Mathematics. — *"On the Surface of the Trisecants of a Twisted Curve which has Multiple Points".* By Dr. G. SCHAAKE. (Communicated by Prof. HENDRIK DE VRIES).

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§ 1. In this paper we shall derive how the formula of CAYLEY:

$$(n-2)\{h-\tfrac{1}{6}\,n\,(n-1)\}$$

for the order of the surface of the trisecants of a twisted curve of the order n that has h apparent double points, must be altered if this curve has one or more multiple points.

After that we shall give an application of the result by determining the order of the surface that is formed by the vertices of the plane pencils which contain three chords of an arbitrary twisted curve.

§ 2. First we consider a twisted curve of the n^{th} order with h apparent double points that has one m-fold point M, and we determine the order of the surface of the trisecants of this curve in a way which is analogous to that in which ZEUTHEN, *Abzählende Methoden* [139], finds the number of quadrisecants of an arbitrary twisted curve.

The chords of the curve k^n, which cut a given straight line l, form a surface Ω of the order $h+\tfrac{1}{2}n\,(n-1)$. For through a point of l there pass h chords of k^n, so that l is an h-fold straight line of Ω, and in a plane through l there lie $\tfrac{1}{2}n(n-1)$ more generatrices of Ω, which are the joins of the n points which k^n has in common with this plane. We assume two points A_1 and A_2 outside k^n and to any point P_1, in which the plane through A_1 and a generatrix b of Ω cuts the curve k^n outside b, we associate all the points P_2 which the plane through A_2 and the same generatrix b has in common with k^n outside b.

The straight lines b corresponding in this way to a point P_1 of k^n, are the generatrices of Ω which cut the line A_1P_1 outside P_1. Now k^n is an $(n-1)$-fold curve of Ω as $n-1$ of the chords of this curve which pass through a point of k^n, and which form a cone of the order $n-1$, cut the line l, and accordingly there pass $n-1$ generatrices of Ω through any point of k^n. Hence there are

$$h+\tfrac{1}{2}n(n-1)-(n-1)=h+\tfrac{1}{2}(n-1)(n-2)$$

straight lines b that cut A_1P_1 outside P_1. Any plane through A_2 and one of these lines b contains $n-2$ points P_2 associated to P_1. Consequently $(n-2)\{h+\frac{1}{2}(n-1)(n-2)\}$ points P_2 are associated to any point P_1.

The same number of points P_1 correspond to a given point P_2 of k^n.

Now we apply the formula of CAYLEY-BRILL:

$$\gamma=\alpha_1+\alpha_2+2kp$$

to the correspondence (P_1P_2). Here γ is the number of coincidences of the correspondence; α_1 and α_2, which indicate resp. how many points P_1 correspond to a given point P_2 and how many points P_2 to a given point P_1, are both $(n-2)\{h+\frac{1}{2}(n-1)(n-2)\}$. Further we may write for p, the genus of the carrier of the correspondence, $\frac{1}{2}(n-1)(n-2)-h-\frac{1}{2}m(m-1)$. Now the validity k of the correspondence must still be determined.

If we call k' the validity of the correspondence between the points P_1 of k^n and the points of intersection P_3 of the corresponding lines b with this curve, we have

$$k+k'=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

as the points P_2 and P_3 corresponding to a point P_1, are the points of intersection of k^n with planes through A_2 which do not pass through P_1. Let further k'' be the validity of the correspondence which is defined by the pairs of points of k^n lying on the lines b. In this correspondence there are conjugated to any point P_3 $n-1$ points, lying on the $n-1$ straight lines b through P_3. These points and the points P_1 corresponding to P_3, are the points of intersection outside P_1 of $n-1$ planes through P_3 with k^n. Hence:

