Mathematics. — "*Representation of the Polar Triangles of a Conic Section on the Points of Space*". By Dr. G. SCHAAKE. (Communicated by Prof. HENDRIK DE VRIES).

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§ 1. Let a one-one correspondence be given between the points P of a conic k^2 lying in the plane α, and the points Q of a rational biquadratic twisted curve k^4 [1]). To a chord k of k^4 joining the two points Q_1 and Q_2 of this curve, we associate the point K of α where the tangents to k^2 at the points P_1 and P_2 corresponding resp. to Q_1 and Q_2, cut each other.

We shall show that the correspondence (K, k) leads us to a representation of the polar triangles of a conic different from k^2 on the points of space and we shall investigate the latter representation.

§ 2. If we choose K on k^2, P_1 and P_2 coincide in K and Q_1 and Q_2 in the point Q associated on k^4 to this point of k^2. The conic k^2 is accordingly the locus of the images K of the tangents to k^4. A tangent to k^2 is the image of the cubic cone which projects k^4 out of the point Q associated to the point of contact P.

The quadruples of points of k^2 that correspond to the quadruples of points of k^4 in which this curve is intersected by planes, form an involution I_3^4 of the fourth order and the third rank. The triples of points which together with a given point of k^2 form quadruples of I_3^4, define a cubic involution I_2^3 of the second rank. The neutral pair of points of I_2^3 forms with P a neutral triple of points of I_3^4. Hence the neutral triples of I_3^4 define a cubic involution of the first rank, I^3. I^3 consists of the triples of k^2 that are associated to the triples of k^4 lying on trisecants of the latter curve. A trisecant of k^4, which may be considered as a triple chord, is represented on the three angular points of the triangle circumscribed to k^2 the sides of which touch k^2 in the points of a triple of I^3. As I^3 has two pairs in common with the quadratic involution of the pairs of

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1) The properties of this curve which we shall mention in this paper, and the corresponding proofs are to be found in the thesis for the doctorate of Dr. D. J. E. SCHREK, "Rationale ruimtekrommen van den vierden graad", Utrecht, 1915.

points of k^2 in which lines cutting each other on a given line l of α, touch k^2, *the images of the trisecants of k^4 form a conic t^2 which is circumscribed to an infinite number of tangent-triangles of k^2*. The conics k^2 and t^2 cut each other in the four double points S_i of I^3, which are the images K of the trisecants t_i of k^4 touching this curve. The other two image points of a trisecant t_i coincide in the point T_i where the tangent to k^2 at S_i cuts the conic t^2 besides. The tangent to t^2 at T_i touches k^2 at the point that is associated to the point of k^4 where t_i cuts this curve.

The pairs of points of k^2 which together with two points coinciding in a given point of this conic, form a quadruple of I_3^4, define an involution with two double points. Accordingly the quadruples of I_3^4 the points of which coincide in pairs, form an involutorial (2,2)-correspondence I (2,2). The pairs of I (2,2) are associated to the pairs of points of k^4 where the double tangent planes σ of this curve touch k^4. The planes σ envelop a developable surface Δ of the fourth class and of the sixth order. For the projection of k^4 out of an arbitrary point has four double tangents and the involutorial (2,2)-correspondence of the pairs of points where generatrices of Δ cut k^4, has six pairs of points in common with the likewise involutorial (3,3)-correspondence of the pairs of points of intersection of k^4 with chords cutting a given straight line. Δ contains k^4 as a nodal curve. The generatrices of Δ are represented on the points K of a conic d^2, which, however, in this case is not circumscribed to a tangent triangle of k^2. The conics d^2 and k^2 cut each other in the four double points D_i of I (2,2), i.e. the four quadruple points of I_3^4 to which the points of k^4 with stationary planes of osculation are associated. A trisecant of k^4 touching k^4, is at the same time a generatrix of Δ as the plane through such a trisecant t_i containing also the tangent to k^4 at the point of intersection of t_i with k^4, touches this curve at the point of contact as well as at the point of intersection of t_i. Hence d^2 and t^2 intersect each other in the four points T_i.

