Mathematics. — *"On the place of conformal and projective geometry in the theory of linear displacements"*. By Prof. J. A. SCHOUTEN. (Communicated by Prof. JAN DE VRIES).

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Introduction. When we try to found differential geometry on the theory of linear displacements, that has developed itself from the idea of pseudoparallel displacement, the difficulty arises, that projective differential geometry (systematically developed by WILCZYNSKI and FUBINI) and conformal differential geometry (systematically developed by CARTAN [1]) [3]) and THOMSEN [2]) can not be reduced to a linear displacement. Then CARTAN has enlarged the idea of displacement by adjoining to each point of the manifold, not, as usual, the differential element of this manifold itself, but quite another manifold with a given group of transformations, in the here considered case a projective resp. a conformal group. He then seems to leave principally the linear displacements. This false appearance however only arises because he uses an unusual symbolism, which has not the same invariance as the Ricci-calculus, so that it does not become clear, that in reality we have to do with linear displacements of a more general art, already introduced by R. KÖNIG in 1920 [4]). These displacements of KÖNIG differ from the ordinary ones only by the number of coordinates in the conjugated manifolds, which is not equal to that, used in the given manifold.

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[1]) Bull. Soc. Math. 45 p. 57—121, (1917).

[2]) Ueber konforme Geometrie, Abh. Math. Seminar Hamburg 3 (23) 31—56.

[3]) *a.* Sur les espaces généralisés etc., C.R. 174 (22) 734—737.
b. Sur les espaces conformes généralisés etc., C.R. 174 (22) 857—860.
c. Sur la connexion projective des surfaces, C.R. 178 (24) 750—752.
d. Sur les variétés à connexion affine etc., Ann. de l'école norm. sup. 40 (23) 325—412, especially 383 a.f.
e. Les espaces à connexion conforme, Ann de la soc. polon. de math. (23) 171—221.

[4]) Beiträge zu einer allgemeinen Mannigfaltigkeitslehre, Jahresber. d. D. M. V. 28 (20) 213—228.

In the following we will show, that it is perfectly possible to master the general projective and the general conformal manifold of CARTAN (both generalisations of the ordinary manifolds) with the aid of these more general displacements. As this can be done with the same simple invariant equations as are known from the other kinds of geometry and also in the same way, the relations become clearer, and so we can show f. i., that the numbers A^{j}_{ikl} of CARTAN (*l.c.*(*e*)) are identical with the components of the quantity of conformal curvature already discovered by WEYL. The far going analogy between conformal and projective geometry becomes very clear, especially in the theorem, proved in the second part, on the relations between affin geometry with invariant geodesic lines (geometry of paths) and projective displacement, a theorem, perfectly analogous to the theorem of CARTAN, demonstrated in the first part, on the relations between conformal RIEMANN geometry and conformal displacement. This second theorem implies of course the quantity of projective curvature, also discovered by WEYL.

I. *The general conformal displacement.*

§ 1. *The euclidean-conformal manifold.* It is known that a hypersphere in R_n [1]) can be given by $n+2$ homogeneous characteristic numbers $v^{\gamma}, \alpha, \beta, \gamma, \delta = b_1, \ldots, b_{n+2}$. For $n = 3$ these 5 numbers are pentaspherical coordinates. The quantities v^{γ} and λv^{γ} will be considered algebraically as different, although they correspond geometrically with the same figure. Then there exists a *fundamental tensor* $G_{\alpha\beta}$, such, that the equations

$$(1) \qquad G_{\alpha\beta}\, v^{\alpha}\, v^{\beta} = 0 \;;\quad G_{\alpha\beta}\, v^{\alpha}\, v^{\beta} = 1 \;;\quad G_{\alpha\beta}\, v^{\alpha}\, w^{\beta} = 0$$

are characteristic for a *point*, resp. a *unitsphere*, resp. *two mutually orthogonal spheres*. $G_{\alpha\beta}$ may not be confused with the fundamental tensor of the euclidian-metrical geometry in R_n, $g_{\lambda\mu}$, $\varkappa, \lambda, \mu, \nu, \omega = a_1 \ldots, a_n$.

When $G_{\alpha\beta}$ is multiplied with a number, nothing is changed in the geometrical interpretation. Such an alteration of $G_{\alpha\beta}$ we do not use however, which does not imply any geometrical restriction. On account of this, there is no difference between covariant and contravariant *quantities*, only between covariant and contravariant *characteristic numbers*, and it is allowed to raise and lower the suffices with the aid of $G_{\alpha\beta}$ and the conjugated tensor $G^{\alpha\beta}$.

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[1]) We denote with R_n an ordinary n dimensional euclidean-metrical manifold.

With the aid of $G_{\alpha\beta}$ we can introduce with THOMSEN [1]) orthogonal characteristic numbers, starting with $n+2$ mutually orthogonal unit-spheres. Simpler character have the characteristic numbers, used by CARTAN [2]), that we will call *normal*. Then we start with a *normal system*, t.i. a system composed of n mutually orthogonal unit-spheres $\underset{i}{u}^{\gamma}, i, j, k, l, m = 1, \ldots, n$ and their two common points $\underset{0}{u}^{\gamma}, \underset{n+1}{u}^{\gamma}$. The equations, characteristic for such a system, are

$$(2) \qquad \underset{0}{u}^{\gamma} \underset{n+1}{u}_{\gamma} = 1, \qquad \underset{i}{u}^{\gamma} \underset{i}{u}_{\gamma} = 1, \text{ all other transvections } = 0.$$

When $\overset{0}{u}_{\alpha}, \overset{k}{u}_{\alpha}, \overset{n+1}{u}_{\alpha}$, is the corresponding reciprocal system, we have:

$$(3) \qquad \overset{n+1}{u}{}^{\gamma} = \underset{0}{u}^{\gamma}, \qquad \overset{0}{u}{}^{\gamma} = \underset{n+1}{u}^{\gamma}, \qquad \overset{i}{u}{}^{\gamma} = \underset{i}{u}^{\gamma}$$

and $G_{\alpha\beta}$ is equal to:

$$(4) \qquad G_{\alpha\beta} = \overset{0}{u}_{\alpha} \overset{n+1}{u}_{\beta} + \overset{n+1}{u}_{\alpha} \overset{0}{u}_{\beta} + \sum_{k} \overset{k}{u}_{\alpha} \overset{k}{u}_{\beta}.$$

For the normal characteristic numbers of an arbitrary sphere v^{γ} we find:

$$(5) \qquad v^{0} = v_{n+1}, \qquad v_{0} = v^{n+1}, \qquad v_{i} = v^{i}$$

and for the transvection $v^{\gamma} w_{\gamma}$:

$$(6) \qquad v^{\gamma} w_{\gamma} = v_{n+1} w_{0} + \sum_{i} v_{i} w_{i} + v_{0} w_{n+1}.$$

A linear transformation, transforming the system $\underset{a}{u}^{\gamma}, a, b, c, d = 0, \ldots, n+1$ into another normal system, transforms all other normal systems into each other and leaves $G_{\alpha\beta}$ invariant. The corresponding pointtransformation is conformal. To the group of transformations, that leave $G_{\alpha\beta}$ invariant, belongs for $n > 2$ the group of conformal transformations of R_n.

