Mathematics. — *"On Generalisations of the Notion of Absolute Convergence"*. By M. J. BELINFANTE. (Communicated by Prof. L. E. J. BROUWER.)

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In an article, entitled: "Sur la sommabilité absolue des séries par les moyennes arithmétiques" [1]) KOGBETLIANTZ has extended the notion of absolute convergence to summable series, and has proved some theorems concerning this so-called "absolute summability", which theorems are generalisations of the well known theorems of CAUCHY and MERTENS about the product of two infinite series.

If we substitute the words "joinable of order p" instead of "summable of order p" in these theorems of KOGBETLIANTZ, then we get generalisations of the theorems of CAUCHY and MERTENS, which I have proved in the articles "A Generalisation of MERTENS' Theorem" [2]), "On the Product and Summability of Infinite Series" [3]) and my dissertation [4]). This may be seen from the following scheme:

Theorems I. The product of a series wich is $\dfrac{\textit{absolutely summable}}{\textit{joinable}}$ *of order* p, *by a series which is* $\dfrac{\textit{absolutely summable}}{\textit{joinable}}$ *of order* q, *is* $\dfrac{\textit{absolutely summable}}{\textit{joinable}}$ *of order* $p+q$ [5]).

Theorems II. The product of a series which is $\dfrac{\textit{absolutely summable}}{\textit{joinable}}$ *of order* p, *by a series, which is summable of order* q, *is summable of order* $p+q$ [6]).

The theorems I are generalisations of CAUCHY's theorem to which both are reduced if we take $p=0$, $q=0$; the theorems II are generalisations of MERTENS' theorem to which both are reduced if we take $p=0$, $q=0$.

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1) Comptes Rendus de l'Academie des Sciences. t. 178, p. 295—298; 1924.

2) These Proceedings. Vol. 26, p. 203—215. The article will be cited as "Art Mert".

3) These Proceedings, Vol. 27, p. 33—45.

4) „Over oneindige reeksen". Noordhoff, Groningen 1923. p. 22—31.

5) KOGBETLIANTZ, l. c. p. 297 theorem IV.
„Over oneindige reeksen", p. 30 theorem 8.

6) KOGBETLIANTZ, l. c. p. 297 theorem V.
„Art. Mert.", p. 204.

We now will prove that the notions of absolute summability and joinability are not equivalent, and that series exist, which are absolutely summable of order p but not joinable of order p and also series which are joinable of order p but not absolutely summable of order p.

A series is joinable of order $p\,(p>0)$ if it is summable of order p and its mean-values of order $p-1$ are finite, or in other words: if it is summable (C, p) and finite $(C, p-1)$. A series is joinable of order zero if it converges absolutely.

A series Σa_n is absolutely summable of order p if the series of differences of two immediately following mean-values $\dfrac{S_n^{(p+1)}}{A_n^{(p+1)}}$ of order p converges absolutely, i.e. if the series

$$\sum_1^\infty \left\{\frac{S_n^{(p+1)}}{A_n^{(p+1)}} - \frac{S_{n-1}^{(p+1)}}{A_{n-1}^{(p+1)}}\right\}$$

converges absolutely [1]).

In order to find out whether the defined properties are dependent from each other, we reduce the n^{th} term $u_n^{(p)}$ of the above series as follows:

$$u_n^{(p)} = \frac{S_n^{(p+1)}}{A_n^{(p+1)}} - \frac{S_{n-1}^{(p+1)}}{A_{n-1}^{(p+1)}} = \frac{A_{n-1}^{(p+1)}\,S_n^{(p+1)} - A_n^{(p+1)}\,S_{n-1}^{(p+1)}}{A_n^{(p+1)}\,A_{n-1}^{(p+1)}}$$

$$= \frac{A_{n-1}^{(p+1)}\,S_n^{(p+1)} - A_{n-1}^{(p+1)}\,S_{n-1}^{(p+1)} + A_{n-1}^{(p+1)}\,S_{n-1}^{(p+1)} - A_n^{(p+1)}\,S_{n-1}^{(p+1)}}{A_n^{(p+1)}\,A_{n-1}^{(p+1)}}$$

$$= \frac{A_{n-1}^{(p+1)}\left[S_n^{(p+1)} - S_{n-1}^{(p+1)}\right] - S_{n-1}^{(p+1)}\left[A_n^{(p+1)} - A_{n-1}^{(p+1)}\right]}{A_n^{(p+1)}\,A_{n-1}^{(p+1)}}$$

