Mathematics. — *"The equivalence in R_n of the n-dimensional simplex star and the spherical neighbourhood."* By WILFRID WILSON. (Communicated by Prof. L. E. J. BROUWER.)

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The object of this paper is to prove the theorems I and II stated beneath. So far as the writer is aware these theorems have not yet been explicitly stated and proved, but have been implicitly used in several topological investigations. Some related theorems are stated and proved by H. KNESER in a paper in these Proceedings, the proof sheets of which I have seen through the intermediation of Prof. BROUWER. [1])

The Simplex star.

In the m-dimensional number space R_m consider a finite number of n-dimensional simplexes, ($n \leqslant m$) of common vertex A_0 and such that:

(a). Any (n—1)-dimensional face of vertex A_0 is common to two and only two n-dimensional simplexes;

(b). Any two n-dimensional simplexes have in common either (1) no point other than A_0, or (2) one p-dimensional face and all its (p—k)-dimensional faces, ($p \leqslant n-1$; $k = 1, 2, \ldots, p$).

The set of points constituting these simplexes and their boundaries is called an n-dimensional simplex star of centre A_0; those (n—p)-dimensional faces, ($p = 1, 2, \ldots, n$), of which A_0 is not a vertex are called the boundary of the star while the remaining points are called the interior.

Regular subdivision of a simplex star. [2])

Let α_i be the number of i-dimensional simplexes, ($i = 0, 1, \ldots, n$), of the n-dimensional star S_n, so that any simplex of S_n may be written a_j^i, ($i = 0, 1, \ldots, n$; $j = 1, 2, \ldots, \alpha_i$), the a_j^0, ($j = 1, 2, \ldots, \alpha_0$), being the vertices. In the interior of a_j^i take an arbitrary point P_j^i, P_j^0 being the vertex a_j^0, and subdivide S_n into a set $\overline{S}$ of n-dimensional simplexes, in the following way:

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[1]) H. KNESER, "Ein topologischer Zerlegungssatz", § 1, Satz 3 (m) and 4 (m), these Proceedings 27, p. 603.

[2]) VEBLEN, Cambridge Colloquium, Analysis Situs, p. 85—86.

(0) The vertices of $\overline{S}$ are the points P_j^i, $(i=0,1,\ldots,n; j=1,2,\ldots,\alpha_i)$.

(1) The 1-dimensional simplexes of $\overline{S}$ are the segments which join the point P_j^i to each vertex of $\overline{S}$ in the boundary of a_j^i, $(i=1,2,\ldots,n; j=1,2,\ldots,\alpha_i)$.

$\vdots$

(k) A k-dimensional simplex of $\overline{S}$ is the set of points on all segments joining a point P_j^i to the points of a $(k-1)$-dimensional simplex of $\overline{S}$ in the boundary of a_j^i, $(i=k,k+1,\ldots,n; j=1,2,\ldots,\alpha_i)$.

$\vdots$

(n) An n-dimensional simplex of $\overline{S}$ is the set of points on all segments joining a point P_j^n to the points of an $(n-1)$-dimensional simplex of $\overline{S}$ in the boundary of a_j^n, $(j=1,2,\ldots,\alpha_n)$.

It follows from these definitions that any n-dimensional simplex of $\overline{S}$ has the form $P_q^0 P_r^1 \ldots P_s^n$ and that the number of simplexes in $\overline{S}$ is finite (being $(n+1)!$ times the number in S_n).

Theorem 1. *Any point P of the interior of an n-dimensional star S_n is the centre of an n-dimensional star S_n' composed of simplexes of a regular subdivision $\overline{S}$ of S_n.*

Let P be in the k-dimensional simplex a_s^k of S_n, $(0 \leqslant k \leqslant n)$. Choose the subdivision $\overline{S}$ so that $P_s^k = P$, and let

$$P_l^0 P_m^1 P_n^2 \ldots P_p^{j-1} P_q^j P_r^{j+1} \ldots P_s^k \ldots P_v^{n-1} P_w^n = \bar{a}_w^n$$

be any n-dimensional simplex of $\overline{S}$ of vertex P_s^k.