The totality of all spheres of R_n with the fundamental tensor $G_{\alpha\beta}$ is called an *euclidean-conformal manifold* $\mathfrak{C}_n^e$. In a $\mathfrak{C}_n^e$ we thus can make a difference between points and spheres, and in every point we know which directions are mutually orthogonal. Straight lines however do not exist in $\mathfrak{C}_n^e$.

§ 2. *The general conformal manifold.* To each point P of an X_n [3]) be conjugated a $\mathfrak{C}_n^e$ in such a way that P itself is also a

[1]) L.c.

[2]) L.c. (e) p. 172 a.f.

[3]) With X_n we denote an n-dimensional manifold without further particular properties.

point of $\mathfrak{C}_n^e$ [1]). Between both $\mathfrak{C}_n^e$ in P and a neighbouring point Q there exist up till now no relations whatever. Such a relation is introduced, by indicating, how a quantity belonging to the $\mathfrak{C}_n^e$ in P can be displaced to Q. When in every point a normal system $\underset{a}{u}^\gamma$ is fixed, such that $\underset{0}{u}^\gamma$ is the point itself, the displacement is defined by the equation:

$$(7) \qquad \delta \underset{a}{u}^\gamma = \Sigma \overset{b}{\Lambda}_{a\mu} \underset{b}{u}^\gamma \, dx^\mu \qquad \begin{cases} a, b, c, d = 0, \dots, n+1 \\ \alpha, \beta, \gamma, \delta = b_1, \dots, b_{n+2} \\ \varkappa, \lambda, \mu, \nu, \omega = a_1, \dots, a_n \end{cases}$$

or in normal characteristic numbers:

$$(8) \qquad \delta \underset{a}{u}^c = \Sigma \overset{b}{\Lambda}_{a\mu} \underset{b}{u}^c \, dx^\mu = \overset{c}{\Lambda}_{a\mu} \, dx^\mu ,$$

in which equation the $\overset{b}{\Lambda}_{a\mu}$ are $n(n+2)^2$ arbitrary parameters. We now introduce the condition, that $G_{\alpha\beta}$ is invariant, that is, that normal systems always are transformed into normal systems. This leads to the equations:

$$(9) \qquad \begin{aligned} &\overset{0}{\Lambda}_{0\mu} = - \overset{n+1}{\Lambda}_{n+1,\mu} \\ &\overset{k}{\Lambda}_{0\mu} = - \overset{n+1}{\Lambda}_{k\mu}, \ \overset{0}{\Lambda}_{i\mu} = - \overset{i}{\Lambda}_{n+1,\mu} \\ &\overset{n+1}{\Lambda}_{0\mu} = \overset{0}{\Lambda}_{n+1,\mu} = 0, \ \overset{k}{\Lambda}_{i\mu} = - \overset{i}{\Lambda}_{k\mu}, \ i \neq k \\ &\overset{i}{\Lambda}_{i\mu} = 0. \end{aligned}$$

From (9) follows

$$(10) \qquad \begin{aligned} \alpha) \quad & \delta \underset{0}{u}^\gamma = (\overset{0}{\Lambda}_{0\mu} \underset{0}{u}^\gamma + \Sigma \overset{j}{\Lambda}_{0\mu} \underset{j}{u}^\gamma) \, dx^\mu \\ \beta) \quad & \delta \underset{i}{u}^\gamma = (\overset{0}{\Lambda}_{i\mu} \underset{0}{u}^\gamma - \overset{i}{\Lambda}_{0\mu} \overset{0}{u}^\gamma + \Sigma \overset{j}{\Lambda}_{i\mu} \underset{j}{u}^\gamma) \, dx^\mu \\ \gamma) \quad & \delta \overset{0}{u}^\gamma = (- \Sigma \overset{0}{\Lambda}_{j\mu} \underset{j}{u}^\gamma - \overset{0}{\Lambda}_{0\mu} \overset{0}{u}^\gamma) \, dx^\mu . \end{aligned}$$

It may be remarked, that, after (10α) the point P will in general *not* be transformed into Q. From (8) follows for the covariant differential of an arbitrary quantity of the first degree:

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[1]) As an example we can take an X_n in R_{n+m}. Then we can consider as $\mathfrak{C}_e^n$ in P the $\mathfrak{C}_n^e$ of the tangential R_n .

$$(11) \qquad \begin{aligned} \delta v^c &= d v^c + \sum_a \Lambda^c_{a\mu} v^a dx^\mu \\ \delta w_a &= d w_a - \sum_c \Lambda^c_{a\mu} w_c dx^\mu . \end{aligned}$$

The corresponding covariant differential quotients are:

$$(12) \qquad \begin{aligned} \nabla_\mu v^c &= \frac{\partial v^c}{\partial x^\mu} + \sum_a \Lambda^c_{a\mu} v^a \\ \nabla_\mu w_a &= \frac{\partial w_a}{\partial x^\mu} - \sum_c \Lambda^c_{a\mu} w_c . \end{aligned}$$

From (11) and (12) follows indeed that $\delta G_{\alpha\beta}$ and $\nabla_\mu G_{\alpha\beta}$ vanish. The displacement, obtained in this way, we call *conformal.* The totality of the points of the X_n with the $\mathfrak{C}^e_n$ conjugated to each point and the introduced conformal displacement we call a *general conformal manifold,* $\mathfrak{C}_n$ [1]). A $\mathfrak{C}^e_n$ is a special case of a $\mathfrak{C}_n$. Here to each point is conjugated the $\mathfrak{C}^e_n$ itself, and the displacement becomes a trivial one, each point passes into itself.

In a $\mathfrak{C}_n$ is known in each point what we understand by mutual orthogonal directions in the conjugated $\mathfrak{C}^e_n$. In each $\mathfrak{C}^e_n$ their exist spheres, spheres in neighbouring points can be compared, spheres in not neighbouring points in general not. Transformation of the normal systems has no influence on the displacement, but has of course influence on the parameters $\Lambda^c_{a\mu}$. We only consider such transformation of the normal systems, that leave $\underset{0}{u}^\gamma$ invariant but for a factor. Such a transformation has the form

$$(13) \qquad {}'\underset{a}{u}^\gamma = \sum_b P^b_a \underset{b}{u}^\gamma , \qquad \underset{a}{u}^\gamma = \sum_b Q^b_a \, {}'\underset{b}{u}^\gamma ,$$

in which equation there exists a number of simple relations between the parameters P^b_a, of which we only need the following:

$$(14) \qquad \begin{cases} P^k_0 = P^{n+1}_0 = P^{n+1}_i = 0 , \quad Q^k_0 = Q^{n+1}_0 = Q^{n+1}_i = 0 \\ P^0_0 \, Q^0_0 = 1 , \quad P^{n+1}_{n+1} = Q^0_0 , \quad P^0_0 = Q^{n+1}_{n+1} \end{cases}$$

$$(15) \qquad \sum_k P^k_i \, P^k_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

[1]) An example of a $\mathfrak{C}_n$ can be obtained when we conjugate to each point P of an X_n in R_{n+1} an arbitrary tangential sphere B_n in R_{n+1}. Then a sphere B_{n-1} in the tangential R_n in P is conjugated to the sphere in the tangential R_n in Q, that is cut out of this R_n by a sphere of R_{n+1}, that is orthogonal to B_n and contains B_{n-1}.