§ 3. If K moves on a straight line r of α, k describes a scroll the generatrices of which cut k^4 in the pairs of the involution I^2 which corresponds to the involution of the pairs of points of k^2 where two straight lines intersecting each other on r, touch k^2. I^2 has three pairs of points in common with the involutorial (3,3)-correspondence in which the chords intersecting a given straight line, cut k^4. Hence *to a straight line r of points K there corresponds a cubic scroll ϱ^3 of chords k, which has k^4 as a single curve.*

The involution I^2 is defined by two pairs of points. Let k_1 and k_2 be the carriers of two pairs of points of I^2. The chords of k^4 cutting k_1, form a cubic surface with k^4 as a single curve as from the surface of the ninth order of the chords of k^4 cutting an arbitrary straight line l, which has k^4 as a triple curve, two cubic cones through k^4 split off. Outside k^4 the chord k_2 cuts one of the generatrices of this cubic surface, so that there is one chord k_ρ of k^4 intersecting k_1 and k_2 outside this curve. The chords of k^4 intersecting k_ρ define an involution on k^4 of which the pairs of points are the intersections of planes through k_ρ with k^4 and which has two pairs of points in common with I^2 and is therefore identical with I^2. *The scroll* ϱ^3 *associated to a straight line* r *of* α, *is accordingly the scroll of the chords of* k^4 *that cut a given chord* k_ρ. Inversely the chords of k^4 intersecting a given chord k_ρ, are represented on the points K of a straight line. This line is the locus of the points of intersection of pairs of tangents touching k^2 in pairs of points of the involution corresponding to the I^2 defined by k_ρ.

To the points of intersection of r with k^2, there correspond the generatrices of ϱ^3 which touch k^4 and lie therefore in the two planes through k_ρ touching k^4 outside k_ρ. The pole K of r relative to k^2 is therefore the image of the line k joining the points of contact of the tangents to k^4 that cut the said k_ρ. The points of intersection with t^2 are images of the trisecants of k^4 through the points of intersection of k_ρ with this curve, and the points of intersection with d^2 are the images K of the generatrices of $\varDelta$ which cut k_ρ outside k^4.

The chord k_ρ is a nodal line of ϱ^3, as two more chords of k^4 pass through a point of k_ρ. If we associate to each other the image points K of two chords of k^4 passing through the same point of k_ρ, we get an involution on r. As this involution has two double points, k_ρ contains two points for which the chords of k^4 through them coincide. For each of the said two points the plane through the two chords passing through a point of k_ρ, touches k^4 in the points of intersection of k^4 with the straight line in which the two chords through such a point coincide, so that this straight line is a generatrix of $\varDelta$. The double points of the involution defined on r, are accordingly the intersections of r with d^2. The images K of two chords of k^4 cutting each other outside k^4, lie on the line r corresponding to the third chord through their point of intersection, considered as a chord k_ρ, and are associated to each other relative to d^2.

Two chords k *of* k^4 *intersecting each other outside* k^4, *are accordingly represented on two points* K *of* α, *which are associated to each other relative to* d^2.

Hence:

The three chords of k^4 passing through an arbitrary point of space outside k^4, are always represented on the angular points of a polar triangle of d^2.

It is easily seen that also the inverse of this theorem holds good.

The angular points of the common polar triangle of d^2 and k^2 are the images of the cardinal chords or double osculation-chords of k^4 lying in the planes of osculation of the points of k^4 where each of the said chords rests on k^4. These chords pass through one point, the so-called cardinal point of the curve.

§ 4. We arrive at the representation announced in § 1 bij associating to each polar triangle φ of d^2 the point F of space where the chords k corresponding to the angular points K_i of φ, cut each other.

We remark that d^2 can coincide with an arbitrary conic of α. For we can also arrive at the correspondence (K, k) by starting from a one-one correspondence between the points of d^2 and the generatrices of the rational scroll Δ and by associating to a chord k of k^4 the point of intersection K of the tangents to d^2 at the points corresponding to the generatrices of Δ which cut k outside k^4.

To a curve of points F there corresponds a system of ∞^1 polar triangles φ. The order of such a system is the number of its polar triangles which have an angular point on a given line of α. The order of a system of ∞^2 polar triangles, which is therefore represented on a surface of points F, is the number of its individuals which have an angular point given in α.

§ 5. For the representation (φ, F) the points of k^4 are singular. If a point F' approaches a point F of k^4, the three chords of k^4 through that point get into the trisecant of k^4 through F and the two chords of k^4 through F lying in a tangent plane to k^4 at F. For a chord through F' for which F' and a point of intersection with k^4 approach each other, must finally coincide with one of the two chords through F which lie in the final position of the plane through F' and the tangent at F and the third chord through F' becomes the trisecant through F. Accordingly there correspond to F ∞^1 polar triangles of d^2 with one common angular point K_1 lying on t^2 whereas the other angular points of each are a pair of the involution defined by d^2 on the polar line r_1 of K relative to d^2. This may also be seen in the following way. The line r_1 is a tangent of k^2 and its points K are therefore the images of the chords of k^4 passing through the point Q of this curve which corresponds to the point of contact P of r_1. Each pair of the involution defined on r_1 by d^2, correponds to two chords lying in tangent planes to k^4 at Q, because the involution arising on r_1 if we associate the image points of each of such a pair of chords to each other, must also have its double points in the points of intersection of r_1 with d^2.

