Astronomy. — "*The Light-Curve of the Cepheids*". By Prof. A. A. NIJLAND.

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It has been tried to establish a connection between the Cepheids and the semi-regular long-period variable stars of the Mira-type. In fact, in some respects the mean curve of the Cepheids may be considered as a reduced copy of that of the long-period variables, assuming for a moment, that there can be question of mean curves for phenomena each presenting so much variety. If for the mean periods of Cepheids and long-period variables 6^d and 300^d is taken, respectively, a proportion of 50 is found, which presents itself also in the range of the light variation, at least when magnitudes are converted into light-intensities. For the ratio 50 corresponds to a difference of $4^1/_4$ magnitudes; and, as a matter of fact, this is about what observation has taught about the difference of range of the mean light-variation of Cepheids (1^m) and long-period variables (5^m). It is to be remarked, however, that the light-intensity curve of the Mira-variables resembles more closely that of e.g. $V\,18 = RZ$ *Camelopardalis* or $V\,18 = RZ$ *Lyrae* [1]) than that of δ *Cephei*.

It is interesting to ascertain whether the asserted analogy goes still further. The semi-regular variables of the Mira-type exhibit irregularities both in period and in amplitude. Moreover the light-curves have every now and then secondary undulations or humps, which often attain a full magnitude, and can be proved to exist beyond any doubt. No unanimity has been reached on the question whether this is the case in the same degree also with the Cepheids. It will be readily admitted that without doubt all these irregularities will occur here too *occasionally*. The secondary wave with *S Sagittae* is incontestable, and the case of η *Aquilae* [2]), which is perfectly confirmed by simultaneous Utrecht observations [3]), speaks for itself. Whether, however, the changes, which according to some astronomers are supposed to continually modify the light-curve of e.g. $V\,49 = XX$ *Cygni* [4]), are all real, may be questioned; and likewise whether this is the case with the numerous secondary inflexions, which many authors draw in the light-curves of the Cepheids observed by them.

I propose to discuss the question whether *as a rule* the light-curves of the Cepheids can be drawn smooth and tense, and whether the period and the extent of the light-variation may *as a rule* be considered so nearly constant that a mean light-curve may be constructed. The question is of importance also for this reason that in the pulsation theory, which is founded on a physical base, secondary undulations are more readily accepted than in the purely geometrical double star theory. It may, however, be pointed out that also in the latter there is every reason to assume physical disturbances of the smooth geometric process, since the spectrum is known to change [5]).

It seemed of importance to discuss the errors occurring in my Cepheid curves, and to compare them with the errors of the Algol variable curves, which for obvious reasons must be drawn smooth and tense.

Too often the term secondary wave is erroneously applied to what is simply an effect of errors of observation; the astronomical literature presents numerous examples of this misunderstanding. In my opinion, this is due to two different causes.

First of all a light-curve is often founded on too few observations. LAU [6]) derives a light-curve of η Aquilae from only 56 estimates; the existence of a deep secondary wave appears to be dependent on the reliability of *two* successive observations, both giving a smaller brightness than corresponds with a smooth curve. KIESS [7]) derives a curve of $V12 =$ *RT Aurigae* from 66 photometric observations, and assumes in it no less than 3 secondary undulations of a depth of $0^m.1$, which cannot but make a somewhat arbitrary impression; the curve would present a totally different aspect when a different grouping to normal places had been chosen. About this point and in general about the way in which the observations have been reduced, KIESS leaves us in the dark. At all events it seems impossible, that these not over-accurate observations (the mean error is $0^m.13$) could say anything with certainty about undulations of $0^m.1$.

In the second place the curve traced follows the observations *too closely*; the amount of the inevitable errors of observation is accordingly not sufficiently taken into account. A striking example is furnished by SPERRA [8]), whose curve of $V23 =$ *SW Draconis* follows the normal places, as if there were no errors of observation at all. In the same number of the Astronomische Nachrichten ICHINOHE publishes a curve of $V26 =$ *SZ Aquilae*, in the descending branch of which he assumes a hump; if, however, the curve is drawn smooth, the deviations are no greater, nay, even smaller than ICHINOHE himself allows in the neighbourhood of the minimum. LEINER speaks of a "scharf ausgeprägte sekundäre Welle" in the light-curve of $V42 =$ *VX Cygni*, while a glance at the plotted normal places leads to the conviction that there is at least as much room for some other inflexions not assumed by LEINER, and that a smooth curve leaves deviations, which are perfectly admissible.

It is self-evident that the mean error must decide the question in every particular case, but, strange enough, this is often not given by the authors, so that any possibility of control is wanting.