$$= \frac{A_{n-1}^{(p+1)}\,S_n^{(p)} - S_{n-1}^{(p+1)}\,A_n^{(p)}}{A_n^{(p+1)}\,A_{n-1}^{(p+1)}}$$

$$= \frac{A_n^{(p)}}{A_n^{(p+1)}}\left\{\frac{S_n^{(p)}}{A_n^{(p)}} - \frac{S_{n-1}^{(p+1)}}{A_{n-1}^{(p+1)}}\right\}$$

$$= \frac{p}{p+n-1}\left\{\frac{S_n^{(p)}}{A_n^{(p)}} - \frac{S_{n-1}^{(p+1)}}{A_{n-1}^{(p+1)}}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; , \; . \; . \qquad (1)$$

[1]) For the notation see these Proceedings, Vol. 27, p. 34.

If Σa_n is joinable of order p then of course $\Sigma u_n^{(p)}$ is convergent; further there exists a positive number M so that

$$\left|\frac{S_n^{(p)}}{A_n^{(p)}}\right| < M \text{ for every } n.$$

The expression $\dfrac{S_{n-1}^{(p+1)}}{A_{n-1}^{(p+1)}}$ which tends to a limit for $n = \infty$, is also finite; hence the expression between brackets in (1) is finite. Therefore:

$$\left|u_n^{(p)}\right| < \frac{p}{p+n-1} \cdot k \text{ or:}$$

$$n\left|u_n^{(p)}\right| < k' \quad \ldots\ldots\ldots\ldots \quad (2)$$

Conversely: from the convergence of $\Sigma u_n^{(p)}$ together with the relation (2) it follows that Σa_n is joinable of order p, as may be seen by resolving (1) for the mean-value of order $p-1$:

$$\frac{S_n^{(p)}}{A_n^{(p)}} = \frac{S_{n-1}^{(p+1)}}{A_{n-1}^{(p+1)}} + \frac{p+n-1}{p \cdot n} \cdot n u_n^{(p)}.$$

Hence we see that the joinability of order p of Σa_n is equivalent with the condition $n\,|u_n^{(p)}| < k$ together with the convergence of $\Sigma u_n^{(p)}$, whereas the absolute summability of order p of Σa_n is by definition equivalent with the absolute convergence of $\Sigma u_n^{(p)}$.

In order to find out whether the absolute summability is a consequence of the joinability or vice-versa, we have only to investigate whether the absolute convergence of a series Σu_n implies $n|u_n| < k$ or conversely whether the absolute convergence follows from the convergence together with the condition $n\,|u_n| < k$. As will be well-known neither need be the case; a simple example that excludes the last possibility is the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ Examples that contradict the first possibility are generally less simple because for monotonic series the condition $|nu_n| < k$ is a consequence of the absolute convergence. The latter examples therefore consist of non-monotonic series; a wellknown example is the series:

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^{2/3}} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^{2/3}} + \ldots$$

It is interesting to observe that the properties of absolute convergence of Σu_n and the convergence of Σu_n together with

$|nu_n| < k$ (which characterize the difference between the notions of absolute summability and joinability) are sufficient to secure the convergence of the product of an arbitrary number of series, provided one of the two properties is valid for each of them [1]).

If a series $\Sigma u_n^{(p)}$ is arbitrarily given, it is possible to calculate the corresponding series Σa_n with the aid of the formulas of our previous article; therefore the examples above give after this calculation examples of series, which are absolutely summable but not joinable, respectively of series which are joinable but not absolutely summable of the same order. We also see that series of the last kind may have a less complicated character than those of the first kind: so the series $1 - 1 + 1 - \ldots$ is joinable of the first order but not absolutely summable of the first order [2]). That the product of this series by a series which is summable of order p is summable of order $p+1$ cannot be proved with the aid of KOGBETLIANTZ' generalisation of MERTENS' theorem, [3]) but follows immediately from the fact that it is joinable [4]) (see theorem II).

Another very important property which also often enables us to reduce the order of summability to a lower degree than that which is given by CESARO's [5]) rule, is the so-called index of summability which has been introduced by CHAPMAN [6]). A series which is summable of order x whatever be $x > p$ has an index of summability which is equal to p when it is not summable of any order $< p$ (the series may or may not be summable of order p). On a former occasion [7]), we have observed that sometimes the theorems concerning the joinability give more information than CHAPMAN's rule that the index of the product of two series cannot exceed the sum of the indices of the series by more than unity.