To a point F of k^4 there corresponds a system of ∞^1 polar triangles of the first order.

The ∞^2 polar triangles associated to points F of k^4, are those for which one angular point lies on t^2 and the subtending side touches k^2. They form a system Σ of the second order.

There exists therefore a rational quartic of singular points F. To each of these points there corresponds a linear system of ∞^1 polar triangles. The polar triangles associated to the points of k^4, form a quadratic system of ∞^2 individuals.

The polar triangles of d^2 circumscribed to k^2 and therefore inscribed in t^2, are singular for our representation. For if we choose for φ one of these triangles, the three chords k_i corresponding to the angular points K_i, coincide in a trisecant of k^4, on which the point F may be chosen at random.

There exists accordingly a quadratic system S of ∞^1 singular polar triangles to each of which there corresponds a straight line of points F. The locus of the points F which are associated to all the individuals of S, is the quadratic surface τ^2 of the trisecants of k^4.

§ 6. The polar triangles φ of a linear system of ∞^1 individuals have one common angular point and the other two lie on the polar line of this point relative to d^2. The point of intersection of the three chords k_i which correspond to the angular points K_i of such a triangle, lies therefore always on the chord k_1 associated to the fixed angle K_1.

A linear system of ∞^1 polar triangles is accordingly represented on a chord of k^4.

The points of intersection of this chord with k^4 are the images of the triangles of the system which have an angular point in one of the points of intersection of the polar line of K_1 relative to d^2, with t^2.

The image points of the polar triangles of d^2 which have an angular point on a given straight line r of α, lie on the chords k of k^4 that are associated to the points K of this straight line. *To such a system of ∞^2 polar triangles there corresponds therefore a cubic scroll ϱ^3 passing through k^4.*

To the points F of a straight line l of space there correspond ∞^1

polar triangles of d^2, which form a system A of the third order, because l has three points of intersection with a surface ϱ^3 and A contains therefore three triangles with an angular point on a given straight line r. To the points of intersection of l and τ^2 there correspond two triangles inscribed in t^2 and circumscribed to k^2. The locus of the angular points of the triangles of A is a cubic λ^3 circumscribed to an infinite number of polar triangles of d^2, but also to an infinite number of complete quadrilaterals the sides of which touch k^2. The six angular points of any of these quadrilaterals are the images K of the chords k of k^4 in a plane through l.

A straight line l of points F is the image of a cubic system A of ∞^1 polar triangles of d^2 which has two individuals in common with S.

If l cuts k^4, a linear system of polar triangles splits off from A. There remains a quadratic system that has one individual in common with S. The locus of the angular points is in this case a conic circumscribed to an infinite number of polar triangles of d^2 but also to an infinite number of tangent-triangles of k^2. The angular points of the former are points K of the chords through an arbitrary point of l, those of the latter are points K of the chords which lie in an arbitrary plane through l but do not intersect l on k^4.

The polar triangles φ which are associated to the points F of a plane V, form a linear system Φ of ∞^2 individuals. For there is one polar triangle that has an angular point in a given point K of α, to wit the triangle which is represented on the point of intersection of the chord k corresponding to K with V. As each trisecant of k^4 has one point in common with V, Φ contains all the polar triangles of d^2 inscribed in t^2.

Consequently a plane V is the image of a linear system of ∞^2 polar triangles that contains S.

§ 7. A twisted curve of the order n intersecting k^4 in m points, has in common with a surface ϱ^3 and with τ^2 resp. $3n—m$ and $2n—m$ points that are not singular for the representation.

Hence:

A curve of the order n cutting k^4 in m points, is the image of a system of ∞^1 polar triangles which has the order $3n—m$ and contains $2n—m$ individuals of S.

If inversely we have a system of ∞^1 polar triangles of the order p which contains q individuals of S, we find if we substitute for p and q resp. $3n—m$ and $2n—m$ and if we resolve n and m out of the equations arising in this way :

A system of ∞^1 *polar triangles of the order* p *which has* q *individuals in common with* S, *is represented on a curve of the order* $p-q$ *cutting* k^4 *in* $2p-3q$ *points.*

The image of a system of ∞^1 polar triangles of the order p which is general relative to S, is accordingly a curve of the order p which cuts k^4 in $2p$ points.

A surface of the order ν containing k^4 as a μ-fold curve, has in common with a bisecant of k^4 $\nu-2\mu$ points and with a trisecant $\nu-3\mu$ points which are not singular for the representation.

