Mathematics. — "*Representation of the Pairs of Points Conjugated relative to a Conic, on the Points of Space.*" By G. SCHAAKE. (Communicated by Prof. JAN DE VRIES.)

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§ 1. Let a conic k^2 in a plane α be the projection on α of a twisted cubic k^3 out of a point A of the latter cubic. If we associate to each point P of k^3 its projection Q on α, there arises a one-one correspondence (P, Q) between the points P of k^3 and the points Q of k^2 where the point of intersection B on k^2 of the tangent a at A to k^3 with α corresponds to the point A of k^3.

Let C_1 and C_2 be two points of a associated to each other relative to k^2. The lines BC_1 and BC_2 cut k^2 outside B resp. in K_1 and K_2. The tangents t_1 and t_2 which may be drawn to k^2 resp. at K_1 and K_2, have their point of intersection C on the line l joining C_1 and C_2. For the pairs (K_1, K_2) which may be derived from the pairs (C_1, C_2) of the involution defined on l by k^2, form an involution on k^2 the double points of which lie in the points of intersection Q_1 and Q_2 of l with k^2 and which, therefore, has l as axis.

The plane (A, l) contains one chord k of k^3 which does not pass through A, the line joining the points P_1 and P_2 of the twisted cubic that correspond to Q_1 and Q_2. We consider the point of intersection S of the lines AC and k as the image of the pair of points (C_1, C_2).

Inversely a pair of points C_1 and C_2 that are associated relative to k^2, may be derived from any point S. With a view to this we draw the chord k of k^3 through S. We project k and S on α out of A and find the line l and the point C on l. We determine the points of contact K_1 and K_2 of the tangents t_1 and t_2 to k^2 through C. C_1 and C_2 are resp. the points of intersection of BK_1 and BK_2 with l.

In this way we have found a representation of the pairs of points C_1 and C_2 associated to each other relative to k^2, on the points S of space.

§ 2. In order to arrive at another way to produce our representation, we examine what corresponds to the system S_1 of the ∞^1 pairs of points associated relative to k^2, for which C_1 is a fixed point of α. In this case K_1, the point of intersection of BC_1 and k^2, is the same point of this conic for all the individuals of S_1 and t_1 is a fixed tangent of k^2. The point S always lies in the tangent plane τ_1 through t_1 to the cone $\varkappa$ which projects k^3 out of A. The chords k which contain the image points S of the pairs of points of S_1, from the scroll ϱ of the chords of k^3 which cut the straight line AC_1, as the projections of these chords on α, i.e. the lines l of S_1, all pass through C_1. The generatrix of ϱ through A lies in the plane ABC_1 through AC_1 and the tangent a, joins A to the point of intersection of that plane and k^3, and is accordingly the line AK_1 in τ_1.

The locus of the points of intersection of the lines of ϱ with τ_1 is consequently a line s_1 of the scroll σ associated to ϱ, which passes through the point of intersection L_1 outside A of AK_1 with k^3, as the quadratic surface φ^2 that contains ϱ and σ, touches the plane τ_1 at L_1. For the tangent plane to φ^2 at L_1 is completely defined by the tangent to k^3 and AK_1.

The system of the ∞^1 pairs of points conjugated relative to k^2 of which one point is given, is accordingly represented on a line s_1 which touches the cone $\varkappa$ on k^3.

§ 3. The straight line s_1 corresponding to a given point C_1 of α according to § 2, may be found in the following way. We consider the scroll ϱ of the chords of k^3 which cut the line AC_1. The generatrices of the corresponding scroll σ which meet k^3 once, all cut the generatrix AK_1 of ϱ through A which passes through the point of intersection K_1 outside B of BC_1 with k^2, and are, therefore, projected on α as straight lines through K_1. Among the lines through K_2 there is one, t_1, which touches k^2, so that σ contains one line s_1 which touches $\varkappa$ on k^3, and that in the point of intersection L_1 outside A of AK_1 and k^3. This line is the line s_1 corresponding to C_1.

The point C_1 of α which corresponds to a line s_1 which touches $\varkappa$ on k^3, is found in the following way. We form the scroll ϱ of the chords k of k^3 which cut s_1. The generatrice through A of the scroll σ associated to ϱ, cuts α in C_1.