$$k'+k''=n-1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We find k'' by applying the formula of CAYLEY-BRILL to the corresponding correspondence. With a view to this we substitute for γ the number of generatrices b of Ω that have two coinciding points of intersection with k^n, hence the rank of k^n, i.e. $n(n-1)-2h-$ $-m(m-1)$, for α_1 and α_2 the number of the straight lines b through a point of k^n, i.e. $n-1$, for p the genus of k^n, i.e. $\frac{1}{2}(n-1)(n-2)-$ $-h-\frac{1}{2}m(m-1)$, and for k the validity k'' of the correspondence in question. If we solve k'' out of the equation arising in this way, we find:

$$k''=1.$$

From this there follows resp. by the aid of the equations (2) and (1):

$$k' = n-2$$
$$k = 2-n.$$

The application of the formula of CAYLEY-BRILL gives, that the correspondence (P_1, P_2) has

$$(n-2)\{4h+m(m-1)\}$$

coincidences.

If the planes (A_1, b) and (A_2, b) coincide, we have a coincidence (P_1, P_2) in each of the $n-2$ points of intersection outside b of the plane (A_1, A_2, b) with k^n. This happens for each of the $h+\frac{1}{2}n(n-1)$ generatrices b of Ω that cut the straight line A_1A_2. We put aside these coincidences. We must therefore diminish the number found above by

$$(n-2)\{h+\tfrac{1}{2}n(n-1)\}$$

and there remain

$$(n-2)\{3h-\tfrac{1}{2}n(n-1)+m(m-1)\}$$

coincidences.

Through the m-fold point M of k^n there pass $n-m$ generatrices b_1 of Ω which cut k^n outside M; they are the generatrices which meet l of the cone of the order $n-m$ which projects k^n out of M. If a point P on k^n approaches M along one of the m branches through M, $n-m$ of the $n-1$ generatrices of Ω through P approach to the lines b_1, which are therefore m-fold generatrices of Ω. The remaining $m-1$ lines b through P are transformed into the $m-1$ lines b_2 through M, which lie in the planes touching the chosen branch of k^n and one of the other $m-1$ branches in M and cutting l. Besides the $n-m$ lines b_1 there pass accordingly through M $\frac{1}{2}m(m-1)$ more single generatrices b_2 of Ω, so that M is a $\frac{1}{2}m(2n-m-1)$-fold point of Ω.

If P gets into M along a branch of k^n, on each of the other $m-1$ branches the two points of $(n-m)$ pairs of points (P_1, P_2), corresponding to the $n-m$ lines b of Ω that are transformed into lines b_1, get into M. In this way the lines b_1 give rise to $(n-m)m(m-1)$ coincidences (P_1, P_2) lying in M. Further any line b_2 gives a coincidence (P_1, P_2) in M on the $m-2$ branches of k^n which do not belong to the leaf of Ω containing b_2. Again we put aside these coincidences and we must therefore also diminish the number of coincidences of the correspondence (P_1, P_2) by

$$(n-m)m(m-1)+\tfrac{1}{2}m(m-1)(m-2).$$

We also find the latter if we investigate by the aid of the rule of ZEUTHEN [1]) how many coincidences of the correspondence (P_1, P_2) lie in M. For if P_1 moves out of M along one of the branches of k^n, $m(n-m)+\frac{1}{2}m(m-1)-(n-1)=(n-m)(m-1)+\frac{1}{2}(m-1)(m-2)$ generatrices of Ω will cut the line P_1A_1 outside P_1, and each gives therefore a point P_2 which originally coincided with P_1, belonging to the same branch of k^n. Hence in all $(n-m)m(m-1)+\frac{1}{2}m(m-1)(m-2)$ coincidences (P_1, P_2) lie in M.

If we put these aside, there remain:

$$(n-2)\{3h-\tfrac{1}{2}n(n-1)\}+\tfrac{1}{2}m(m-1)(m-2)$$

coincidences of the correspondence (P_1, P_2).