A surface of the order ν *of which* k^4 *is a* μ*-fold curve, is therefore the image of a system of* ∞^2 *polar triangles of the order* $\nu-2\mu$ *which contains* S $(\nu-3\mu)$ *times.*

Inversely it is easily seen that:

A system of ∞^2 *polar triangles of the order* π *containing* S $\varkappa$ *times, is represented on a surface of the order* π *of which* k^4 *is a* $(\pi-\varkappa)$*-fold curve.*

Consequently a system of ∞^2 polar triangles of the order π which is general relative to S, is associated to a surface of the order 3π which contains k^4 π times.

§ 8. The images of a system of ∞^1 polar triangles of the order p that is general relative to S, and of a similar system of ∞^2 polar triangles of the order π, which are resp. a curve of the order p cutting k^4 in $2p$ points, and a surface of the order 3π with a π-fold curve in k^4, have in common $3p\pi-2p\pi=p\pi$ points that are not singular for the representation.

Accordingly a system of ∞^1 *polar triangles of the order* p *and a system of* ∞^2 *polar triangles of the order* π *relative to the same conic, have* $p\pi$ *polar triangles in common.*

The images of two general systems of ∞^2 polar triangles of the order π and π' have in common besides k^4 a curve of the order $9\pi\pi'-4\pi\pi'=5\pi\pi'$ cutting k^4 in $10\pi\pi'$ points, because a point of intersection outside k^4 of this curve and τ^2 would cause these two surfaces to have a trisecant of k^4 in common so that the corresponding systems would contain the same polar triangle of S.

Hence:

Two systems of the order π *and* π' *of* ∞^2 *polar triangles of the same conic have in common a system of* ∞^1 *individuals of the order* $5\pi\pi'$.

The polar triangles of a system Π of ∞^2 individuals which have one angular point on a given straight line r and which therefore also belong to a linear system of ∞^2 polar triangles, form accordingly a system of ∞^2 individuals of the order 5π.

The locus of the angular points of the polar triangles of the latter system consists of the line r, counted π times, and a curve of the order 4π.

Consequently if a point K_1 describes a straight line r, the angular points different from K_1 of the polar triangles of π that have an angular point in K_1, describe a curve of the order 4π.

From the former of the two theorems derived in this § there follows:

Three systems of ∞^2 polar triangles of the same conic which have resp. the order π, π', and π'', have $5\pi\pi'\pi''$ individuals in common.

§ 9. A system Π of ∞^2 polar triangles of the order π gives an involutorial $(2\pi, 2\pi)$-transformation in α if we associate the angular points of the same triangle of Π to each other. This transformation contains two kinds of branch-points.

In the first place we have a single branch-point E_1 if the two angular points different from E_1 of a polar triangle of Π which has an angle in E_1 coincide. In this case the coinciding angular points lie on d^2. Such polar triangles are represented on points for which two of the three chords of k^4 through them coincide; these points lie therefore on the developable surface Δ which according to § 2 is of the sixth order and has k^4 as a nodal curve.

The image of Π has in common with the surface Δ outside k^4 a curve of the order $18\pi - 8\pi = 10\pi$, which cuts k^4 in 20π points so that the polar triangles of Π with two coinciding angular points, form a system of the order 10π. Each point of k^2 contains two angular points of π of these triangles, so that k^2 splits off 2π times from the locus of the angular points of the triangles of the latter system. There remains accordingly a curve of the order 6π for the angular points that do not lie on k^2.

The latter may also be seen in the following way. From the curve of intersection of the surface Δ with a surface ϱ^3 there split off k^4, counted double, and the two straight lines of Δ that are torsal lines of ϱ^3, also twice each, because along these lines Δ and ϱ^3 have the same tangent planes. There remains accordingly a curve of the sixth order which cuts k^4 twelve times and which is the image of the system of polar triangles of d^2 with two coinciding angles which have their third angular point on a given straight line r. This curve intersects the surface corresponding to Π in 6π points not singular for the representation, so that Π contains 6π single branch-points lying on a given straight line.

Consequently the locus of the single branch-points of the involutorial *transformation which is defined by a system of* ∞^2 *polar triangles, is a curve of the order* 6π.

The polar triangles of which the three angular points coincide, which can only happen in a point of d^2, are represented on points for which the three chords of k^4 passing through them, coincide. The projection of k^4 out of such a point on an arbitrary plane has a node of osculation in the point of intersection of the line in which the three chords coincide, with the plane of projection. In the tangent at this point there coincide three double tangents of the projection [1]). Hence the polar triangles with three coinciding angular points are represented on the points of the cuspidal curve σ of Δ. This curve passes through the points of the nodal curve k^4 of Δ that have two coinciding tangent planes to Δ. For such a point the involution in which planes through the tangent to k^4 cut k^4, becomes parabolic. In this case all the pairs of points must have one point in common; hence the tangent to k^4 in a common point of σ and k^4 must cut the latter curve in one more point. As we saw in § 2 there are four points on k^4 that have this property. As the two generatrices of Δ through such a point coincide in the tangent to k^4, σ and k^4 have there the same tangent. Therefore σ must have a cusp in such a point [2]). A projection of σ is a rational curve of the sixth class with six double tangents and four inflectional tangents. According to the second formula of PLÜCKER the order of σ is therefore equal to:

$$6 \times 5 - 2 \times 6 - 3 \times 4 = 6.$$

For the determination of this order we have not made use of the number of cusps. We can also determine this number by the aid of the first or the third formula of PLÜCKER if we remark that the number of nodes of a projection of σ is ten. Both formulas give indeed that σ must have four cusps.

