consequently D is the mean density within the surface β, expressed in the same unit [1]). For the outer surface we have of course

$$D_1 = 1 \quad , \quad S_1 = \frac{5}{3} J \quad , \quad T_1 = 0.$$

The comparison of (7) and (9) gives for the condition that (7) shall be an equipotential surface:

$$D \left(\varepsilon' + \frac{2}{7} \varepsilon^2 - \frac{1}{2} \varrho \right) - \frac{3}{5} (S + T) = \frac{4}{21} \varepsilon (\varrho_1 - 3T). \quad . \quad (10)$$

The right hand member of (10), which is of the second order, has been simplified by means of the left hand member equated to zero.

Putting further

$$\zeta = -\frac{\beta}{D} \frac{dD}{d\beta} = -\frac{\beta}{\varrho} \frac{d\varrho}{d\beta},$$

we have

$$\zeta = 3 \left(1 - \frac{\delta}{D} \right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

This equation is now *rigorous* in consequence of the introduction of the mean, instead of the equatorial radius as argument.

Differentiating (10), and putting

$$\eta' = \frac{\beta}{\varepsilon'} \frac{d\varepsilon'}{d\beta},$$

we find, after reduction

$$\eta' \left[D \left(1 + \frac{4}{7} \varepsilon \right) - \frac{4}{21} \varrho_1 + \frac{4}{7} T \right] = 3 D \left(1 + \frac{2}{7} \varepsilon \right) - 3 \frac{S}{\varepsilon'}. \quad (13)$$

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[1]) It should be remarked that, adopting these units, we have $M_1 = \frac{4}{3} \pi r_1^3 D_1 = \frac{4}{3} \pi$.

and differentiating again, we find the differential equation for η:

$$\beta \frac{d\eta'}{d\beta} + \eta'^2 + 5\,\eta' - 2\,\zeta\,(1+\eta') - \frac{4}{21}\zeta\, Q = 0 \quad . \quad . \quad . \quad (14)$$

where

$$Q = 7\,\varrho\,(1+\eta') - 3\,\varepsilon\,(1+\eta')^2 - 4\,\varepsilon. \quad . \quad . \quad . \quad . \quad (15)$$

Omitting the term with Q (14) is the well known differential equation correct to the first order. The introduction of ε' for ε has removed from (14) a term of the form $\varepsilon\eta'^2$, thus having all small terms multiplied by ζ. DARWIN, using the equatorial instead of the mean radius, finds the same equation, but in his value of Q the numerical coefficients of the second and third term are larger than here.

From (14) we derive, as was first done by RADAU:

$$D\beta^5 \sqrt{1+\eta'} = 5\int_0^\beta D\beta^4\, F(\eta')\, d\beta, \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where

$$F(\eta') = \frac{1 + \frac{1}{2}\eta' - \frac{1}{10}\eta'^2 + \frac{2}{105}\zeta\, Q}{\sqrt{1+\eta'}} \quad . \quad . \quad . \quad . \quad (17)$$

The value of $F(\eta')$ is always very near to unity. If by $1+\lambda$ we denote a certain average value of it over the range of integration, we have

$$\int_0^\beta D\beta^4\, d\beta = \frac{D\beta^5\sqrt{1+\eta'}}{5\,(1+\lambda)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Now we have, to the required order of accuracy

$$C = \frac{8}{15}\pi\int_0^1 \delta\frac{d}{d\beta}\left[\beta^5\left(1+\frac{2}{3}\varepsilon\right)\right]d\beta = \frac{8}{3}\pi\int_0^1 \delta\beta^4 d\beta + \frac{2}{3}(C-A),$$

and since in our units $M_1 b^2 = \frac{4}{3}\pi\left(1+\frac{2}{3}\right)$, we have by (3)

$$q = 3\left(1-\frac{2}{3}\varepsilon\right)\int_0^1 \delta\beta^4\, d\beta + \frac{2}{3}J. \quad . \quad . \quad . \quad . \quad (19)$$