These lie apparently in those points of k^n through which there pass trisecants of this curve cutting l. The last found number is therefore equal to three times the number of trisecants of k^n cutting l. Hence:

The trisecants of a twisted curve of the order n with h apparent double points which has an m-fold point and is for the rest general, form a surface of the order:

$$(n-2)\{h-\tfrac{1}{6}n(n-1)\}+\tfrac{1}{6}m(m-1)(m-2).$$

If we consider therefore a twisted curve with a double point, the formula of CAYLEY gives the right result. In this case the cone of the order $n-2$ projecting the curve out of the double point, splits off from the surface of the trisecants and for such a curve h is one less and the result of the above mentioned formula $n-2$ less than for a twisted curve of the same genus without a double point.

If k^n has more than one multiple point, namely: v_3 triple points, v_4 quadruple points, etc. .... v_m m-fold points, we find in the same way that the surface of the trisecants is of the order

$$(n-2)\{h-\tfrac{1}{6}n(n-1)\}+\sum_{i=3}^{i=m}\tfrac{1}{6}v_i\,i(i-1)(i-2).$$

For a curve which has only triple points we must therefore add the number of these points to the result of the formula of CAYLEY.

If we take e.g. a k^4 consisting of three straight lines l_1, l_2, l_3 passing through one point, and an arbitrary straight line l_4, $h=3$ and we find three for the order of the surface of the trisecants. Indeed this surface consists of the three plane pencils which lie in the planes that may be passed through each pair of the lines l_1, l_2 and l_3, and which have the points of intersection of l_4 with these planes for vertices. For a k^6 consisting besides of l_1, l_2 and l_3 of a similar

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[1]) Abzählende Methoden, [116]2.

triple of lines l_4, l_5 and l_6, $h = 9$ and we find for the order of the surface of the trisecants 18. In this case there lie indeed three plane pencils of trisecants in each of the six planes which may be passed through two intersecting lines l. If finally we have a k^9 consisting of three triples of straight lines through one point, $h = 9$ and the result of our formula is 108. In this case we have in each of the nine planes through two intersecting lines l six plane pencils and besides 27 scrolls of trisecants.

§ 3. The formula derived in § 2 does not hold good if in a multiple point M k^n has three branches touching at that point the same plane μ. For let us imagine a generatrix b of Ω cutting two of these branches in points lying at distances from M that are infinitely small of the first order. We can always choose l in such a way that such a generatrix of Ω exists. The distances from μ of the two points mentioned are infinitely small of the second order whereas the planes (A_1, b) and (A_2, b) form a finite angle. The straight line b forms accordingly with μ an angle that is infinitely small of the second order, the same as the angle of the lines along which μ is cut by the planes (A_1, b) and $A_2, b)$. Consequently the latter two planes cut the tangent at M to the third branch in two points, the distance of which is infinitely small of the second order and their distances to M are infinitely small of the first order if l is chosen arbitrarily. The same holds also good for the two points P_1 and P_2 in which the third branch is cut resp. by the planes (A_1, b) and (A_2, b). According to the rule of ZEUTHEN, already used in § 2 the coincidence counts twice instead of once on each of the three branches to which the line b_2 [1]) belonging to the other branches, gives rise and the result of § 2 must therefore be diminished by one, as we find the order of the scroll of the trisecants of k^n by dividing the remaining number of coincidences by three.

Consequently the formula derived in § 2, must be diminished by one for each triple of branches passing through the same multiple point of k^n and touching there at the same plane.

From this there ensues that for a curve with only multiple points where all branches have the same tangent plane, the formula of CAYLEY may be applied. This is the case if the curve is the intersection of two surfaces if in each multiple point of this curve at least one of the surfaces has a single point.

Let us take for instance a k^6 composed of a twisted cubic k^3 and

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[1]) The three straight lines b_2 corresponding to the three pairs of branches, coincide here in a triple generatrix of Ω.

two lines l_1 and l_2 passing through the same point M of k^5. For this curve $h = 5$. By the aid of the formula of § 2 we find six for the order of the surface of the trisecants. This surface consists of the two scrolls formed by the chords of k^5 which cut l_1 or l_2, and of the two plane pencils in the plane (l_1, l_2) that have the points of intersection outside M of k^5 with this plane as vertices. If k^5 touches the plane (l_1, l_2) in M, according to the result of this § the order of the surface of the trisecants of k^5 must be five. In this case there is indeed only one plane pencil of trisecants.