From (13) and (14) follows for the transformation of the orthogonal characteristic number:

$$
(16)\qquad
\begin{aligned}
'v_0 &= 'v^{n+1} = \overset{0}{P_0}\, v_0 \\
'v_i &= 'v^i \quad = \overset{0}{P_i}\, v_0 \quad + \sum_j \overset{j}{P_i}\, v_j \\
'v^0 &= 'v_{n+1} = \overset{0}{P}_{n+1}\, v_0 \quad + \sum_j \overset{j}{P}_{n+1}\, v_j + \overset{0}{Q_0}\, v^0
\end{aligned}
$$

and for the transformation of $\overset{0}{A}_{0\mu}$ and $\overset{k}{A}_{0\mu}$:

$$
(17)\qquad
\begin{aligned}
&\alpha)\quad '\overset{0}{A}_{0\mu} = \overset{0}{A}_{0\mu} + \frac{\partial}{\partial x^\mu} \log \overset{0}{P_0} + \overset{0}{P_0} \sum_j \overset{0}{Q_j}\, \overset{j}{A}_{0\mu} \\
&\beta)\quad '\overset{k}{A}_{0\mu} = \overset{0}{P_0} \sum_j \overset{k}{Q_j}\, \overset{j}{A}_{0\mu}.
\end{aligned}
$$

From (17^β) and (15) follows, that the n^2 parameters $\overset{k}{A}_{0\mu}$ can be considered as covariant characteristic numbers of n vectors of X_n, which are transformed orthogonally but for a factor with the transformation (13). When we thus write $\overset{k}{A}_{0\mu} = \overset{k}{i}_\mu$, and

$$
(18)\qquad g_{\lambda\mu} = \sum_k \overset{k}{i}_\lambda\, \overset{k}{i}_\mu\,,
$$

then the tensor $g_{\lambda\mu}$ of X_n rests invariant with (13) but for a factor:

$$
(19)\qquad 'g_{\lambda\mu} = \sigma g_{\lambda\mu} \quad , \quad \sigma^{1/2} = \overset{0}{P_0}\,.
$$

When we thus introduce $g_{\lambda\mu}$ as fundamental tensor of the X_n, the conformal displacement of the $\mathfrak{C}_n$ fixes in the corresponding X_n a Riemann geometry but for a conformal transformation. When we write the equation (10^α) in the form:

$$
(10^\alpha)\qquad \delta \underset{0}{u^\gamma} = \overset{0}{A}_{0\mu}\, \underset{0}{u^\gamma}\, dx^\mu + \sum_j \overset{j}{i}_\mu\, dx^\mu\, \underset{j}{u^\gamma}\,,
$$

we see, that with a transformation dx^μ with the orthogonal characteristic numbers dx_i corresponds a displacement of $\underset{0}{u^\gamma}$ orthogonal to the system of spheres $\alpha \underset{0}{u^\gamma} + \beta \sum_i \underset{i}{u^\gamma}\, dx_i$. Hence the directions of X_n in P are conjugated in a one-to-one way to the directions of the $\mathfrak{C}_n^e$ in P, in such a manner, that the orthogonality fixed by $g_{\lambda\mu}$ in X_n corresponds to that fixed by $G_{\alpha\beta}$ in $\mathfrak{C}_n^e$.

From the equation (17^α) in the form

$$
(17^\alpha)\qquad '\overset{0}{A}_{0\mu} = \overset{0}{A}_{0\mu} + \frac{\partial}{\partial x^\mu} \log \overset{0}{P_0} + \overset{0}{P_0} \sum_j \overset{0}{Q_j}\, \overset{j}{i}_\mu
$$

follows that it is always possible to choose the normal systems in such a way, that $\overset{0}{\Lambda}_{0\mu}$ always vanishes. Hence we put

$$(20) \qquad \overset{0}{\Lambda}_{0\mu} = 0$$

and only permit such transformations of the normal systems, that leave (20) invariant. The conditions are:

$$(21) \qquad \sum_j \overset{0}{Q}_j \overset{j}{i}_\mu = -\overset{0}{Q}_0 \frac{\partial}{\partial x^\mu} \log \overset{0}{P}_0 = \frac{\partial}{\partial x^\mu} \overset{0}{Q}_0 ,$$

or, when $\overset{\nu}{e}_i$ are the orthogonal characteristic numbers of the covariant measuring vectors $\overset{\nu}{e}_\lambda$ of the X_n:

$$(22) \qquad \overset{0}{Q}_i = \overset{\mu}{e}_i \frac{\partial}{\partial x^\mu} \overset{0}{Q}_0 .$$

§ 3. *The quantity of curvature of* $\mathfrak{C}_n$. In the X_n itself no displacement whatever is defined. Hence the expression $\nabla_\mu w_\lambda$, in which w_λ is a covariant vector of the X_n, has no meaning. Without defining a displacement in the X_n we can however define already the *alternating* part of $\nabla_\mu w_\lambda$:

$$(23) \qquad \nabla_{[\mu} w_{\lambda]} = {}^1/_2 \left(\frac{\partial w_\lambda}{\partial x^\mu} - \frac{\partial w_\mu}{\partial x^\lambda} \right) \; {}^{1)}$$

from wich equation follows:

$$(24) \qquad \nabla_{[\omega} \nabla_\mu w_{\lambda]} = 0.$$

With the aid of (23) we can now build the second alternated differential quotient $\nabla_{[\omega}\nabla_{\mu]} v^\gamma$ of a quantity v^γ of the $\mathfrak{C}_n$. For, when we write $\nabla_\mu v^\gamma$ as a product of two ideal factors $m_\mu n^\gamma$, m_μ is a quantity of X_n, n^γ a quantity of $\mathfrak{C}_n$, and thus we find in relation to (23) and (11):