As σ has three coinciding points of intersection with any tangent plane to k^4 in each of the four points which this curve has in common with k^4, σ and the surface corresponding to Π, must have 6π points of intersection that are not singular for the representation.

Accordingly a system of ∞^2 *polar triangles of the order* π *has* 6π *triangles with three coinciding angular points.*

For the linear system of the polar triangles that have an angular point on a given straight line, these six triangles coincide apparently in groups of three in the points of intersection of r with d^2.

[1]) SALMON-FIEDLER, Höhere ebene Kurven, Leipzig 1882, p. 279, 3.

[2]) ZEUTHEN, Abzählende Methoden, p. 143.

Hence the cubic scrolls through k^4 osculate the cuspidal curve of Δ twice outside k^4.

§ 10. The involutorial transformation defined by Π, has a double branch-point in a point K of α, if two of the π polar triangles of Π that have an angular point in K, coincide. The chord k of k^4 corresponding to K, must touch in this case the image surface ω of Π outside k^4. We ask how many of such chords there pass through a given point O of k^4.

We associate to each other the points in which a chord through O cuts the surface ω outside k^4. We get in this way ∞^1 pairs of points (P, Q) to which we applie the formula of SCHUBERT [1])

$$\varepsilon = p + q - g.$$

Here p an q are the numbers of pairs (P, Q) for which resp. P and Q lie in a given plane. The cubic cone projecting k^4 out of O, has besides k^4 an intersection with ω of the order 5π. To each of the 5π points in which a given plane cuts this curve, there correspond $\pi-1$ points which together with such a point of intersection form a pair (P, Q). Hence $p = q = 5\pi(\pi-1)$. Further g is the number of pairs (P, Q) for which the carriers cut a given straight line g. Each of the three lines of the cubic cone projecting k^4 out of O, carries $\pi(\pi-1)$ pairs of points. Hence $g = 3\pi(\pi-1)$.

We find that the number ε of the coincidences (P, Q) is equal to $7\pi(\pi-1)$. Some of these coincidences lie apparently in the points O_1 and O_2 which the trisecant through O has besides in common with k^4. The intersection outside k^4 of the cubic cone and ω has apparently in each of these points a π-fold point. A plane infinitely near to O_1 contains accordingly π points P infinitely near to O_1, to each of which there correspond $\pi-1$ similar points Q. Consequently in each of the points O_1 and O_2 there lie $\pi(\pi-1)$ coincidences (P,Q). The carriers of the remaining $5\pi(\pi-1)$ coincidences are chords of k^4 through O which touch ω outside k^4.

The point K corresponding to one of these chords, is the angular point of one polar triangle belonging to the linear system that corresponds to O. K lies accordingly on the straight line of the free angular points of this system. Hence:

The locus of the double branch-points of the involutorial transformation that is defined by a system of ∞^2 polar triangles of the order π, is a curve of the order $5\pi(\pi-1)$.

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[1]) **Kalkül der abzählenden Geometrie, p. 44.**

§ 11. We assume again a system Π of ∞^2 polar triangles of the order π. If a point K_1 describes a straight line r, the angular points different from K_1 of the polar triangles of Π which have an angular point in K_1, describe a curve $r^{4\pi}$, according to § 9. The pairs of points of this curve which correspond to the same point K_1 of r, lie on straight lines through the pole R of r. $r^{4\pi}$ has a 2π-fold point in R as the π triangles of Π that have an angular point in R, have 2π more angular points on r. [1])

$r^{4\pi}$ has further π-fold points in the points of intersection of r with d^2, because the triangles corresponding to these angular points have a second angular point in such an intersection [1]).

Accordingly

$$4\pi\,(4\pi-1)-2\pi\,(2\pi+1)-2\pi\,(\pi-1)=10\pi^2-10\pi$$

tangents may be drawn out of R to $r^{4\pi}$. The points of contact of 6π of these lines are coinciding angular points of polar triangles of Π and lie therefore on d^2. The remaining $10\pi\,(\pi-1)$ coincide in pairs in $5\pi\,(\pi-1)$ double tangents and are sides of individuals of Π in which two triangles associated to the same point of r, coincide.