The lines s which are represented in this way on the points of α, form a congruence Σ (2, 3). For the lines s through a point S of space have their points of contact in the two points of intersection outside A of the polar plane of S relative to $\varkappa$ with k^3, and the lines s lying in a given plane φ are the tangents that may be drawn in the points of intersection of φ with k^3 to the conic along which φ cuts the cone $\varkappa$.

Consequently the pair of points (C_1, C_2) corresponding in the representation of § 1 to a given point S of space, is formed by the two points C_1 and C_2 that are associated to the lines s_1 and s_2 through S of Σ. Inversely the image S of a pair of points (C_1, C_2) associated relative to k^2, is the point of intersection of the lines s_1 and s_2 corresponding to C_1 and C_2.

§ 4. If we choose S on the cone $\varkappa$, s_1 and s_2 coincide in the generatrix AS of $\varkappa$ through S which cuts k^3 besides in A in L, where L_1 and L_2, the points in which s_1 and s_2 touch the cone $\varkappa$ on k^3, coincide.

The scroll ϱ and the associated scroll σ are both formed by the generatrices of $\varkappa$. The generatrix of σ passing through the point A of k^3, is the tangent a to k^3 at A which cuts α in B. Hence C_1 and C_2 coincide in B for any point S of $\varkappa$.

Inversely it appears that an infinite number of lines s correspond to B, the generatrices of k.

Accordingly the representation has one cardinal pair of points, the pair for which C_1 and C_2 coincide in B. To this pair there correspond the points of the quadratic cone $\varkappa$.

If we choose for carrier of this pair of points a definite straight line through B which cuts k^2 besides in K, according to the construction of § 1 there are only ∞^1 image points, which form the generatrix AK of $\varkappa$.

Through making use of the way of producing our representation indicated in § 1, we should perhaps think that the pairs of points (C_1, C_2) for which C_1 lies in B, hence C_2 on the tangent b at B to k^2, are singular. Indeed, for such a pair of points K_1 may be chosen at random on k^2 and K_2 lies in B. There are an infinite number of points C, the points of b. The chord k becomes the tangent a to k^3 at A. If we choose C in B, AC lies along a and we find that any point of a may be considered as the image point of (C_1,C_2).

That this is not right, however, appears if we use the way of producing our representation indicated in § 3. The line s_1 corresponding to the point C_1 lying in B, may be chosen along an arbitrary generatrix of $\varkappa$. The line AC_2 belongs to the plane pencil that has A as vertex and of which the plane is the tangent plane to $\varkappa$ along AB, i. e. the plane of osculation of k^3 at A. The generatrices of this pencil are straight lines of $\Sigma(2,3)$. Hence AC_2 is just the line of the scroll σ that may be derived from C_2 which belongs to Σ, that is the line s_2 corresponding to C_2. This line cuts $\varkappa$ only in A, which point is, therefore, the only image point of all the pairs (C_1,C_2) that are associated relative to k^2 of which one point lies in B.

Hence A is a singular point for our representation. A has this property in common with all the points of k^3. For if we choose S on k^3 we have, instead of two lines s_1 and s_2, a plane pencil of lines s which has S for vertex and which lies in the tangent plane τ to $\varkappa$ at S. The generatrices through A of the scrolls σ corresponding to these lines s, form the plane pencil that has A for vertex and that lies in the plane ABS which cuts α along the line BK, if K is the projection of S on α lying on k^2. For AK is always the generatrix through A of the scrolls ϱ, and accordingly the projections of all the generatrices of the scrolls σ pass through K. But the projections of the generatrices through A of the latter scrolls also pass through B because the projecting planes pass through a. Hence the point S must be considered as the image point of all the pairs of BK that are associated relative to k^2.

Consequently the points of k^3 are singular for our representation. To *a point S of k^3 there correspond the pairs of points of the line BK through B that are associated relative to k^2. The pairs of points of which the image lies on k^3, are the ∞^2 of which the carriers pass through B.*

If we choose S in A, there is, besides a plane pencil of lines s, a cone of similar lines, $\varkappa$. Hence the pairs of points (C_1, C_2) of the line b are associated to A, a result already found, on the understanding, however, that the carrier of the coincidence in B may be chosen arbitrarily through that point.