$$\begin{aligned}
\nabla_{[\omega} \nabla_{\mu]} v^c &= \nabla_{[\omega} m_{\mu]} n^c = n^c \nabla_{[\omega} m_{\mu]} + m_{[\mu} \nabla_{\omega]} n^c = \\
&= {}^1/_2 n^c \left(\frac{\partial m_\mu}{\partial x^\omega} - \frac{\partial m_\omega}{\partial x^\mu} \right) + m_{[\mu} \left(\frac{\partial}{\partial x^{\omega]}} n^c + \sum_b \overset{c}{\Lambda}_{|b|\omega]} n^b \right) \\
(25) \qquad &= \frac{\partial}{\partial x^{[\omega}} (m_{\mu]} n^c) + \sum_b \overset{c}{\Lambda}_{b[\omega} m_{\mu]} n^b \\
&= \frac{\partial}{\partial x^{[\omega}} \left(\frac{\partial}{\partial x^{\mu]}} v^c + \sum_a \overset{c}{\Lambda}_{|a|\mu]} v^a \right) + \sum_b \overset{c}{\Gamma}_{b[\omega} \left(\frac{\partial}{\partial x^{\mu]}} v^b + \sum_a \overset{b}{\Lambda}_{|a|\mu]} v^a \right) = \\
&= \sum_a \left(\frac{\partial}{\partial x^{[\omega}} \overset{c}{\Lambda}_{|a|\mu]} + \sum_b \overset{c}{\Lambda}_{b[\omega} \overset{b}{\Lambda}_{|a|\mu]} \right) v^a .
\end{aligned}$$

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1) We fix with this equation, that the displacement in the X_n, in which way it may be specified, always is a *symmetrical* one.

When now we write:

(26) $$\mathfrak{C}_{\omega\mu a}^{\cdot\cdot\cdot c} = -2 \frac{\partial}{\partial x^{[\omega}} \Lambda^{c}_{|a|\mu]} - 2 \sum_{b} \Lambda^{c}_{b[\omega} \Lambda^{b}_{|a|\mu]},$$

we have

(27) $$2 \nabla_{[\omega} \nabla_{\mu]} v^c = -\sum_{a} \mathfrak{C}_{\omega\mu a}^{\cdot\cdot\cdot c} v^a .$$

$\mathfrak{C}_{\omega\mu a}^{\cdot\cdot\cdot c}$ is the *quantity of curvature* of $\mathfrak{C}_n$. When a quantity v^c is moved in the sense of the accepted displacement along the boundary of an X_2-element $f^{\omega\mu} d\sigma$, the difference between its begin and end-value is

(28) $$D v^c = f^{\omega\mu} d\sigma \sum_{a} \mathfrak{C}_{\omega\mu a}^{\cdot\cdot\cdot c} v^a .$$

Hence a $\mathfrak{C}^e_n$ is characterised by the property that $\mathfrak{C}_{\omega\mu a}^{\cdot\cdot\cdot c}$ vanishes. From (26) follows for the normal components of $\mathfrak{C}_{\omega\mu a}^{\cdot\cdot\cdot c}$:

(29) $$\begin{aligned}
\mathfrak{C}_{\omega\mu 0}^{\cdot\cdot\cdot 0} &= \mathfrak{C}_{\omega\mu 0\, n+1} = -\mathfrak{C}_{\omega\mu\, n+1}^{\cdot\cdot\cdot\, n+1} = -\mathfrak{C}_{\omega\mu\, n+1\, 0}\\
\mathfrak{C}_{\omega\mu 0}^{\cdot\cdot\cdot k} &= \mathfrak{C}_{\omega\mu 0 k} = -\mathfrak{C}_{\omega\mu k}^{\cdot\cdot\cdot n+1} = -\mathfrak{C}_{\omega\mu k 0}\\
\mathfrak{C}_{\omega\mu 0}^{\cdot\cdot\cdot n+1} &= \mathfrak{C}_{\omega\mu 0 0} = \mathfrak{C}_{\omega\mu\, n+1}^{\cdot\cdot\cdot\, 0} = \mathfrak{C}_{\omega\mu\, n+1\, n+1} = 0\\
\mathfrak{C}_{\omega\mu i}^{\cdot\cdot\cdot 0} &= \mathfrak{C}_{\omega\mu i\, n+1} = -\mathfrak{C}_{\omega\mu\, n+1}^{\cdot\cdot\cdot\, i} = -\mathfrak{C}_{\omega\mu\, n+1\, i}\\
\mathfrak{C}_{\omega\mu i}^{\cdot\cdot\cdot j} &= \mathfrak{C}_{\omega\mu i j} = -\mathfrak{C}_{\omega\mu j}^{\cdot\cdot\cdot i} = -\mathfrak{C}_{\omega\mu j i} .
\end{aligned}$$

Thus $\mathfrak{C}_{\omega\mu i j}$ is not only alternating in $\omega\mu$, but also in ij:

(30) $$\mathfrak{C}_{\omega\mu i j} = \mathfrak{C}_{[\omega\mu] i j} = \mathfrak{C}_{\omega\mu [i j]}.$$

These identities have the same form as the *first* and *third* identity valuable for the quantity of curvature of a Riemann geometry.

§ 4. *Symmetrical conformal displacements.* $\underset{0}{u}^{\gamma}$ is invariant but for a factor. Hence the equation

(31) $$\underset{0}{u}^a \mathfrak{C}_{\omega\mu a}^{\cdot\cdot\cdot c} = \mathfrak{C}_{\omega\mu 0}^{\cdot\cdot\cdot c} = 0$$

is invariant. From (28) follows, that (31) expresses, that the point P returns in P when P is moved along a closed infinitesimal curve in the sense of the displacement. A displacement with this property is called *symmetrical*[1]). From (31) follows:

$$\tfrac{1}{2}\, \mathfrak{C}_{\omega\mu 0}^{\cdot\cdot\cdot 0} = - \sum_{j} \overset{j}{i}_{[\omega} \overset{0}{A}_{|j|\mu]} = 0 \tag{32}$$

$$\tfrac{1}{2}\, \mathfrak{C}_{\omega\mu 0}^{\cdot\cdot\cdot k} = - \nabla_{[\omega} \overset{k}{i}_{\mu]} - \sum_{j} \overset{k}{A}_{j[\omega} \overset{j}{i}_{\mu]} = 0 \tag{33}$$

or, in normal characteristic numbers:

$$\overset{0}{A}_{[ij]} = 0 \tag{32^a}$$

$$\nabla_{[m} \overset{k}{i}_{l]} = - \overset{k}{A}_{[lm]}\,. \tag{33^a}$$

The remaining characteristic numbers of $\mathfrak{C}_{\omega\mu a}^{\cdot\cdot\cdot c}$ are transformed with (13) in the following way:

$$'\mathfrak{C}_{\omega\mu i}^{\cdot\cdot\cdot 0} = \overset{0}{Q}_{0} \sum_{j} \overset{j}{P}_{i}\, \mathfrak{C}_{\omega\mu j}^{\cdot\cdot\cdot 0} + \sum_{j,k} \overset{j}{P}_{i}\, \overset{0}{Q}_{k}\, \mathfrak{C}_{\omega\mu j}^{\cdot\cdot\cdot k} \tag{34}$$

$$'\mathfrak{C}_{\omega\mu i}^{\cdot\cdot\cdot k} = \sum_{j,l} \overset{j}{P}_{i}\, \overset{l}{P}_{k}\, \mathfrak{C}_{\omega\mu j}^{\cdot\cdot\cdot l}\,. \tag{35}$$

Hence the characteristic numbers $\mathfrak{C}_{\omega\mu i}^{\cdot\cdot\cdot k}$ are transformed into themselves and in such a way, that they can be considered as the characteristic numbers of a quantity of the X_n.