In this way we have found a check on the results of the two preceding §§ and we have found at the same time:

The curve corresponding in a system Π to a straight line r, is of the genus $5\pi^2-4\pi+1$.

We remark that our results satisfy the formula of ZEUTHEN for the genus, applied to the correspondence between the points of r and of $r^{4\pi}$.

§ 12. The polar triangles that have an angular point on a curve o^{π} of the order π given in α, form a system of ∞^2 individuals of the order π. For the polar line of an arbitrary point K_1 of α cuts o^{π} in π points and each of these points forms with K_1 a pair of angular points of a polar triangle belonging to the said system. The image is a scroll consisting of the chords of k^4 which correspond to the points of o^{π}. According to § 8 it is a surface of the order 3π that has a π-fold curve $\omega^{3\pi}$ in k^4.

The systems corresponding in this way to two curves o^{π} and $o^{\pi'}$, have in common according to § 9 a system of ∞^1 individuals, which is of the order $5\pi\pi'$. The $\pi\pi'$ linear systems of the polar triangles that have an angular point in one of the points of intersection of the given curves, split off from this system. The locus of the angular

[1]) This may also be seen from the representation.

points of the remaining system of the order $4\pi\pi'$ contains o^{π} and $o^{\pi'}$, resp. π' and π times. There remains therefore a curve of the order $2\pi\pi'$.

For the polar triangles that have an angular point on each of two given curves of the order π and π', the locus of the third angular points is a curve of the order $2\pi\pi'$.

Hence:

There are $2\pi\pi'\pi''$ polar triangles that have an angular point on each of three given curves of the order π, π' and π''.

§ 13. If we take for o^{π} a conic o^2, there corresponds to it a scroll ω^6 that has k^4 as a nodal curve. Let us now assume that o^2 is circumscribed to one polar triangle of d^2. To the angular points of this polar triangle there correspond three chords k through the same point F_1 which is a triple point of ω^6. A plane through an arbitrary generatrix of ω^6 has also in common with this surface a curve of the fifth order which cuts this generatrix besides in the point of contact of the plane and in the two points of intersection with k^4, in two more points.

Two intersecting generatrices k of ω^6 correspond to two points K of o^2 associated to each other relative to d^2, and to each point of o^2 there correspond indeed two points of the same conic associated to it relative to d^2. Consequently besides k^4 ω^6 has another nodal curve. Let us assume a point F_2 on this curve. Among the ∞^6 cubic surfaces through k^4 there is one that has a conical point in F_1 and that contains the straight line $F_1 F_2$. The tangent plane at F_2 to this surface cuts it besides along a conic that passes through F_1 and F_2 and that cuts k^4 four times. We can at once point out thirteen points of intersection of this conic and ω^6 so that the conic lies entirely on ω.

Now the surface of the chords of k^4 that cut this conic outside k^4, has the order six, as it intersects the surface of the ninth order of the chords of k^4 that meet a given straight line, in six points outside k^4. Our surface ω^6 is accordingly identical with the surface of the chords of k^4 which cut a conic intersecting k^4 four times, and contains therefore a conic of triple points. To the three chords through a triple point of ω^6 there correspond three points of o^2 that are the angular points of a polar triangle of d^2. *By means of our representation we have found a proof of the theorem that a conic that is circumscribed to one polar triangle of another one, has an infinite number of inscribed polar triangles of the latter conic.*

We see that the property according to which the angular points of two polar triangles of the same conic lie on a conic, is equivalent to the one according to which there passes through two given points a conic that cuts k^4 four times. According to § 8 there corresponds to this conic a quadratic system that contains the two polar triangles which are represented on the given points.

§ 14. The triples of angular points of the individuals of a linear system Π of ∞^3 polar triangles of α, are triples of a cubic involution of the first rank in α. The triples of such an involution may be defined on α by the points of intersection with the twisted cubics of a congruence of REIJE which pass all through five given points. The involution has been investigated by Prof. VAN DER WOUDE[1]) independently from this definition.

According to § 8 the image of an arbitrary linear system of ∞^3 polar triangles is a cubic surface through k^4. There are ∞^6 of such surfaces.

A linear system of ∞^3 polar triangles relative to a given conic is therefore defined by six of its triangles.

If the conic is not given we have:

In a given plane there lie ∞^{11} linear systems of ∞^3 polar triangles.