Replacing in the integral δ by its value from (12), then integrating by parts, and using (5), of which only the terms of the lowest order are here needed, we find

$$q = 1 - \frac{1}{3}\varrho_1 - 2\left(1 - \frac{2}{3}\varepsilon\right)\int_0^1 D\beta^4\, d\beta, \quad . \quad . \quad . \quad . \quad (20)$$

or by (18)

$$q = 1 - \frac{1}{3}\varrho_1 - \frac{2}{5}\left(1 - \frac{2}{3}\varepsilon\right)\frac{\sqrt{1+\eta'_1}}{1+\lambda_1}. \quad . \quad . \quad . \quad (21)$$

From the equation (13) we find for the outer surface

$$\varepsilon'\eta'_1 = \frac{5}{2}\varrho_1 - 2\varepsilon' + \frac{10}{21}\varrho^2 + \frac{4}{7}\varepsilon^2 - \frac{6}{7}\varepsilon\varrho, \quad . \quad . \quad . \quad (22)$$

from which, with $\varepsilon = {}^1/_{296\cdot 50} = \cdot 0033727$, corresponding to $\varepsilon' = {}^1/_{296\cdot 60} + \Delta\varepsilon^{-1}$ we find

$$\eta'_1 = \cdot 55893 + \cdot 00863\, \Delta\, \varepsilon^{-1}.$$

Therefore

$$q = \cdot 50053 - \cdot 00140\, \Delta\, \varepsilon^{-1} + [\cdot 4983 + \cdot 0014\, \Delta\, \varepsilon^{-1}]\frac{\lambda_1}{1+\lambda_1} \quad . \quad (23)$$

3. To derive a probable value for λ_1, and to ascertain the limits within which this value may be considered as trustworthy, we must discuss the function $F(\eta')$ given by (17). The first part of this function is the same as discussed by RADAU, POINCARÉ and others, viz.:

$$F_1(\eta) = \frac{1 + \frac{1}{2}\eta - \frac{1}{10}\eta^2}{\sqrt{1+\eta}}.$$

For convenience I drop the accent, which is of no importance here. The function $F_1(\eta)$—1 is zero for $\eta = 0$, rises to a maximum of $+ .00074$ for $\eta = \frac{1}{3}$, and then decreases again, becoming zero for $\eta = \cdot 528\, (= 5 - 2\sqrt{5})$. For the surface $\eta_1 = \cdot 56$ it is $-.00029$. It is thus positive for practically the whole range, and is larger than $+ .00050$ from $\eta = \cdot 19$ to $\eta = \cdot 44$. Since η increases continually from zero at the centre to $\eta_1 = \cdot 56$ at the surface, we can take as a plausible value

$$F_1(\eta) = 1\cdot 00050 \pm \cdot 00015 \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

The "probable error" attached to this value is meant to express the belief that there is an even chance that the true average value of $F_1(\eta)$ is included within a range of .00030, the adopted value 1.00050 being somewhere inside this range, and probably not far from the middle.

To discuss the value of Q we put

$$\varepsilon = \varrho(1-\nu) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

We then have

$$Q = Q_1 + Q_2,$$

$$Q_1 = (\eta - 3\eta^2)\,\varrho,$$

$$Q_2 = (7 + 6\eta + \eta^2)\,\varrho\nu.$$

Consider first the first part Q_1. This varies between the limits $+\cdot 08\,\varrho$, for $\eta = \frac{\cdot 1}{6}$ and $-\cdot 36\,\varrho$ at the surface. At the centre it is zero. The value of ϱ is always smaller than ϱ_1, $\sqrt{1+\eta}$ is always larger than 1, and it is probable that ζ never considerably exceeds its surface value $\zeta_1 = 1.5$. We can thus take

$$\frac{2}{105}\,\frac{\zeta\varrho}{\sqrt{1+\eta}} < \frac{2}{105}\,\zeta_1\,\varrho_1 = \cdot 00010.$$

Consequently the value of $\dfrac{2}{105}\,\dfrac{\zeta Q_1}{\sqrt{1+\eta}}$ is certainly comprised between the limits $+.000008$ and $-.000036$. We can take as a plausible average

$$F_2(\eta) = -\cdot 000014 \pm .000010 \quad . \quad . \quad . \quad . \quad . \quad (26)$$