Since a series, whose index is equal to p is certainly summable of order $p+1$, the question arises whether there is some connection between the absolute summability, respectively the joinability of order $p+1$ and the index p. It may be seen from the following examples that these properties do not follow from each other.

1. *Example of a series with an index p which is not absolutely summable of order $p+1$.*

[1]) Proc. Lond. Math. Soc., Ser 2, Vol. 11, 1913 (p. 464).

[2]) KOGBETLIANTZ, l. c. p. 296.

[3]) KOGBETLIANTZ, l. c. p. 297.

[4]) „Art. Mert." p. 211.

[5]) Sur la multiplication des séries. Bulletin des Sciences Mathématiques, 2e série, t. 14, p. 114–120.

[6]) Proc. Lond. Math. Soc., Ser. 2, Vol. 9, p. 369—409, 1911.

[7]) „Art. Mert." p 211.

The series $1 - 1 + 1 - 1 + \dots$ has an index [1] zero, but is not absolutely summable of order 1.

2. *Example of a series which is absolutely summable of order* $p+1$, *but whose index exceeds* p.

The series

$$2-1+1-\frac{3}{4}+0+0+\frac{8}{9}-\dots+0+\frac{2r}{r^2}-\frac{2r-1}{r^2}+\overbrace{0+\dots+0}^{2^r-2 \text{ zeros}}+\frac{2r+1}{(r+1)^2}+.$$

is, as mentioned by Kogbetliantz without proof, absolutely summable of the first order but not summable of any order <1; the index therefore exceeds 0.

3. *Example of a series whose index is* p, *but which is not joinable of order* $p+1$.

The series

$$1-\left[1+\frac{1}{2}\right]+\left[1+\frac{1}{2}+\frac{1}{3}\right]-\left[1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right]+\dots$$

has an index equal to zero; indeed it is the product-series of

$$1-1+1-1+\dots \text{ and } 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots$$

which have the indices 0 and -1 respectively [2]; hence the index of the product does not exceed $0-1+1=0$, and since the series does not converge the index must be 0. The series is not joinable of order 1, since the partial sums:

$$1, -\frac{1}{2}, \quad 1+\frac{1}{3}, \quad -\frac{1}{2}-\frac{1}{4}, \quad 1+\frac{1}{3}+\frac{1}{5}, \dots$$

grow infinite.

I have not found an example of a series which is joinable of order $p+1$, but whose index would exceed p. Hence it might occur that each series which is joinable of order $p+1$ would have an index not less than p.

We will now give a condition which when satisfied for a series with an index p guarantees that the series is joinable of order $p+1$.

That the index of the series Σa_n is p, is expressed by the following relation:

$$\lim_{n=\infty} . C_n^{(p+i)} = s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

if $i>0$.

($C_n^{(k)}$ is the n^{th} mean-value of order k).

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[1]) Chapman, l. c., p. 378.

[2]) Proc. Lond. Math. Soc., Ser. 2, Vol. 11, p. 462, 1913.

It is not allowed to infer from the existence of (3) that there exists a number M such that:

$$\left|C_n^{(p+i)}\right| < M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

for $i > 0$. (In that case we call the mean-values of order $p+i$ uniformously limited for $i > 0$).

Indeed, although it is possible to calculate a number M_i for every $i > 0$ such that (4) is satisfied, it might happen that the values of M_i could increase to ∞ with decreasing i. It will be seen that this is the case with the series of example (3).

We now prove that *if* the relation (4) is satisfied whatever be $i > 0$, it is also satisfied for $i = 0$ if M is replaced by another finite number M', i.e. we prove that it is possible to calculate a finite number M' so that

$$\left|C_n^{(p)}\right| < M' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

whatever be n, which implies that $\Sigma\, a_n$ is joinable of order $p+1$.