§ 5. Bianchi's *identity*. From (27) follows:

$$\begin{aligned} 2\nabla_{[\xi} \nabla_{\omega} \nabla_{\mu]} v^c &= - \sum_{a} \nabla_{[\xi}\, \mathfrak{C}_{\omega\mu]a}^{\cdot\cdot\cdot c}\, v^a = \\ &= - \sum_{a} v^a\, \nabla_{[\xi} \mathfrak{C}_{\omega\mu]a}^{\cdot\cdot\cdot c} - \sum_{a} (\nabla_{[\xi}\, v^a)\, \mathfrak{C}_{\omega\mu]a}^{\cdot\cdot\cdot c}. \end{aligned} \tag{36}$$

But from (27) also follows, with respect to (24)

$$2\nabla_{[\xi} \nabla_{\omega} \nabla_{\mu]}\, v^c = 0 - \sum_{a} \mathfrak{C}_{[\xi\omega|a|}^{\cdot\cdot\cdot c} \nabla_{\mu]}\, v^a\,, \tag{37}$$

so that:

$$\nabla_{[\xi} \mathfrak{C}_{\omega\mu]a}^{\cdot\cdot\cdot c} = 0. \tag{38}$$

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[1]) The mentioned property is for a common linear displacement the criterion for symmetry, th. i. for the vanishing of $\Gamma_{[\lambda\mu]}^{\nu}$. Cartan uses the expression "sans torsion" that gives rise however to the false supposition that there is a relation with the second curvature of space curves.

This is BIANCHI's *identity* for a conformal displacement [1]). From (38) follows:

$$(39)\quad \sum_{b,d} \nabla_{[\xi}\, \mathfrak{C}_{\omega\mu]b}^{\;\cdot\;\cdot\;\cdot\;d} \overset{b}{\underset{a}{u}}\, \overset{c}{u}_d = \sum_{b,d} \mathfrak{C}_{[\omega\mu|b|}^{\;\cdot\;\cdot\;\cdot\;d} (\nabla_{\xi]} \underset{a}{u^b}) \overset{c}{u}_d + \sum_{b,d} \mathfrak{C}_{[\omega\mu|b|}^{\;\cdot\;\cdot\;\cdot\;d} (\nabla_{\xi]} \overset{c}{u}_d)\, \underset{a}{u^b} =$$

$$= \sum_b \mathfrak{C}_{[\omega\mu|b}^{\;\cdot\;\cdot\;\cdot\;c} \Lambda_{a|\xi]}^{b} - \sum_b \mathfrak{C}_{[\omega\mu|a}^{\;\cdot\;\cdot\;\cdot\;b} \Lambda_{b|\xi]}^{c} .$$

For a symmetrical displacement follows from this equation for $a=0$:

$$(40)\qquad \sum_b \mathfrak{C}_{[\omega\mu|b|}^{\;\cdot\;\cdot\;\cdot\;c}\, \overset{b}{i}_{\xi]} = 0$$

or in orthogonal characteristic numbers:

$$(41)\qquad \mathfrak{C}_{[mli]}^{\;\cdot\;\cdot\;\cdot\;c} = 0,$$

an equation with the form of the second identity valuable for the quantity of curvature of a RIEMANN geometry.

§ 6. *On the* $\mathfrak{C}_n$ *fixed by a* RIEMANN *geometry given but for conformal transformations.* In X_n be given a fundamental tensor but for a scalar factor. Then we can ask to construct a conformal displacement transforming the orthogonal directions with respect to the fundamental tensor always into orthogonal directions. We fix the fundamental tensor in some way and choose n mutually perpendicular unit-vectors. The desired condition is then and only then fulfilled when these vectors can be found but for a factor from the $\overset{k}{i}_\lambda$ of § 2 by an orthogonal transformation. By choosing the coordinate systems and fixing the fundamental tensor in a right way, we can thus always attain that these vectors are identical with the $\overset{k}{i}_\lambda$. Then for the fundamental tensor the equation (18) holds.

From (33^a) the Λ_{lm}^{k} now can be solved. For there is one and only one symmetrical displacement in X_n that leaves $g_{\lambda\mu}$ invariant. For this displacement however holds:

$$(42)\qquad \nabla_m \overset{k}{i}_l = - \nabla_m \overset{l}{i}_k ;$$

hence (33^a) can with respect to (9) be transformed into:

$$(43)\qquad \nabla_m \overset{k}{i}_l + \nabla_l \overset{m}{i}_k = - \Lambda_{lm}^{k} - \Lambda_{kl}^{m},$$

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[1]) CARTAN finds equations corresponding to (38), which he calls: "théorème de la conservation de la courbure et de la torsion" l.c. (e) p. 183.

from which equation follows:

$$(44) \qquad \overset{k}{A}_{lm} = -\nabla_m \overset{k}{i_l} .$$

The conformal displacement would now be known if also $\overset{0}{A}_{i\mu}$ were known. When we substitute the values obtained in (44) into (26) for $a = i$, $c = k$, we obtain:

$$(45) \qquad \begin{aligned} {}^1/_2\, \mathfrak{C}_{\omega\mu i}^{\cdot\cdot\cdot k} &= \frac{\partial}{\partial x^{[\omega}} \nabla_{\mu]} \overset{k}{i_i} - \sum_j \overset{j}{A}_{i[\omega} \nabla_{\mu]} \overset{k}{i_j} - \overset{k}{i}_{[\omega} \overset{0}{A}_{|i|\mu]} + \overset{i}{i}_{[\omega} \overset{0}{A}_{|k|\mu]} = \\ &= \nabla_{[\omega} \nabla_{\mu]} \overset{k}{i_l} - \overset{k}{i}_{[\omega} \overset{0}{A}_{|i|\mu]} + \overset{i}{i}_{[\omega} \overset{0}{A}_{|k|\mu]} = {}^1/_2\, K_{\omega\mu i}^{\cdot\cdot\cdot k} - 2 g_{[k[\omega} \overset{0}{A}_{i]\mu]} \end{aligned}$$

or in orthogonal characteristic numbers:

$$(46) \qquad \mathfrak{C}_{mlik} = K_{mlik} + 4\, g_{[i[m} \overset{0}{A}_{k]\,l]} \text{ [1]},$$

in which K_{mlik} is the quantity of curvature of the RIEMANN geometry belonging to $g_{\lambda\mu}$.