To discuss the function Q_2 we must consider the possible values of ν. Differentiating (25) we find

$$\frac{\beta}{1-\nu}\,\frac{d\nu}{d\beta} = \zeta - \eta,$$

from which, since ζ is always larger than η, it follows that ν increases continuously from the centre outwards. For the earth the surface value is $\nu_1 = +\cdot 02$. For ν_0 we find upper and lower limits from the formula (42) given by TISSERAND, *Méc. Cél.* II, p. 227, which depend on the density at the centre σ_0. These are:

ρ_0	Limits of $\frac{\nu_0 \rho_0}{\rho_1}$	
1.28	—.25	—.25
1.5	—.26	—.22
2.0	—.33	—.22
2.5	—.39	—.21
3.0	—.42	—.21

We can thus take with certainty

$$-\cdot 40 < \frac{\nu_0\, \varrho_0}{\varrho_1} < -\cdot 21. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (27)$$

Now put for abbreviation

$$\frac{Q_2}{\sqrt{1+\eta}} = \omega\, \varrho_1.$$

For the surface we have

$$\omega_1 = +\cdot 17,$$

and for the centre, where $\eta = 0$, $Q_2 = 7\nu_0\varrho_0$, we have

$$-2\cdot 8 < \omega_0 < -1\cdot 5,$$

Consequently the product $\zeta\omega$, as we proceed from the surface to the centre, will with some rough approximation follow the diagonal of the following table, starting from the left hand top corner, and reaching the bottom somewhere between the last two colums.

Values of the product $\zeta . \omega$

ζ \ ω	.17	0	—.5	—1.0	—1.5	—2.8
1.5	+.25	0	—.75			
1.25	+.21	0	—.62	—1.25		
1.00	+.17	0	—.50	—1.00	—1.50	
.75		0	—.38	— .75	—1.12	—2.10
.50			—.25	— .50	—0.75	—1.40
.25				— .25	—0.38	— .70
.00					.00	.00

It thus appears probable that the average value of this product over the range of integration is included between comparatively narrow limits, say between —.20 and —1.20. We can take as a plausible value

$$\overline{\zeta\omega} = -\cdot 70 \pm \cdot 30 ,$$

from which

$$F_2(\eta) = \frac{2}{105} \cdot \overline{\zeta\omega} \cdot \varrho_1 = -\cdot 000046 \pm \cdot 000020 \quad . \quad . \quad (28)$$

Adding together the three partial values (24), (26) and (28), we find

$$\lambda_1 = + \cdot 00044 \pm \cdot 00015 \; . \; . \; . \; . \; . \; . \quad (29)$$

It will be seen that the effect of the second order term with Q is only about one tenth of the whole.

Substituting this value of λ_1 in (23) we find

$$q = \cdot 50075 \pm \cdot 00008 - \cdot 00140 \, \Delta\varepsilon^{-1} + \cdot 499 \, \Delta\lambda_1 \; . \; . \quad (30)$$

4. We must now consider the question whether these results, which have been derived from the theory of CLAIRAUT, are applicable to the actual earth.

The actual surface of the earth is, of course, neither a surface of equal density nor an equipotential surface. We can however safely assume that up to a certain distance from the centre the material out of which the earth is made up is — so far as secular forces are concerned — in hydrostatic equilibrium, and consequently satisfies the conditions of the theory of CLAIRAUT. The last surface for which this theory is applicable is called the *isostatic* surface, and will be denoted by S_0. Above this there exist, of course, further equipotential surfaces, but these are not as a rule surfaces of equal density, and not necessarily spheroids. The actual land surface is, of course, not such a surface, but the undisturbed surfaces of the different oceans can be assumed to form parts of one and the same equipotential surface, which is called the *geoid*. This geoid is determined from geodetic measures on the continents, and from determinations of the intensity of gravity on the continents and on the oceans. It is found that it deviates only very little from an ellipsoid of revolution. The ellipsoid of revolution, or rather the spheroid, best fitting the geoid is called the *normal* surface, and is denoted by S. The differences between the geoid and the normal surface never amount to more than a few tens of meters. This fact has led to the well known theory of isostasy, which asserts that within any cylinder erected over a (not too small) surface element ω of the isostatic surface there is the same mass as there would be with a certain ideal distribution, which we can take to be in accordance with the theory of CLAIRAUT. The upper surface that would result if the conditions of this theory were satisfied throughout will be called the *ideal* surface, and will be denoted by S_1.