From the definitions (0), (1), ..., (n) it follows that

$P_l^0 P_m^1 P_n^2 \ldots P_p^{j-1} P_q^j P_r^{j+1} \ldots P_s^k \ldots P_v^{n-1} P_w^n$ is in the simplex a_w^n of S_n,

$P_l^0 P_m^1 P_n^2 \ldots P_p^{j-1} P_q^j P_r^{j+1} \ldots P_s^k \ldots P_v^{n-1}$ is in the $(n-1)$-dimensional face a_v^{n-1} of a_w^n,

$P_l^0 P_m^1 P_n^2 \ldots P_p^{j-1} P_q^j P_r^{j+1} \ldots P_s^k$ is in the k-dimensional face a_s^k of a_v^{n-1},

$P_l^0 P_m^1 P_n^2 \ldots P_p^{j-1} P_q^j P_r^{j+1}$ is in the $(j+1)$-dimensional face a_r^{j+1} of a_s^k,

$P_l^0 P_m^1 P_n^2 \ldots P_p^{j-1} P_q^j$ is in the j-dimensional face a_q^j of a_r^{j+1},

$P_l^0 P_m^1 P_n^2 \ldots P_p^{j-1}$ is in the $(j-1)$-dimensional face a_p^{j-1} of a_q^j,

$P_l^0 P_m^1 P_n^2$ is in the 2-dimensional face a_n^2 of a_p^{j-1}

$P_l^0 P_m^1$ is in the edge a_m^1 of a_n^2 and

P_l^0 is the vertex a_l^0 of a_m^1.

Since P^k_s is an interior point of S_n, the (n—1)-dimensional face a^{n-1}_v containing it, must have the centre of S_n as a vertex and therefore by (a) of the definition of a star, a^{n-1}_v is common to two and only two n-dimensional simplexes a^n_w and $a^n_{w'}$ of S_n.

By the concluding remark of the previous paragraph the number of the simplexes $\bar{a}^n_w$ is finite. We require to prove that they satisfy conditions (a) and (b) of the definition of a simplex star.

(a) The (n—1)-dimensional faces of $\bar{a}^n_w$ of vertex P^k_s are

(1) $P^1_m P^2_n \ldots P^{j-1}_p P^j_q P^{j+1}_r \ldots P^k_s \ldots P^{n-1}_v P^n_w$

(2) $P^0_l P^1_m P^2_n \ldots P^{j-1}_p P^{j+1}_r \ldots P^k_s \ldots P^{n-1}_v P^n_w$,

$(j = 1, 2, \ldots k-1, k+1, \ldots, n-1)$, and

(3) $P^0_l P^1_m P^2_n \ldots P^{j-1}_p P^j_q P^{j+1}_r \ldots P^k_s \ldots P^{n-1}_v$.

(1) Let P^0_l and $P^0_{l'}$ be the vertices of a^1_m. Then:

By definition (1) of Subdivision, $P^0_{l'} P^1_m$ is an edge of $\overline{S}$, hence by definition (2) of subdivison, $P^0_{l'} P^1_m P^2_n$ is a 2-dimensional simplex of $\overline{S}$. Applying definitions (3), (4), ..., (n—1) and (n) in succession we prove that $P^0_{l'} P^1_m P^2_n \ldots P^{j-1}_p P^j_q P^{j+1}_r \ldots P^k_s \ldots P^{n-1}_v P^n_w$ is an n-dimensional simplex of $\overline{S}$. Thus $P^1_m P^2_n \ldots P^{j-1}_p P^j_q P^{j+1}_r \ldots P^k_s \ldots P^{n-1}_v P^n_w$ is common to two n-dimensional simplexes of $\overline{S}$. Any other n-dimensional simplex having the face $P^1_m P^2_n \ldots P^n_w$ must be of the form $P^0_{l''} P^1_m \ldots P^n_w$ and by definition (1) of the regular subdivision, $P^0_l P^1_m$ and $P^0_{l'} P^1_m$ are the only edges of $\overline{S}$ of the form $P^0_{l''} P^1_m$ incident with P^1_m. Therefore l'' must be l or l'. Thus

$$P^1_m P^2_n \ldots P^{j-1}_p P^j_q P^{j+1}_r P^k_s P^{n-1}_v P^n_w$$

is incident with two and only two n-dimensional simplexes of $\overline{S}$.

(2) The (j—1)-dimensional face a^{j-1}_p is incident with two j-dimensional faces a^j_q and $a^j_{q'}$ of a^{j+1}_r. Therefore by definition (j) of regular subdivision there are two j-dimensional simplexes $P^0_l \ldots P^{j-1}_p P^j_q$ and $P^0_l \ldots P^{j-1}_p P^j_{q'}$ of $\overline{S}$ and by applying successively the definitions (j), (j+1), ..., (n) of regular subdivision we obtain two simplexes $P^0_l \ldots P^{j-1}_p P^j_q P^{j+1}_r \ldots P^n_w$ and $P^0_l \ldots P^{j-1}_p P^j_{q'} P^{j+1}_r \ldots P^n_w$ of $\overline{S}$ having $P^0_l \ldots P^{j-1}_p P^{j+1}_r \ldots P^n_w$ as common (n—1)-dimensional face. Any other n-dimensional simplex having this face must be of the form:

$P_l^0 \dots P_p^{j-1} P_{q''}^j P_r^{j+1} \dots P_w^n$, where by definition $(j+1)$ of regular subdivision $a_{q''}^j$ is a j-dimensional face of a_r^{j+1} incident with a_p^{j-1}, that is, $a_{q''}^j$ is either a_q^j or $a_{q'}^j$ and thus $P_{q''}^j$ is either P_q^j or $P_{q'}^j$. Thus $P_l^0 \dots P_p^{j-1} P_r^{j+1} \dots P_w^n$ is an $(n-1)$-dimensional face of two and only two n-dimensional simplexes of $\overline{S}$.