Now $\overset{0}{A}_{kl}$ can be fixed by the assumption that $\sum_k \mathfrak{C}_{klik}$ vanishes. Then we find

$$(47) \qquad \overset{0}{A}_{kl} = \frac{1}{n-2} K_{kl} - \frac{1}{2(n-1)(n-2)} K g_{kl}; \quad K_{kl} = \sum_i K_{ikli} \qquad K = \sum_i K_{ii} ,$$

which equation can be solved because $\overset{0}{A}_{[kl]}$ vanishes after (32^a). Then $\mathfrak{C}_{mlik}$ becomes equal to the so-called *quantity of conformal curvature* C_{mlik}, belonging to all RIEMANN geometries that can be obtained by a conformal transformation of $g_{\lambda\mu}$, and hence $\mathfrak{C}_{mlik}$ is a conformal invariant.

Hence we have obtained the theorem:

When in X_n a fundamental tensor is fixed but for a factor, there exists one and only one symmetrical conformal displacement that possesses the orthogonality fixed by this fundamental tensor and for which the equation $\sum_k \mathfrak{C}_{klik} = 0$ holds [2]).

At the same time we get:

The characteristic numbers $\mathfrak{C}_{mlik}$ are the same as those of the quantity of conformal curvature C_{mlik} of X_n.

II. *The general projective displacement.*

§ 1. *The euclidean-projective manifold.* With $\mathfrak{P}_n^e$ we denote an

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[1]) For $\Lambda^0_{i\mu} = 0$ the conformal displacement passes into a Riemann one.

[2]) CARTAN, l.c. (*e*) p. 184.

euclidean-projective manifold (ordinary "projective space" of n dimensions). A point in $\mathfrak{P}_n^e$ can be given by $n+1$ homogeneous contravariant characteristic numbers $v^\gamma, \alpha, \beta, \gamma, \delta = b_1, \ldots b_{n+1}$. The quantities λv^γ and v^γ will be considered algebraically as different although they represent geometrically the same point. Each point can be written

$$(1) \qquad v^\gamma = v^\alpha \underset{\alpha}{u}^\gamma ,$$

in which equations $\underset{\alpha}{u}^\gamma$ are the *measuring points*. In the same way every hyperplane can be given by $n+1$ covariant characteristic numbers, w_α. These general characteristic numbers will be immediately changed into others. We choose $n+1$ arbitrary points $\underset{a}{u}^\gamma$, $a, b, c, d = 0, \ldots, n$, and call the $n+1$ hyperplanes of their $(n+1)$-edron $\overset{c}{u}_\alpha$; $\underset{a}{u}^\alpha \overset{a}{u}_\alpha = 1$. The characteristic numbers of a point v^γ or hyperplane w_α with respect to this measuring system are denoted by v^c resp. w_a:

$$(2) \qquad v^\gamma = \sum_c v^c \underset{c}{u}^\gamma \quad , \quad w_\alpha = \sum_a w_a \overset{a}{u}_\alpha .$$

We now only allow those measuring systems, that arise from the system $\underset{a}{u}^\gamma$, $\overset{c}{u}_\alpha$ by a linear homogenous transformation with a determinant $+1$. These systems are called *normal systems*.

Hence the quantities

$$(3) \qquad \begin{aligned} U^{c_0 \ldots c_n} &= \underset{0}{u}^{[c_0} \ldots \underset{n}{u}^{c_n]} \\ U_{a_0 \ldots a_n} &= \overset{0}{u}_{[a_0} \ldots \overset{n}{u}_{a_n]} \end{aligned}$$

are invariant. This does not contain any geometrical restriction, but the calculations become simpler. The group of transformations of the measuring systems into themselves has the geometrical meaning of the group of projective transformations of the $\mathfrak{P}_n^e$.

§ 2. *The general projective manifold.* To each point of an X_n be conjugated a $\mathfrak{P}_n^e$, in such a way that P itself is also a point of $\mathfrak{P}_n^e$ [1]). A relation between the $\mathfrak{P}_n^e$ in P and in a neighbouring point Q does not exist. It can be introduced by fixing in each point an

[1]) As an example we can take an X_n in $\mathfrak{P}_{n+m}^e$. Then we can consider as $\mathfrak{P}_n^e$ in P the $\mathfrak{P}_n^e$ tangential in P.

arbitrary normal system $\underset{a}{u}^c$ in such a way that $\underset{0}{u}^c$ is the point itself. Then a displacement can be defined by the equations

(4)
$$\delta \underset{a}{u}^c = \sum_{o} \Lambda_{a\mu}^{b}\, \underset{b}{u}^c\, dx^\mu = \Lambda_{a\mu}^{c}\, dx^\mu$$
$$\delta \overset{c}{u}_a = - \sum_{b} \Lambda_{b\mu}^{c}\, \overset{b}{u}_a\, dx^\mu = - \Lambda_{a\mu}^{c}\, dx^\mu,$$

in which $\Lambda_{a\mu}^{c}$ are $n(n+1)^2$ arbitrary parameters. We will now introduce the condition that normal systems always will pass into normal ones. This condition is

(5)
$$\sum_a \Lambda_{a\mu}^{a} = 0.$$

Then the equations for the measuring points are

(6)
$$\alpha)\quad \delta \underset{0}{u}^c = (\Lambda_{0\mu}^{0}\, \underset{0}{u}^c + \sum_j \Lambda_{0\mu}^{j}\, \underset{j}{u}^c)\, dx^\mu$$
$$\beta)\quad \delta \underset{i}{u}^c = (\Lambda_{i\mu}^{0}\, \underset{0}{u}^c + \sum_j \Lambda_{i\mu}^{j}\, \underset{j}{u}^c)\, dx^\mu$$
$$i, j, k, l = 1, \ldots, n.$$

It must be remarked, that the point P after (6^α) does *not* in general pass into Q. From (4) follows for the covariant differential of an arbitrary contravariant or covariant quantity of the first degree:

(7)
$$\delta v^c = dv^c + \sum_a \Lambda_{a\mu}^{c}\, v^a\, dx^\mu$$
$$\delta w_a = dw_a - \sum_c \Lambda_{a\mu}^{c}\, w_c\, dx^\mu\,.$$

The corresponding covariant differential quotients are

(8)
$$\nabla_\mu\, v^c = \frac{\partial v^c}{\partial x^\mu} + \sum_a \Gamma_{a\mu}^{c}\, v^a$$
$$\nabla_\mu\, w_a = \frac{\partial w_a}{\partial x^\mu} - \sum_c \Gamma_{a\mu}^{c}\, w_c\,.$$

From (3) follows that the differentials of $U^{c_0 \ldots c_n}$ and of $V_{a_0 \ldots a_n}$ vanish.