To this surface S_1 the equations (5) and (2), with the value (21) of q, are applicable. The normal surface S on the other hand is not an equipotential surface, but it is the spheroid best fitting the geoid, which is an equipotential surface. For the condition of the "best fitting" we can take that in the developments of both surfaces in series of spherical harmonics the coefficients of the harmonics of the orders zero, two and four are the same. Then the equation (5) is applicable if for ε we take the compression of the normal surface, and for J its actual value for the real earth. We can, of course, again write down the equation (2), taking for H also its actual value, but now q is determined by (3) and not by (21) and the problem before us is to find the difference between these two values of q.

If the earth were entirely constituted according to the theory of CLAIRAUT, it would be covered by an ocean of an average depth of about 2.4 km., of which the upper surface would be the ideal surface S_1, and the bottom would also be an equipotential surface, which we will call S_b. The true distribution of mass differs from this ideal one on the one hand by an excess of mass in the continents and the shallow seas, and a defect in the deep oceans, and on the other hand by the isostatic compensations of these excesses and defects.

This compensation is assumed to be equally distributed over the layer between the surfaces S_1 and S_b or, in the case of the deep oceans, between S_1 and the bottom of the ocean.

The formulas have been worked out by me in 1915 [1]).

Correcting a mistake in the formulas, and treating the layer between S_1 and S_b somewhat more carefully than was done there, we find in units of the seventh decimal place: (see table p. 243).

If there were no isostatic compensation, these numbers would be increased about 55 times. The approximation of the computations is such that each of the partial numbers is correct to a few percents of its amount. The sums may thus easily be a unit or more in error.

We have thus as the result of this computation that the difference

[1]) These Proceedings, XVII, p. 1295.

$C—C_1$ is entirely negligible, and consequently

$$q = q_1$$

Consequently we can use the value (21) or (30) of q to derive

	Parts of the world	$H—H_1$	$\frac{C—C_1}{C}$
1.	North Polar regions	+1.7	+ .2
2.	Europe	— .1	+ 2.4
3.	Asia	+ .8	+14.5
4.	North-America	—1.0	+ 5.2
5.	Northern Atlantic Ocean	—1.2	— 4.3
6.	South-America	+2.3	+ 9.4
7.	Southern Atlantic Ocean	+1.2	—10.7
8.	Africa	+2.9	+14.3
9.	Indian Ocean	+1.2	— 9.2
10.	Indian Archipelago	— .7	+ 2.0
11.	Australia and New Guinea	+ .5	+ 3.7
12.	Pacific Ocean	—9.5	—31.5
13.	South Polar regions	—6.6	+ .6
	Total	—8.5	— 3.4

ε from H for the actual earth. The difference $q—q_1$ would still be negligible, if there were no isostatic compensation.

For H we find:

$$H - H_1 = - \cdot 0000008,$$

from which

$$\varepsilon - \varepsilon_1 - \cdot 0000004 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

This of course is entirely negligible. It means that the polar semi-diameter of the ideal surface is 1.8 meters shorter, and the equatorial radius 0.9 meters longer, than of the normal surface. If there were no isostatic compensation however, the difference between ε and ε_1 would be of the order of two units in the denominator.