§ 4. The formula derived in § 2 may be applied in order to determine the order of the surface consisting of the points through which there pass three chords in one plane of a curve k^n with h apparent double points.

In the communication: "*On the plane pencils Containing Three straight lines of a Given Algebraic Congruence of Rays*" [1]) the order of the surface formed by the vertices of the plane pencils containing three rays of a congruence Γ (α, β) of the rank r has been determined. This has been done by the aid of a representation of a special linear complex with axis a on the points of space, through which the surface Ω of the straight lines of Γ that cut a, is transformed into a twisted curve γ. The plane pencils with their vertices on a containing three straight lines of Γ, are represented on the trisecants of γ which cut a β-fold line of intersection of this curve.

If we take for Γ the congruence of the chords of k^n, the number of triple points of γ is the same as the order t of the surface of the trisecants of k^n. For in this case the trisecants of k^n cutting a are triple generatrices of Ω.

As a rule γ does not have one tangent plane at these triple points. For in that case there would exist a bilinear congruence with directrix a which in a trisecant d cutting a, has two coinciding straight lines in common with the three leaves through d of Ω. If this congruence were not special, the second directrix would touch the three leaves of Ω in a point of d. But then the three leaves of Ω would have the same tangent plane in each point of d, as two of these leaves already touch each other in the point of intersection of d and a and in one of the points of intersection of d with k^n. The same holds good if the bilinear congruence is special. For in that case the two points of d where two of the leaves of Ω through d have the same tangent plane, would coincide in the

[1]) These Proceedings vol. XXVI, p. 522.

point of intersection of d with a, and as their tangent planes also coincide in a point of intersection of d with k^n, these leaves would touch each other in each point of d. But then the straight line a would apparently touch in its point of intersection with d the quadratic surface which is defined by the three tangents to k^n in the points of intersection of d with this curve, hence the surface of the trisecants of k^n, which may be avoided by a suitable choice of a.

The bilinear congruence composed of the plane (a, d) and the sheaf (a, d) satisfies the above mentioned condition, but it is represented on a pair of planes which both pass through the triple point of k^n and consequently it does not give a tangent plane to k^n at this point.

We find accordingly the order of the surface of the trisecants of γ by adding t to the result of the formula of CAYLEY. Hence for a congruence Γ consisting of the chords of a twisted curve k^n the order of the surface of the points through which there pass three straight lines of Γ in one plane, is found by adding the order of the surface of the trisecants of k^n to the general result mentioned above.

Accordingly the locus of the vertices of the plane pencils containing three chords of a twisted curve, the bisecants of which form a congruence (α, β) *of the rank* r *and the trisecants of which form a scroll of the order* t, *is a surface of the order*

$$\tfrac{1}{6}(\alpha-2)\{6r-(\alpha-1)(3\beta-\alpha)\}+t.$$

To express this result in the order n and the number of apparent double points of the twisted curve k^n, we must substitute resp. h, $\frac{1}{2}n(n-1)$ and $(n-2)\{h-\frac{1}{6}n(n-1)\}$ for α, β and t. We find the rank r of the congruence of the chords of k^n in the following way. We choose a straight line l, draw the h bisecants of k^n through a point P_1 of l and pass planes through l and these h lines. In each of these planes there lie $\frac{1}{2}(n-2)(n-3)$ chords of k^n that do not cut on k^n the chord in this plane through P_1. We associate to P_1 the points of intersection P_2 of these chords with l. In these same way we find the points P_2 corresponding to P_1. The correspondence (P_1, P_2) has $h(n-2)(n-3)$ coincidences, which, however, coincide in pairs, for if P_1 moves out of a coincidence, two points P_2 become different from P_1. Hence the rank r of the congruence of the chords of k^n is equal to $\frac{1}{2}h(n-2)(n-3)$.