The displacement obtained in this way is called a *projective* displacement. The totality of the points of X_n with the $\mathfrak{P}_n^e$ conjugated to each point and the introduced displacement is called a *general* *projective manifold* $\mathfrak{P}_n$ [1]). A $\mathfrak{P}^e_n$ is a particular case of a $\mathfrak{P}_n$. Here to each point is conjugated the $\mathfrak{P}^e_n$ itself and the displacement transforms each point into itself.

Transformation of normal systems has no influence on the displacement, but has influence on the parameters $\Lambda^c_{a\mu}$. We only consider such transformations that leave $\underset{0}{u}^\gamma$ invariant but for a factor. Such a transformation has the form:

$$(9) \qquad {}'\underset{a}{u}^c = \Sigma_b P_a^b \underset{b}{u}^c \quad ; \quad \underset{a}{u}^c = \Sigma_b Q_a^b \, {}'\underset{b}{u}^c ,$$

in which there exists between the parameters P_a^b a number of simple relations of which we only need the following:

$$(10) \quad \begin{aligned} &\alpha) \quad P_0^k = Q_0^k = 0 \qquad\qquad P_0^0 \, Q_0^0 = 1 \\ &\beta) \quad P_{[i_1}^{[k_1} \dots P_{i_n]}^{k_n]} = P_0^0 \, A_{[i_1}^{[k_1} \dots A_{i_n]}^{k_n]} \end{aligned}$$

$$A_i^k = \begin{cases} 1, i = k \\ 0, i \neq k \end{cases} .$$

From (9) and (10) follows for the transformation of the normal characteristic numbers:

$$(11) \quad \begin{aligned} &{}'v_0 = P_0^0 \, v_0 & &{}'w^0 = Q_0^0 \, w^0 + \Sigma_j Q_j^0 \, w^j \\ &{}'v_i = P_i^0 \, v_0 + \Sigma_j P_i^j \, v_j & &{}'w^k = \Sigma_j Q_j^k \, w^j \end{aligned}$$

and for the transformation of $\Lambda^c_{0\mu}$:

$$(12) \quad \begin{aligned} &\alpha) \quad {}'\Lambda^0_{0\mu} = \Lambda^0_{0\mu} + \frac{\partial}{\partial x^\mu} \log P_0^0 + P_0^0 \, \Sigma_j Q_j^0 \, \Lambda^j_{0\mu} \\ &\beta) \quad {}'\Lambda^k_{0\mu} = P_0^0 \, \Sigma_j Q_j^k \, \Lambda^j_{0\mu}. \end{aligned}$$

From (12^β) follows, that the n parameters $\Lambda^k_{0\mu}$ can be considered as characteristic numbers of n covariant vectors of X_n, that are transformed linear with the transformation (9), and from (10^β) follows,

[1]) As an example we can change an X_n in $\mathfrak{P}^e_{n+1}$ into a $\mathfrak{P}^e_n$ by conjugating to each point P of X_n an arbitrary point C of $\mathfrak{P}^e_{n+1}$ *not* situated in the tangential $\mathfrak{P}^e_n$. A point in the tangential $\mathfrak{P}^e_n$ in P is then conjugated to its central projection from C on the tangential $\mathfrak{P}^e_n$ in a neighbouring point Q. CARTAN l.c. (c).

that for $P_0^0 = 1$ the determinant of the transformation is $+1$. For these n vectors and their reciprocals we write $\overset{k}{a}_\mu \; \underset{i}{a}^\nu$:

$$(13) \qquad \overset{k}{a}_\mu = \Gamma^k_{0\mu} \quad ; \quad \underset{i}{a}^\gamma \overset{k}{a}_\gamma = \begin{cases} 1, i = k \\ 0, i \neq k \end{cases}.$$

Then follows from (6^α) that with a displacement dx^μ with the characteristic numbers dx^j with respect to the measuring system $\underset{j}{a}^\gamma$ corresponds a displacement of $\underset{0}{u}^c$ in the direction of the point $\sum_j \underset{j}{u}^c dx^j$. Hence the directions of X_n in P are conjugated in a one-to-one way to those of the $\mathfrak{P}_n^e$ in P. From the equations (12^α) follows that it is always possible to choose the normal systems in such a way that $'\overset{0}{\Lambda}_{0\mu}$ everywhere vanishes. We thus assume $\overset{0}{\Lambda}_{0\mu}$ to vanish and only permit such transformations of normal systems that leave $\overset{0}{\Lambda}_{0\mu}$ zero, th. i. for which the equations hold:

$$(14) \qquad \sum_j Q_j^0 \overset{j}{a}_\mu = - Q_0^0 \frac{\partial}{\partial x^\mu} \log P_0^0 = \frac{\partial}{\partial x^\mu} Q_0^0 .$$

§ 3. *The quantity of curvature of* $\mathfrak{P}_n$. As in § 3 we assume from the displacement in X_n only that it is a symmetrical one. Then the equations (I 23) and (I 24) hold and in the same way as in I § 3 we find:

$$(15) \qquad 2 \nabla_{[\omega} \nabla_{\mu]} v^c = - \mathfrak{P}_{\omega\mu a}{}^{\cdot\cdot\cdot c} v^a ,$$

in which equation

$$(16) \qquad \mathfrak{P}_{\omega\mu a}{}^{\cdot\cdot\cdot c} = - 2 \frac{\partial}{\partial x^{[\omega}} \overset{c}{\Lambda}_{|a|\mu]} - 2 \overset{c}{\Lambda}_{b[\omega} \overset{b}{\Lambda}_{|a|\mu]} .$$

$\mathfrak{P}_{\omega\mu a}{}^{\cdot\cdot\cdot c}$ is the quantity of curvature of $\mathfrak{P}_n$. A $\mathfrak{P}_n^e$ is characterized by the vanishing of this quantity.

From (5) and $\Lambda^0_{0\mu} = 0$ follows:

$$(17) \qquad \begin{aligned} \sum_a \mathfrak{P}_{\omega\mu a}{}^{\cdot\cdot\cdot a} &= - \sum_a \Gamma^a_{0[\omega} \Gamma^0_{|a|\mu]} - \sum_{a,j} \Gamma^a_{j[\omega} \Gamma^j_{|a|\mu]} \\ &= - \sum_j \Gamma^j_{0[\omega} \Gamma^0_{|j|\mu]} - \sum_j \Gamma^0_{j[\omega} \Gamma^j_{|0|\mu]} = 0, \end{aligned}$$

an equation, which has the form of the condition of conservation of volume in an affine displacement.