5. We have from (5) and (2)

$$\varepsilon = \cdot 0017287 + q\,H \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

The value of H can be derived with great accuracy from the constant of precession. Adopting for the reciprocal of the mass of the moon

$$\mu^{-1} = 81 \cdot 50 \pm \cdot 07 + \Delta \mu^{-1},$$

I find

$$H = \cdot 0032774 + \cdot 0000270 \, \Delta \mu^{-1}.$$

The probable error of H is made up of ± 19 in the seventh decimal place due to the uncertainty of μ, and ± 0.6 due to the constant of precession. Since it has been shown that the value of q derived from the theory of CLAIRAUT may be used for the actual earth, we can substitute (32) into (31). Then taking

$$\varepsilon = \cdot 0033727 + \Delta \varepsilon,$$

we find [1]):

$$597 \, \Delta \varepsilon = - \cdot 0000029 + \cdot 0000135 \, \Delta \mu^{-1} + \cdot 00163 \, \Delta \lambda_1 \,. \quad (33)$$

from which

$$\frac{1}{\varepsilon} = 296{\cdot}92 \pm {\cdot}136 - 1{\cdot}99 \, \Delta\mu^{-1} - 152 \, \Delta \lambda_1 . \quad . \quad . \quad (34)$$

The probable error of $1/\varepsilon$ is made up as follows:

from the precessional constant	$\pm \cdot 004$,
from μ	$\pm \cdot 132$,
from λ_1	$\pm \cdot 035$.

The remaining uncertainty of ε is thus due almost entirely to that of the mass of the moon.

The most important and trustworthy determinations of ε by other methods have already been quoted in my paper of 1915. Shortly after the publication of that paper HELMERT [2]) has published a new determination of ε from the intensity of gravity, which is

$$\varepsilon^{-1} = 296{\cdot}7 \pm 0{\cdot}6$$

The most reliable geodetic determination is that by HAYFORD:

$$\varepsilon^{-1} = 297{\cdot}0 \pm 1{\cdot}2$$

Both agree with (34), but both are very much less accurate. When by the Eros campaign of 1930 the mass of the moon will be better known than it is now, the determination of the compression from the precessional constant will become still more accurate.

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[1]) In my paper of 1915 the equation for $\Delta \varepsilon$ was not derived independently, but adopted from DARWIN. There is however exactly at this point a numerical mistake in DARWIN's work, in consequence of which the value of ε derived from his formula is erroneous. Moreover DARWIN's computations are based on ROCHE's hypothesis, which gives an incorrect value for q. The formula (33) in the text must thus be used.

[2]) Neue Formeln für den Verlauf der Schwerkraft im Meeresniveau beim Festlande, Sitzungsberichte Berlin, 1915, p. 676.

6. The actual distribution of density within the earth is unknown. We can make hypotheses regarding that distribution, and by their aid compute the different integrals occurring in the theory of CLAIRAUT, and thus arrive at values of the different quantities, which can be determined by observation, such as ε, J, H and the surface density δ_1. The equation (5) is independent of the inner constitution. Consequently any hypothesis, which will reproduce any one of the two quantities ε and J will also give the correct value for the other. There are thus three conditions to be satisfied by any hypothesis on the distribution of mass.

ROCHE's hypothesis

$$\delta = \delta_0 (1 - k \beta^2)$$

contains only two constants, and can thus not be expected to satisfy the conditions. This expectation is confirmed by DARWIN's computations, which are based on this hypothesis. If we interpolate in his table (*Scient. Papers,* III, p. 112) for the correct value $\delta_1 = \cdot 495$, we find $\varepsilon^{-1} = 288{\cdot}1$ and $q = \cdot 5132$, which are entirely outside the limits of possibility.

WIECHERT's hypothesis, according to which the earth consists of a core and a crust, each of constant density, separated by a surface of discontinuity at which the density changes abruptly, contains three parameters, viz. the two densities and the radius of the surface of discontinuity. It is thus theoretically possible by this hypothesis to satisfy the three conditions, but it remains to be seen whether the values of the parameters, by which this is effected, are otherwise acceptable.