In order to arrive at an impartial judgment, four conditions should in my opinion be observed as accurately as possible:

1. The algebraic sum of the deviations δ (normal point minus curve) should be 0;

2. an equal number of plus and minus signs should be found in the series of deviations, and likewise

3. an equal number of permanences of sign and of variations;

4. the mean error ε_0 derived *a posteriori* from the deviations δ should not be much greater than the m. e. ε_1, which is to be expected *a priori* on account of the internal agreement of the normal places, and which follows from the m. e. ε of one observation through division by $\sqrt{m}$ (where m is the number of observations of a normal place).

If on these conditions a tense curve can be drawn, it is improbable that real significance should be assigned to the secondary waves, and it is preferable to leave them alone for the present, and direct one's full attention to the main problem.

Probably — but unfortunately this cannot be verified — ε_0 is considerably *smaller* than ε_1 in the cases cited above, and this would be irreconcilable with the circumstance, that in the derivation of ε_0 always new systematic errors occur, so that necessarily ε_0 must be found greater than ε_1.

In a total of 19 light-curves (17 Cepheids)[10]) I met with only two cases (*S Sagittae* and *η Aquilae*), where secondary waves could not be dispensed

	Cepheids	Algol-stars
m. e. ε of an observation	$0^m.104$	$0^m.125$
m. e. ε_1 of a normal place	0.029	0.036
m. e. ε_0 (normal place minus curve)	0.031	0.046
number of plus signs	135	353
„ noughts	86	124
„ minus signs	129	363
„ permanences	111	373
„ variations	153	343

with. The table shows in how far the above-mentioned conditions are satisfied. For a comparison the results of a discussion of 36 Algol-stars *) have been added (see further on).

Two remarks about the Cepheids should be made here. First of all ε_0 appears to be about equal to ε_1, though in consequence of the appearance of new systematic errors it might be expected that ε_0 would come out considerably greater. It should further be noted that the number of permanences is much smaller than the number of variations. It follows from both remarks that, though only twice a secondary wave was

*) Besides these 36 there are still ten ready for publication, but it will not be easy to find the necessary funds to publish this bulky material *in originali*.

assumed, yet in general the curves followed the normal points still too closely, and should have been drawn still tenser. The smoothing down of the curves gives rise to permanences, since the inflexions, that are removed, are based on at least two normal places: indeed, if an inflexion should be founded on only one normal place, it will not be accepted as real, for fear of falling into the error committed by LAU.

This is in my opinion the cause of the secondary waves which many observers, erroneously as it seems to me, assume to be a real phenomenon in their light-curves, whereas they should simply be taken as permanences of sign, which must necessarily occur according to the laws of chance, even when the phenomenon under examination should have to be represented by a perfectly smooth curve: every curve based on observations must exhibit secondary waves, even though the phenomenon observed does not in any way give rise to them.

By way of check it seemed interesting to examine the available Algol-stars with respect to the errors of their light-curves, since these must certainly be drawn tense. These stars have been observed in exactly the same way, with the same instruments and by the same observer as the Cepheids. The results are placed side by side with those of the Cepheids. That the number of permanences is equal to that of the variations suggests a normal distribution of errors.

For the Algol-stars the great value of the mean errors should be noted.

a. First of all the m.e. ε of one observation (hence also the m.e. ε_1 of a normal place, generally formed from 12 observations) is considerably greater with the Algol-stars than with the Cepheids. I am inclined to attribute this to a greater influence of the systematic errors. The normal place has been derived from observations of somewhat different phases, and this may greatly increase the m.e., especially where the curve has a very steep course. It is true, that just as with the Cepheids [11]) many normal places have been excluded from the discussion for this reason, but probably I did not go far enough in this procedure, so that a number of normal places were allowed to contribute to the result, whose m.e. are vitiated, i.e. increased by the systematic error indicated just now.

It may be further noted that in watching an Algol-star the observer is unfortunately often prejudiced to a certain extent; it is practically impossible to prepare for the minimum without an at least approximate knowledge of the epoch of the expected phenomenon, and it is exceedingly difficult to emancipate oneself entirely from this knowledge in making the observations.

b. In the second place it will be noticed that the ratio $\frac{\varepsilon_0}{\varepsilon_1}$ is 1.28 for the Algol-stars, and only 1.07 for the Cepheids. In consequence of the inevitable systematic errors ε_0 must be greater than ε_1, and I am inclined to ascribe these errors in the case of the Algol-stars chiefly to the necessity of drawing the curves smooth. Probably the m.e. ε_0 is not too great for the Algol-stars, but on the contrary too small for the Cepheids: and thus we arrive again at the conclusion that the curves of the Cepheids seem to follow the normal places too closely, and should have been drawn tenser.