We find:

The locus of the vertices of the plane pencils containing three *chords of a twisted curve of the order n with h apparent double points, is a surface of the order*

$$\tfrac{1}{12}\{2h^3+3h^2(n^2-9n+10)-h(3n^3-63n+92)-2n(n^2-1)\}.$$

This result cannot be applied to a conic as on each straight line there lies the vertex of a plane pencil containing an infinite number bisecants of the curve.

If we substitute successively for n and h resp.: 3 and 1, 4 and 2, 4 and 3, 5 and 4, 5 and 5, 5 and 6, we always find the result zero. Indeed in all these cases on an arbitrary straight line there does not lie a vertex of a plane pencil containing three bisecants of the curve, because the plane of this pencil would have six points in common with the curve. If we did not add the order of the surface of the trisecants, in the last four cases we should find a negative result, and we cannot conclude in any other way from the derivation why the formula does not hold good for these cases.

§ 5. We notice that the following more general theorem may be pronounced:

If we have a congruence (α, β) of the rank r with an infinite number of triple rays which form a scroll of the order t, the locus of the vertices of the plane pencils containing three lines of this congruence, is a surface of the order

$$\tfrac{1}{6}(\alpha-2)\{6r-(\alpha-1)(3\beta-\alpha)\}+t.$$

The following example gives a verification of this formula. We imagine three crossing straight lines a_1,a_2, and a_3, and two straight lines b_1 and b_2, which cut a_1, a_2 and a_3. For the congruence C, consisting of the straight lines which cut two crossing individuals of the five given lines and which is therefore composed of four bilinear congruences, $\alpha=\beta=4$. A plane pencil with its vertex outside the given lines which contains three lines of C, must contain one line which cuts b_1 and b_2 and two lines which each meet a pair of the lines a. Suppose that the latter two lines of C both cut the line a_i. In this case the plane of the pencil passes through a_i but also through a line b, because besides a_i this plane contains another transversal of b_1 and b_2 and therefore an infinite number of transversals of b_1 and b_2. If we choose e.g. a point P in the plane through a_i and b_1, three straight lines of C through P belong to a plane pencil. For the transversals through P of a_i and one of the two other straight lines a lie in the plane (a_i, b) together with the transversal of b_1 and b_2. Accordingly the locus of the vertices of the plane pencils that have three straight lines in common with C, consists of *six* planes.

We find the rank r of the congruence C by remarking that the quadratic surface of the lines cutting an arbitrary line l, b_1 and b_2, touches each of the three quadratic surfaces consisting of the lines which cut l and two of the lines a, in two points on l. Consequently there are six points on l through which there pass two lines of C lying in a plane through l. The scroll wich has a_1, a_2 and a_3 as directrices, consists of triple generatrices of C; hence $t = 2$. If we substitute resp. 4, 4, 6 and 2 for α, β, r and t in the formula indicated above, we find indeed the result *six*. Also in this case it appears that it is really necessary to add the order of the surface of the triple generatrices to the result of the general formula.

The five given straight lines form a special case of the twisted curve for which $n = 5$ and $h = 4$. The application of our formula to this curve gave the result zero whereas in the special case we find a surface of the sixth order. This is not in contradiction with the principle of the conservation of the number. For in this special case there lie on each straight line the vertices of six plane pencils which each contains an infinite number of bisecants of the degenerate quintic. They are the points of intersection of l with the six planes (a_i, b_k). However we found a finite result because C consists of only a part of the bisecants of the degenerate quintic.

The following theorem may be proved which indicates the cases that the theorem of this § does not hold good:

If a congruence contains a scroll of triple rays for which the six plane pencils that have the foci for vertices and the corresponding focal planes for planes, always lie in a special bilinear congruence, the order of the said scroll must not be added to the result of the formula which holds good for an arbitrary congruence.