In order to test this and other hypotheses we must, as has been already said, compute the different integrals occurring in the theory of CLAIRAUT. As this theory is only applicable below the isostatic surface, we must take the radius of this surface as upper limit of the integrals. For the layers above it we can however replace the actual distribution of mass by an ideal one according to the theory of CLAIRAUT. The parts contributed to the integrals by the masses above the isostatic surface are then the integrals from S_0 to S_1. These have been computed taking for the density between S_1 and S_b: $\delta' = .186$ and below S_b: $\delta_1 = .495$, increasing regularly to δ_s at S_0. For the mean radii of S_s and S_b, which are called s and b, I take $s = .98200$, $b = .99962$. The three conditions can then be enounced as follows

$$D_s = \frac{3}{s^3} \int_0^s \delta \beta^2 \, d\beta = 1{\cdot}04214 - {\cdot}02724\, \delta_s, \quad . \quad . \quad . \quad (35)$$

$$\frac{1}{s^5\,\varepsilon''_s}\int_0^s \delta \frac{d}{d\beta}(\beta^5\,\varepsilon'')\,d\beta = \cdot 87189 - \cdot 05008\,\delta_s\,, \quad . \quad . \quad (36)$$

where we have put

$$\varepsilon'' = \varepsilon' + \frac{2}{7}\,\varepsilon^2$$

and for the isostatic surface we take the value corresponding to the adopted value of ε [1]), viz.:

$$\varepsilon''_s = {}^1/_{299\cdot 62},$$

finally

$$5\int_0^s D\beta^4\,d\beta = 1\cdot 16142 + \cdot 00077\,\delta_s - 1\cdot 250\,\frac{\lambda_1}{1+\lambda_1}. \quad . \quad . \quad (37)$$

As an example I have applied these formulas to the theory of WIECHERT. The density above the surface of discontinuity thus is δ_s and below it $\delta_0 = \delta_s + \Delta$. The radius is taken $\beta_2 = ps$. The first two conditions give

$$p^3\,\Delta = 1\cdot 04214 - 1\cdot 02724\,\delta_s\,, \quad . \quad . \quad . \quad . \quad (38)$$

$$p^5\,\Delta\,(1-\xi) = \cdot 87189 - 1\cdot 05008\,\delta_s\,, \quad . \quad . \quad . \quad (39)$$

where we have put

$$\varepsilon''_2 = \varepsilon''_s\,(1-\xi).$$

For the determination of ξ I use the equation (10) for the surface of discontinuity $\beta_2 = ps$. This gives, using (38) and (39),

$$\Delta\,(1-\xi) - \left[\frac{5}{2}\delta_s - \cdot 00162\right]\xi + \cdot 0024\,\delta_s\,\xi^2 = 1\cdot 29750 - \cdot 99296\,\delta_s\,, \quad (41)$$

From the equations (38), (39) and (41) we can determine p, Δ, ξ, if δ_s is assumed. The computations have been carried out with two values of δ_s and the result is

for	$\delta_s = 0\cdot 5$:	$p = \cdot 8325$,	$\Delta = \cdot 9160$,	$\xi = \cdot 0531$,
	$0\;6$:	$\cdot 7841$,	$\cdot 8831$,	$\cdot 0762$.

[1]) The computation was first carried out for the approximate value $\varepsilon''_s = {}^1/_{300}$. This led to:

for	$\delta_s = 0.5$:	$\lambda_1 = + -.0007$,
	0.6:	$-.0006$.

The computation was then repeated with the exact value of ε''_s. In the original Dutch communication only the first approximate computation was included in the text, and the final one mentioned in a footnote. For more details regarding these computations the reader is referred to *B. A. N.* 55.

Finally we have

$$5\int_0^s D\beta^4\,d\beta = s^5\left[\delta_s + \frac{5}{2}\Delta p^3 - \frac{3}{2}\Delta p^5\right],$$

from which we find by (37)

for $\quad \delta_s = 0 \cdot 5: \quad \frac{\lambda_1}{1+\lambda_1} = +.0003,$

$\qquad\qquad 0 \cdot 6: \qquad\qquad +.0004.$

Both these values agree with the adopted value $\lambda_1 = +.00044$ within the limits of the uncertainty of the data on which the computation is based. We must thus conclude that WIECHERT's hypothesis even in its simplest form, with only one surface of discontinuity and constánt densities below it and between it and the surface of discontinuity, represents a possible constitution of the earth.