(3) The $(n-1)$-dimensional face a_v^{n-1} being common to two and only two n-dimensional simplexes a_w^n and $a_{w'}^n$ of S_n, it follows from definition (n) of regular subdivision that the $(n-1)$-dimensional face $P_l^0 P_m^1 \dots P_v^{n-1}$ of $\overline{S}$ is common to two and only two simplexes $P_l^0 P_m^1 \dots P_v^{n-1} P_w^n$ and $P_l^0 P_m^1 \dots P_v^{n-1} P_{w'}^n$ of $\overline{S}$. Thus the simplexes of $\overline{S}$ of vertex P_s^k satisfy condition (a) of the definition of a star.

(b) Consider first the simplexes of $\overline{S}$ in a_w^n.

By definition (1) of regular subdivision any two edges of the subdivision of a_w^n have either no point or a vertex in common. Hence by definition (2) any two 2-dimensional faces of the subdivision of a_w^n have either no point or one vertex or one edge in common. Hence by definitions (3), (4), ..., $(n-1)$ any two $(n-1)$-dimensional faces of the subdivision of a_w^n have either no point or one p-dimensional face in common, $(0 \leqslant p \leqslant n-2)$. Finally by definition (n), any two n-dimensional simplexes of vertex P_w^n of the subdivision of a_w^n have one p-dimensional face in common, $(0 \leqslant p \leqslant n-1)$.

Consider now two n-dimensional simplexes $\overline{a}_1^n$ and $\overline{a}_2^n$ of $\overline{S}$ in the simplexes a_1^n and a_2^n of S_n. Then if a_1^n and a_2^n have no common point, $\overline{a}_1^n$ and $\overline{a}_2^n$ have no common point. If a_1^n and a_2^n have in common a p-dimensional face, $(0 \leqslant p \leqslant n-1)$, then by definition (n), $\overline{a}_1^n$ and $\overline{a}_2^n$ have either no point or a q-dimensional face in common, $(0 \leqslant q \leqslant p)$ that is $(0 \leqslant q \leqslant p \leqslant n-1)$.

Thus any two n-dimensional simplexes of $\overline{S}$ have either no point or a p-dimensional face in common, $(0 \leqslant p \leqslant n-1)$ and in particular any two n-dimensional simplexes of $\overline{S}$ of vertex P_s^k have either no point other than P_s^k or a p-dimensional face, $(1 \leqslant p \leqslant n-1)$, in common.

Thus the simplexes $\overline{a}_w^n$ of vertex P_s^k constitute a simplex star of centre P_s^k.

Theorem II. *In R_n, any n-dimensional simplex star of centre A_0 contains an n-dimensional spherical region of centre A_0.*

The proof falls into two parts:

(1) If the theorem is true for $n = (p-1)$, it is true for $n = p$.

(2) The theorem is true for $n = 1$.

(1) We assume then that in R_{p-1}, any $(p-1)$-dimensional simplex star of centre P_n^k contains a $(p-1)$-dimensional spherical region of centre P_n^k. Consider in R_p a p-dimensional star S_p of centre A_0 and let $U(A_0)$ be a p-dimensional spherical neighbourhood of centre A_0 and radius r, where r is less than the distance of A_0 from any point of the boundary of S_p. Let P_1 be any point of $U(A_0)$ in the simplex a^p of S_p and P_2 any point of $U(A_0)$ not in a^p and not in the line $P_1 A_0$. We require to prove that P_2 is in S_p.

Let $a_l^i, a_m^j, \ldots, a_n^k$ $(i, j, \ldots, k \leqslant p-1)$, be the set of all simplexes, finite in number each of which contains one and only one point of the segment $P_1 P_2$. Let $P_1 P_2$ intersect $a_l^i, a_m^j, \ldots, a_n^k$ in the points $P_l^i, P_m^j, \ldots, P_n^k$ respectively and let P_n^k be the nearest of these points to P_2. (Assume P_n^k to be different from P_2, for if $P_n^k = P_2$, then P_2 is in S_p). Since $U(A_0)$ contains no boundary points of S_p the simplex a_n^k containing P_n^k must be of the form $A_0 A_1 \ldots A_k$. Let $a_n^k = A_0 A_1 \ldots A_k$ be a k-dimensional face of the simplex $A_0 \ldots A_k \ldots A_p$ of S_p and let $A_1', A_2', \ldots, A_{p-1}'$ be points of $A_0 A_1, A_0 A_2, \ldots, A_0 A_{p-1}$ respectively such that the simplex $A_1' A_2' \ldots A_k'$ contains P_n^k but such that the R_{p-2} determined by $A_1', A_2', \ldots, A_{p-1}'$, does not contain P_2. Then the R_{p-1} determined by $A_1', A_2', \ldots, A_{d-1}', P_2$ contains the segment $P_n^k P_2$ and intersects $A_0 A_1$ in one point A_1' only, so that A_0 and A_1 are on opposite sides of R_{p-1} in R_p.