Finally it seemed desirable to test the distribution of the errors of observation by the exponential law of GAUSS. As error of observation the deviation "observation minus normal place" was taken, increased by the deviation "normal place minus curve". Both for the Cepheids and the Algol-stars the negative and the positive errors proved to follow exactly the same law, so that the frequency curves could be drawn symmetrically. Both curves present the typical deviation from the exponential curve, consisting in an excess of the very small and of the very great errors, whereas the number of the moderately great errors is too small.

CEPHEIDS.

Number of errors: 4288 m. e. ε of an observation: $0^{m}.102$					$h = \frac{1}{\varepsilon\sqrt{2}} = 6.932$				
a	n	n_w	n_E	$n_w - n_E$	a	n	n_w	n_E	$n_w - n_E$
± 0.00	192	192	167	+ 25	± 0.22	39	31	33	− 2
01	386	382	333	+ 49	23	22	25	26	− 1
02	388	374	328	+ 46	24	19	22	22	0
03	353	360	321	+ 39	25	12	19	18	+ 1
04	318	340	310	+ 30	26	21	16	14	+ 2
05	323	315	297	+ 18	27	13	14	12	+ 2
06	296	291	282	+ 9	28	6	12	9	+ 3
07	275	266	265	+ 1	29	12	11	7	+ 4
08	241	236	246	− 10	30	14	9	5	+ 4
09	181	206	227	− 21	31	9	8	3	+ 5
10	167	177	207	− 30	32	4	7	2	+ 5
11	150	152	187	− 35	33	2	6	1	+ 5
12	152	135	168	− 33	34	3	5	1	+ 4
13	117	120	149	− 29	35	6	5	1	+ 4
14	109	105	131	− 26	36	5	4	1	+ 3
15	105	93	114	− 21	37	0	3	0	+ 3
16	82	80	98	− 18	38	4	3	0	+ 3
17	70	69	84	− 15	39	0	2	0	+ 2
18	47	60	71	− 11	40	0	2	0	+ 2
19	50	50	59	− 9	41	0	1	0	+ 1
20	48	43	49	− 6	42	1	1	0	+ 1
21	46	36	40	− 4		4288	4288	4288	

$x_1 = 0^{m}.070 = 0.65\,\varepsilon$

$x_2 = 0\ .240 = 2.36\,\varepsilon$

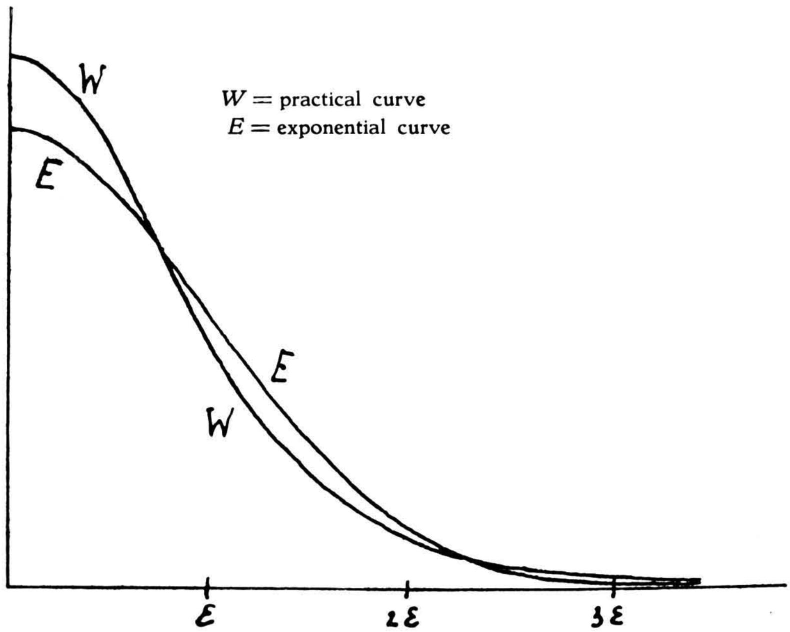
ALGOL-STARS.