To prove this property (which is not self-evident), we introduce the quantities $A_n^{(-k)}$ for $0 < k < 1$ by the following definitions:

$$\left.\begin{aligned} &A_1^{(-p)} A_1^{(p)} = 1 \\ &A_n^{(-p)} A_1^{(p)} + A_{n-1}^{(-p)} A_2^{(p)} + \ldots + A_1^{(-p)} A_n^{(p)} = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad (I)$$

The quantities A with negative upper-indices satisfy the same kind of relations as the A's with positive indices. In particular we have:

$$A_1^{(p+i)} A_n^{(-i)} + A_2^{(p+i)} A_{n-1}^{(-i)} + \ldots + A_n^{(p+i)} A_1^{(-i)} = A_n^{(p)} \quad . \quad . \quad (II)$$

$$S_1^{(p+i)} A_n^{(-i)} + S_2^{(p+i)} A_{n-1}^{(-i)} + \ldots + S_n^{(p+i)} A_1^{(-i)} = S_n^{(p)} \quad . \quad (III)$$

$$\left.\begin{aligned} &A_1^{(-p)} = 1 \\ &A_n^{(-p)} = \frac{(-p)\cdot(-p+1)\ldots\ldots(-p+n-2)}{1\,.\,2\ldots\ldots(n-1)} \end{aligned}\right\} \quad . \quad . \quad (IV)$$

These relations are proved by induction from the definitions (I) and the formulas for positive indices [1]). From (IV) it is evident that the A's with negative upper-indices are all negative except $A_1^{(-p)}$.

Now, if $\left|C_n^{(p+i-1)}\right| < M$ or $\left|\dfrac{S_n^{(p+i)}}{A_n^{(p+i)}}\right| < M$

whatever be n, then it follows from (III):

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[1]) These Proceedings, Vol. 27, p. 34.

$$\left| S_n^{(p)} \right| < \left| S_1^{(p+i)} A_n^{(-i)} + S_2^{(p+i)} A_{n-1}^{(-i)} + \ldots + S_{n-1}^{(p+i)} A_2^{(-i)} \right| + \left| S_n^{(p+i)} A_1^{(-i)} \right|$$
$$< M \left| A_1^{(p+i)} A_n^{(-i)} + \ldots + A_{n-1}^{(p+i)} A_2^{(-i)} \right| + M A_n^{(p+i)}$$
$$< M \left| A_1^{(p+i)} A_n^{(-i)} + \ldots + A_n^{(p+i)} A_1^{(-i)} - A_n^{(p+i)} A_1^{(-i)} \right| + M A_n^{(p+1}$$
$$< M \left| A_n^{(p)} - A_n^{(p+i)} \right| + M A_n^{(p+i)}$$
$$< M A_n^{(p)} + 2\, M A_n^{(p+i)}$$

$$\left| \frac{S_n^{(p)}}{A_n^{(p)}} \right| < M + 2\, M \frac{A_n^{(p+i)}}{A_n^{(p)}}.$$

Since this relation is valid whatever be i, we may take i for a given n so that $\dfrac{A_n^{(p+i)}}{A_n} < 2$. Indeed

$$\frac{A_n^{(p+i)}}{A_n^{(p)}} = \left(1 + \frac{i}{p}\right)\left(1 + \frac{i}{p+1}\right) \ldots \left(1 + \frac{i}{p+n-2}\right) < \left(1 + \frac{i}{p}\right)^n.$$

So, if we take i less than $p(\sqrt[n]{2} - 1)$ we have $\dfrac{A_n^{(p+i)}}{A_n^{(p)}} < 2$. Hence

$\left| \dfrac{S_n^{(p)}}{A_n^{(p)}} \right| < 5M$ or $\left| C_n^{(p-1)} \right| < 5M$ whatever be n.

We have proved that a series which is summable of order p and whose mean-values of order $p-1+i$ are uniformously limited for $i > 0$ is joinable of order p. If the index of the series is $p-1$, then the mean-values are limited for $i > 0$ but need not be uniformously limited. A sufficient condition that a series with index $p-1$ shall be joinable of order p is the condition that the mean-values of order $p+i-1$ are uniformously limited for $i > 0$. As the series of example 3 had an index 0 but was not joinable of order 1, it is clear that we cannot infer from the fact that the index is p that the mean-values of order $p+i$ are uniformously limited for $i > 0$.

Note to the article "On the Product and Summability of Infinite Series." (These Proceedings, Vol. 27 p. 33—45).

The formulas (3) and (5) on page 34 may only be deduced from the formulas (A) and (B) on page 35 if p or q or one of them are integer. If p and q are not integer then the proof cannot be given in that way; then however the proof is superfluous, as the equality of coefficients has already been inferred from the former case. See my dissertation p. 6.