Consider now the intersection of R_{p-1} and any p-dimensional simplex $A_0 A_1 A_{s_2} \ldots A_{s_p}$ of S_p of edge $A_0 A_1$. Then R_{p-1} intersects one of the edges $A_0 A_{s_i}, A_{s_i} A_1$ in a point A_{s_i}'. Thus R_{p-1} intersects the simplex $A_0 A_1 A_{s_2} \ldots A_{s_p}$ in a $(p-1)$-dimensional simplex $A_1' A_{s_2}' \ldots A_{s_p}'$ of vertex A_1'. The set of such simplexes as $A_1' A_{s_2}' \ldots A_{s_p}'$ form a $(p-1)$-dimensional star S_{p-1} of centre A_1' in R_{p-1}, for they are finite in number and satisfy the conditions (a) and (b) in the definition of a star. Thus:

(a) Because S_p is a simplex star of centre A_0, the $(p-1)$-dimensional face $A_0 A_1 A_{s_2} \ldots A_{s_{p-1}}$ is common to two and only two

p-dimensional simplexes $A_0 A_1 A_{s_2} \ldots A_{s_{p-1}} A_{s_q}$ and $A_0 A_1 A_{s_2} \ldots A_{s_{p-1}} A_{s_p}$ and thus the $(p-2)$-dimensional face $A_1' A_{s_2}' \ldots A_{s_{p-1}}'$ is common to two and only two $(p-1)$-dimensional simplexes $A_1' A_{s_2}' \ldots A_{s_{p-1}}' A_{s_p}'$ and $A_1' A_{s_2}' \ldots A_{s_{p-1}}' A_{s_q}'$. Thus the simplexes $A_1' A_{s_2}' \ldots A_{s_p}'$ satisfy condition (a).

(b) Any two simplexes of S_p of edge $A_0 A_1$ have in common either no point other than the edge $A_0 A_1$, or one k-dimensional face, $(k = 2, 3, \ldots, p-1)$. Therefore any two of the $(p-1)$-dimensional simplexes $A_1' A_{s_2}' \ldots A_{s_p}'$ have in common either no point other than A_1', or one k-dimensional face, $(k = 1, 2, \ldots, p-2)$, for if $A_0 A_1 A_{s_2} \ldots A_{s_k}$ is common to two simplexes of S_p, then $A_1' A_{s_2}' \ldots A_{s_k}'$ is common to the two corresponding simplexes in R_{p-1}.

Thus, from (a) and (b) the simplexes $A_1' A_{s_2}' \ldots A_{s_p}'$ form a $(p-1)$-dimensional star S_{p-1} in R_{p-1} of centre A_1', and we have seen that P_n^k is in the simplex $A_1' A_2' \ldots A_k'$ of S_{p-1}. Therefore by Theorem I, P_n^k is the centre of a $(p-1)$-dimensional star S_{p-1}' in S_{p-1}. Therefore by hypothesis, S_{p-1}' being in R_{p-1}, there is a $(p-1)$-dimensional spherical neigbourhood $U(P_n^k)$ of centre P_n^k in S_{p-1}'.

If P_s be in $U(P_n^k)$ it is in S_p and our theorem is proved. Consider the case when P_s not in $U(P_n^k)$. Since $P_n^k P_s$ and $U(P_k^n)$ are in R_{p-1}, the segment $P_n^k P_s$ intersects the boundary of $U(P_n^k)$ in a point Q and the segment $P_n^k Q$ is in $U(P_n^k)$ and thus in S_p. Let Q be in the simplex a^q of S_p and note that $P_s Q$ contains none of the points $P_l^i, P_m^i, \ldots, P_n^k$. Since P_s is in the R_q containing a^q, and Q is in a^q, P_s must be in a^q for otherwise $P_s Q$ would intersect the boundary of a^q in one point which is impossible (since $P_s Q$ contains none of the points P_l^i, $P_m^j, \ldots, P_n^k$).

Thus P_s is in S_p. Therefore $U(A_0)$ is in S_p and the Theorem II is true for $n = p$, if it is true for $n = (p-1)$.

(2) The Theorem is true for $n = 1$, for a 1-dimensional star of centre A_0 in R_1, is a segment of R_1, and A_0 is an inner point of the segment.

Thus Theorem II is true for any finite n.