If Π contains the polar triangles of d^2 inscribed in t^2, the system is represented on a plane. The properties of an arbitrary system Π may be derived from this representation. For we can always consider the cubic surface representing a general system Π, as belonging to a tissue of cubic surfaces that has as base curve k^4 and a conic p^2 which cuts k^4 four times. Now there exists a cubic transformation (F, F') which represents these surfaces on the planes of space. For k^4 and p^2 form together a possible degeneration of a twisted curve of the order six and the genus three which is the base curve of the tissue of the cubic surfaces on which the planes are represented through a cubic transformation[2]). Among the points F' there are two curves of singular points. To the points of one of them there correspond the trisecants of k^4, to the points of the other one the chords of k^4 that intersect p^2. As an arbitrary cubic surface of the tissue contains two trisecants of k^4 and four chords of k^4 which cut p^2, the singular curve corresponding to the surface of trisecants of k^4 is a conic p'^2, the other one is a rational quartic k'^4 which has four points in common with the singular conic, to wit the points corresponding to the trisecants of k^4 through the points of

[1]) These Proceedings Vol. XII, p. 751.

[2]) Cf. STURM: Geometrische Verwandtschaften, p. 370 and 392.

intersection of p^2 and k^4. Together with the representation on a plane of a cubic surface ω^3 of the tissue in consideration, there is also given the representation on this plane of the system Π corresponding to ω^3, which representation has only singular points in the points of intersection of the plane with the quartic of singular points F', as to the points of intersection with the singular conic trisecants of k^4, hence polar triangles of S, are associated. A closer examination shows that this representation is of the same nature as the representation on a plane of a system Π containing S, arising from the correspondence (φ, F).

The representation (φ, F') differs from (φ, F) only in this respect that in the former the quadratic system of ∞^1 polar triangles associated to p^2, takes the place which S has in the latter.[1])

We find accordingly that t^2 may be chosen along an arbitrary conic circumscribed harmonically to d^2 (and consequently k^2 along an arbitrary conic inscribed harmonically in d^2).

§ 15. Let us now consider a system Π that is represented on a plane ω. The four points of intersection T_i of ω with k^4 and the six chords of k^4 in ω give ten linear systems of polar triangles belonging to Π, each of which is defined by a point S and a straight line s, so that S is an angular point of all the polar triangles of such a system and the other angular points lie on the straight line s. The four points S_i of the systems corresponding to the points of intersection of ω with k^4, lie on t^2. To these there correspond four lines s_i touching k^2. A straight line $T_i T_k$ of ω represents the linear system of Π that has a triangle in common with each of the systems corresponding to T_i and T_k and of which the point S_{ik} lies accordingly in the intersection of s_i and s_k, the straight line s_{ik} along $S_i S_k$.

The line s_{14} of the system corresponding to $T_1 T_4$ passes through S_1 and S_4 but also through S_{23}, because $T_1 T_4$ and $T_2 T_3$ have a point of intersection that is not singular for the representation. In the same way it appears that $S_2 S_4$ joins the points S_2 and S_{13} and $S_3 S_4$ the points S_3 and S_{12}, and that these straight lines pass through S_4. Accordingly the triangle $S_1 S_2 S_3$ and the trilateral $s_1 s_2 s_3$ associated to it by a polar correspondence, lie perspectively. The joins of corresponding angular points are lines s of the linear systems belonging

[1]) As one trisecant and two chords intersecting p^2 pass through a point of k^4 there corresponds to this point a chord of k'^4 intersecting p'^2. To a point of p^2 through which there pass no trisecant and three bisecants, there corresponds a trisecant of k'^4.

to Π, the points of intersection of the corresponding sides, i.e. the poles of these lines s, are the corresponding points S. The ten points S are accordingly the angular points of the triangles $S_1 S_2 S_3$ and $s_1 s_2 s_3$, the points of intersection of the corresponding sides of these triangles, and the centre of perspectivity S_4; the ten straight lines s are the sides of the said triangles, the joins of corresponding angular points, and the axis of perspectivity s_4.

Of the cubic involution defined by Π the points S are apparently singular points, the lines s singular lines. They form a *configuration of* DESARGUES.

As each point of the plane ω belongs to a conic of the pencil that has the points T_i as base points, Π consists of the individuals of the quadratic systems corresponding to the pencil in question. The loci of the angular points of these systems are the conics of the pencil that has S_1, S_2, S_3 and S_4 as base points, which form a polar quadrangle of d^2. S_{12} is e.g. the pole of $S_1 S_2$ and S_{12} lies on the subtending side $S_3 S_4$ of the said quadrangle. *We see therefore that we can get any involution that may be derived from a linear system, by the aid of a pencil of conics that has the angular points of a polar quadrangle as base points, by associating to any a point of a the points where the polar line of the chosen point cuts the conic through it of the pencil in question.*

Inversely in this way we always get an involution of which the triangles of the point triples form a system Π because all the conics of a pencil that has the angular points of a polar quadrangle of d^2 as base points, are harmonically circumscribed to d^2.