§ 4. *Symmetrical projective displacements.* $\underset{0}{u}^c$ is invariant but for a factor. Hence the equation

$$(18) \qquad \underset{0}{u^a} \mathfrak{P}_{\omega\mu a}^{\cdot\cdot\cdot c} = \mathfrak{P}_{\omega\mu 0}^{\cdot\cdot\cdot c} = 0$$

is invariant. From (I 28), which equation holds also for $\mathfrak{P}_{\omega\mu a}^{\cdot\cdot\cdot c}$, follows, that (18) expresses that the point P returns in P when P is moved along a closed infinitesimal curve in the sense of the displacement. A displacement with this property is called *symmetrical.*

From (18) follows:

$$(19) \qquad {}^1/_2\, \mathfrak{P}_{\omega\mu 0}^{\cdot\cdot\cdot 0} = -\sum_j \overset{j}{a}_{[\omega} \overset{0}{A}_{|j|\mu]} = 0$$

$$(20) \qquad {}^1/_2\, \mathfrak{P}_{\omega\mu 0}^{\cdot\cdot\cdot k} = -\nabla_{[\omega} \overset{k}{a}_{\mu]} - \sum_j \overset{k}{A}_{j[\omega} \overset{j}{a}_{\mu]} = 0,$$

or, in normal characteristic numbers:

$$(19^a) \qquad \overset{0}{A}_{[ij]} = 0$$

$$(20^a) \qquad \nabla_{[m} \overset{k}{a}_{l]} = \overset{k}{A}_{[lm]}\,.$$

The remaining characteristic numbers are transformed in the following way:

$$(21) \qquad {}'\mathfrak{P}_{\omega\mu i}^{\cdot\cdot\cdot 0} = Q_0^0 \sum_j P_i^j \mathfrak{P}_{\omega\mu j}^{\cdot\cdot\cdot 0} + \sum_{jk} P_i^j Q_k^0 \mathfrak{P}_{\omega\mu j}^{\cdot\cdot\cdot k}$$

$$(22) \qquad {}'\mathfrak{P}_{\omega\mu i}^{\cdot\cdot\cdot k} = \sum_{j,l} P_i^j P_l^k \mathfrak{P}_{\omega\mu j}^{\cdot\cdot\cdot l}\,.$$

Hence the characteristic numbers $\mathfrak{P}_{\omega\mu i}^{\cdot\cdot\cdot k}$ are transformed into themselves and in such way, that they can be considered as the characteristic numbers of a quantity of X_n.

§ 5. BIANCHI's *identity*. In the same way as in I § 5 we get:

$$(23) \qquad \nabla_{[\xi} \mathfrak{P}_{\omega\mu]a}^{\cdot\cdot\cdot c} = 0.$$

This is BIANCHI's *identity* for a projective displacement. For a symmetrical displacement we can deduce from this equation in the same way as in I § 5:

$$(24) \qquad \mathfrak{P}_{[mli]}^{\cdot\cdot\cdot c} = 0.$$

For a symmetrical displacement can be found from (17) and (24):

$$(25) \qquad \sum_i \mathfrak{P}_{i[kl]}^{\cdot\cdot\cdot i} = 0.$$

§ 6. *On the $\mathfrak{P}_n$ that is fixed by the geodesic lines of an affine displacement.* We can now ask when a symmetrical projective displacement determines in the X_n the same geodesic lines as a given symmetrical affine displacement. The affine displacement be given with respect to the vectors $\underset{i}{a}^k$:

$$(26) \qquad \nabla_\mu \underset{i}{a}^k = \Sigma_j \Gamma^j_{i\mu} \underset{j}{a}^k = \Gamma^k_{i\mu}.$$

It can easily be proved that the projective displacement has then and only then the same geodesic lines as the affine displacement when

$$(27) \qquad A^k_{i\mu} = \Gamma^k_{i\mu} + p_i A^k_\mu + p_\mu A^k_i ,$$

in which p_λ is an arbitrary vector. Of all symmetrical affine displacements with the same geodesic lines only one has the properties of conservation of volume and of leaving invariant just the quantity $\underset{1}{a}^{[\nu_1} \ldots . \underset{n}{a}^{\nu_n]}$. If we assume that $\Gamma^k_{i\mu}$ are the parameters of this displacement, then $\Sigma_i \Gamma^i_{i\mu} = 0$ and from (5) and (27) follows $p_\lambda = 0$ or:

$$(28) \qquad A^k_{i\mu} = \Gamma^k_{i\mu} = \nabla_\mu \underset{i}{a}^k .$$

Substituting the value obtained in (28) into (16) for $a = i, c = k$, we find:

$$(29) \qquad \mathfrak{P}_{\omega\mu i}^{\cdot\cdot\cdot k} = R_{\omega\mu i}^{\cdot\cdot\cdot k} - 2 \overset{k}{a}_{[\omega} \overset{0}{A}_{|i|\mu]} ,$$

or, in normal characteristic numbers:

$$(30) \qquad \mathfrak{P}_{mli}^{\cdot\cdot\cdot k} = R_{mli}^{\cdot\cdot\cdot k} - 2 A^k_{[m} \overset{0}{A}_{|i|l]} \text{ [1]},$$

in which $R_{mli}^{\cdot\cdot\cdot k}$ is the quantity of curvature of the affine displacement determined by $\Gamma^k_{\mu i}$. Now $\overset{0}{A}_{il}$ can be fixed by the condition that $\Sigma_k \mathfrak{P}_{kli}^{\cdot\cdot\cdot k}$ vanishes. Then we find

$$(31) \qquad \overset{0}{A}_{il} = \frac{1}{n-1} R_{il} \quad , \quad R_{il} = \Sigma_k R_{kil}^{\cdot\cdot\cdot k},$$

which equations can be fulfilled because of the vanishing of $\overset{0}{A}_{[il]}$ and $R_{[il]}$ after (19^a) and because of the conservation of volume of the affine displacement. Then $\mathfrak{P}_{mli}^{\cdot\cdot\cdot k}$ becomes identical with the so-called

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[1]) For $\overset{0}{\Lambda}_{il} = 0$ the projective displacement passes into an affine one.

quantity of projective curvatare $P_{mli}^{\cdot\cdot\cdot k}$ belonging to all affine displacements with the same geodesic lines, and hence $\mathfrak{P}_{mli}^{\cdot\cdot\cdot k}$ is invariant with all transformations of the affine displacement, that leave geodesic lines invariant.

We thus have obtained the theorem:

When in an X_n *of an affine displacement only the geodesic lines are given, there exists one and only one symmetrical projective displacement, whose geodesic lines are identical with those given lines and for which the equation* $\sum_k \mathfrak{P}_{kli}^{\cdot\cdot\cdot k} = 0$ *holds* [1]).

At the same time we have found:

The characteristic numbers $\mathfrak{P}_{mli}^{\cdot\cdot\cdot k}$ *are the same as those of the quantity of projective curvature* $P_{mli}^{\cdot\cdot\cdot k}$ *of* X_n.

[1]) During the correction of the proofs of this communication Mr. CARTAN kindly send me proof of his new paper "Sur les variétés à connexion affine", Bull. Soc. Math. 1924. In this paper Mr. CARTAN has also given a proof of the here proved theorem.