Number of errors: 9955 m. e. ε of an observation: $0^m.132$					$h = \frac{1}{\varepsilon\sqrt{2}} = 5.348$				
a	n	n_w	n_E	$n_w - n_E$	a	n	n_w	n_E	$n_w - n_E$
± 0.00	335	340	300	+ 40	± 0.34	29	32	22	+ 10
01	712	676	600	+ 76	35	26	28	18	+ 10
02	632	670	593	+ 77	36	26	24	15	+ 9
03	588	650	585	+ 65	37	28	21	12	+ 9
04	589	622	574	+ 48	38	16	18	9	+ 9
05	601	595	559	+ 36	39	11	16	8	+ 8
06	543	566	542	+ 24	40	12	14	6	+ 8
07	597	530	523	+ 7	41	6	12	5	+ 7
08	517	492	499	— 7	42	14	11	4	+ 7
09	458	457	477	— 20	43	6	9	3	+ 6
10	426	421	451	— 30	44	15	8	2	+ 6
11	400	385	425	— 40	45	6	6	2	+ 4
12	335	349	397	— 48	46	3	4	2	+ 2
13	336	315	371	— 56	47	6	4	1	+ 3
14	263	287	344	— 57	48	7	4	1	+ 3
15	292	263	314	— 51	49	2	3	1	+ 2
16	252	239	289	— 50	50	8	3	1	+ 2
17	228	219	265	— 46	51	2	3	0	+ 3
18	175	201	237	— 36	52	1	3	0	+ 3
19	179	183	214	— 31	53	2	2	0	+ 2
20	179	165	191	— 26	54	4	2	0	+ 2
21	150	150	171	— 21	55	1	2	0	+ 2
22	147	136	150	— 14	56	2	2	0	+ 2
23	116	123	132	— 9	57	1	2	0	+ 2
24	113	110	116	— 6	58	0	1	0	+ 1
25	84	98	101	— 3	59	1	1	0	+ 1
26	89	87	87	0	60	0	1	0	+ 1
27	80	78	74	+ 4	61	4	1	0	+ 1
28	57	69	64	+ 5	62	1	1	0	+ 1
29	73	61	54	+ 7	63	0	1	0	+ 1
30	49	53	46	+ 7	64	0	0	0	0
31	36	47	39	+ 8	65	1	0	0	0
32	39	41	32	+ 9		9955	9955	9955	
33	44	36	27	+ 9					

$$x_1 = 0^m.076 = 0.58\,\varepsilon$$

$$x_2 - 0\ .260 = 1.97\,\varepsilon$$

In the tables (see p. 146 and p. 147) the actually observed numbers n are given, graphically smoothed down to n_w; they have been compared with the values n_ε following from the exponential law. The errors 0.00,



0.01, 1.02 etc. are supposed to lie between the limits 0.00 and 0.00^5, 0.00^5 and 0.01^5, 0.01^5 and 0.02^5 etc.

The tables show first that, in accordance with the greater m.e., the greater extreme cases occur with the Algol-stars. It may seem strange that it is possible to make errors of $0^m.5$ and $0^m.6$, but according to the law of the great numbers such extreme cases, more than four times the m.e., must now and then occur in a series of almost 10.000 observations. In the second place we may remark, that for the Cepheid series the abscissae*) of the points of intersection of the exponential with the actually observed curve almost entirely satisfy the remarkable equation [12]):

$$\left(\frac{x}{\varepsilon}\right)^2 - 6\left(\frac{x}{\varepsilon}\right) + 3 = 0.$$

From this it follows:

$$\left.\begin{aligned} x_1 &= 0.74\,\varepsilon = 0^m.076 \\ x_2 &= 2.33\,\varepsilon = 0^m.238 \end{aligned}\right\} \text{for } \varepsilon = 0^m.102,$$

the observed points of intersection lying at $0^m.070$ and $0^m.240$. For the Algol-stars $\varepsilon = 0^m.132$, hence the points of intersection would be expected at

*) Here the positive values are only considered.

$0^m.098$ and $0^m.308$, whereas they are really found at $0^m.076$ and $0^m.260$. That both points lie so much nearer the Y-axis than follows from the above equation may be ascribed — as is evident from the figure — to a too small number of small errors (smaller than that actually observed, though still much greater than the theoretical number) and a too great number of great errors. This may readily be ascribed to the influence of the systematic errors "normal place minus curve", which give rise to considerable corrections in the Algol-curves, since they must necessarily be drawn smooth.

CONCLUSION.

Apart from two exceptional cases there is no urgent reason, why the curves of the examined Cepheids should not be drawn perfectly smooth and tense. The remaining errors appear to behave — in a still greater degree than those of the Algol-curves — entirely as accidental errors. If, as many observers assert, there is hardly question of a mean curve at all for the Cepheids on account of the numerous irregularities of all kinds, these irregularities are entirely hidden in the errors of observation; they do not spoil their accidental character, nor do they appreciably increase their amount.

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