We find that a system Π may be produced in five such ways. For to a plane pencil of rays in the plane ω that has a point T_i for vertex, there corresponds also a pencil of quadratic systems of Π. The conics corresponding in this way to the lines of the plane pencil that has for instance T_1 for vertex, all pass through S_1, S_{23}, S_{24} and S_{34}, through the latter three points because a straight line of this plane pencil cuts $T_2 T_3$, $T_2 T_4$ and $T_3 T_4$ in points that are not singular for the representation [1]).

§ 16. If a point K_1 describes a straight line r of a, the angular points K_2 and K_3 corresponding to K_1 in a linear system Π, describe according to § 12 a curve r^4, of the order four and the genus two

[1]) A straight line of ω is the image of the system of inscribed polar triangles of a cubic that is circumscribed to the complete quadrilateral (s_1, s_2, s_3, s_4). This gives a way of producing Π which has been indicated by Prof. JAN DE VRIES. These Proceedings XXI, p. 295. There are also five such ways.

that has a node in the pole R of r and that passes through the points of intersection of r with d^2. This curve also passes through the points S of the configuration of DESARGUES corresponding to Π, because whenever K_1 lies in a point of intersection of a straight line s with r, one of the two points K_2 and K_3 gets into the corresponding point S. The tangents of r^4 at R are sides of the polar triangle of Π that has R as an angular point and they cut r in the other two angular points of this triangle, through which r also passes, because to one of these points, considered as a point K_1, the other point corresponds as a point K_2 or K_3. The curve r^4 has therefore the property that the nodal tangents at the node R cut it in two more points the join of which has two more points in common with r^4, which separate the former two harmonically. It has been considered by Prof. V. D. WOUDE but not in this connection [1]).

The system of polar triangles of Π having an angular point on r, is represented on the intersection of the surface corresponding to Π and the surface ϱ^2 associated to r, that is a curve of the order five, k^6, which has ten points in common with k^4. We may consider the curve which is composed of k^4 and k^6, as the base curve of a pencil of cubic surfaces to which there corresponds a pencil of systems Π that have in common the polar triangles with one angular point on r and two on r^4. As in all these systems r is associated to the same curve r^4, r^4 passes through the points S of all the configurations of DESARGUES that are defined by the singular elements of these systems.

To any point K of r^4 there corresponds a chord k of k^4 which cuts k^6 and does not lie on ϱ^2. In the pencil in consideration there is one cubic surface which contains k. For the system Π of which this surface is the image, a point S lies in K. We see from this that any point of r^4 belongs to one of the configurations of DESARGUES inscribed in this curve.

Through a point of k^6 there passes one chord of k^4 belonging to ϱ^2, and there are two chords through this point that do not lie on ϱ^2. These chords are associated resp. to the angular point on r and to the angular points on r^4 of a polar triangle. As the chords through the same point of k^6 do not lie on the same cubic surface of the pencil, the angular point K_3 on r^4 of the polar triangle that has one singular point in a given point K_2 of r^4 and another one on r, is not a part of the configuration of DESARGUES to which K_2 belongs. Three

[1]) These Proceedings XXII, p. 645.

of the points of intersection of the polar line of K_2 with r^4 belong therefore to the same configuration of DESARGUES that is inscribed in r^4 as K_2, the fourth is, the same as K_2, an angular point of the polar triangle of which the third angle lies on r.

A similar curve may be produced by the aid of a pencil of conics that have the angular points of a polar quadrangle as base points. For it is locus of the points of intersection of the polar lines of the points of a straight line r with the conics of the pencil passing through these points. We may start from any pencil of conics as every quadrangle is a polar quadrangle relative to ∞^2 conics.

If we choose two arbitrary surfaces ω^3, the intersection k^6 is the image of a system of ∞^1 polar triangles of the fifth order because the pencil defined by these surfaces, contains no scroll. This system consists of the polar triangles that are inscribed in a curve r^5 of the fifth order. This curve has no double points as k^6 and k^4 have no chord in common that cuts k^6 only outside k^4, because this chord would lie on all surfaces of the pencil.

The curve r^5 contains the points S of all the individuals of the pencil of systems Π that is associated to the chosen pencil of surfaces ω^3, as any chord of k^4 lying on one of the surfaces of the latter pencil, intersects k^6. In the same way as in the preceding case we see that any point of r^5 is a part of a configuration of DESARGUES inscribed in this curve, and that three of the points of intersection of the polar line of a point K of r^5 with this curve belong to the same configuration of DESARGUES inscribed in r^5, as K, and that the other two are, together with K, the angular points of a polar triangle inscribed in r^5.

We have proved the existence of a curve r^5 of the order five and the genus six that has an infinite number of configurations of DESARGUES *inscribed in it.* The curve composed of r^4 and r is a special case of this curve.