The latter is also true for any point S of k^3 relative to the coincidence in K. This is, for instance, easily seen by the aid of the construction of § 1. If we choose an arbitrary line through K as carrier, the corresponding line k is a chord of k^3 through S. The points K_1, K_2 and C all coincide in K. The image point of the chosen coincidence, i.e. the intersection of k and AC, in fact always lies in S.

The curve k^3 is, therefore, in particular the locus of the image points of the coincidences among the pairs of points that are associated relative to k^2.

§ 5. The most simple systems of ∞^1 pairs of points (C_1, C_2) are the system λ of which all the individuals belong to a given line l of α, and the system π of the pairs of which one point is a given point P of α and the other point may be chosen at random in the polar line p of P relative to k^2.

To all the individuals of a system λ there corresponds the same chord k of k^3, the chord of this curve that does not pass through A of which the projection on α coincides with l. This chord contains the image points of all the pairs of points of λ.

The system λ of the ∞^1 pairs of points of a line l that are associated relative to k^2, are represented on a chord k of k^3.

The points of intersection of k and k^3 are the image points of the coincidences of λ that lie in the points of intersection of l and k^2.

If, in particular, l is a tangent to k^2, the involution defined on l by k^2 becomes parabolic and λ contains only one coincidence. In this case λ is represented on a tangent to k^3.

In § 2 we found:

A system π is represented on a straight line s that touches $\varkappa$ on k^3.

The intersection of s and k^3 is the image point of the pair of points of π of which the carrier passes through B. The two points of intersection outside k^3 of s and the biquadratic surface of the tangents of this curve are the images of the individuals of π for which the point on p lies in one of the points of intersection of p and k^2.

If P is a point of k^2, the system π is at the same time a system λ, (save the indefiniteness of the carrier of the coincidence in the point of contact), and is represented on a tangent to k^3.

§ 6. Of the systems of ∞^2 pairs of points associated relative to k^2, we shall first treat the system Π of the pairs (C_1, C_2) of which the carriers pass through a given point P of α. The straight lines k of these pairs form the scroll of the chords of k^3 that cut AP.

The system Π of the ∞^2 pairs of points the carriers of which pass through a given point P, i. e. a projective inversion with center P and base k^2, is accordingly represented on a quadratic surface Φ through k^3.

Inversely a quadratic surface through k^3 is the image of a similar system, that for which P lies in the point of intersection with α of the line through A of this surface that cuts k^3 only in A.

If P lies on k^2, the image of Π is the cone which projects k^3 out of the point of intersection outside A of AP and k^3.

Two systems Π and Π' corresponding resp. to the points P and P', are represented on two quadratic surfaces Φ and Φ' which, besides k^3, have a chord of this curve in common; it is the image of the system of pairs (C_1,C_2) of the line PP'. Φ and Φ' define a pencil of which each surface is the representation of the system of pairs associated relative to k^2 of which the carriers pass through a given point of PP'.

Let us further investigate the representation of the system Λ of the pairs of which one point lies on a given straight line l. This system consists of the ∞^1 systems π of the points P of l and is, therefore, represented on a surface Ψ which consists of the ∞^1 corresponding lines s. As each of the said systems π contains one individual of the system λ of the pairs (C_1,C_2) of l, Ψ consists of the lines s that cut the chord k of k^3 which corresponds to l.

Ψ has k as nodal line. The two lines s through a point S of k are the images of the systems π for which P lies in one of the two points of the pair (C_1,C_2) that has its image in S. A plane through k contains only one generatrix of Ψ, the line which touches $\varkappa$ in the point of intersection outside k of this plane and k^3. As all the generatrices of Ψ touch the cone $\varkappa$ on k^3, we find:

The system Λ of the ∞^2 pairs of points (C_1,C_2) of which one point lies on a given line l of α, is represented on a cubic scroll Ψ which touches $\varkappa$ on k^3.

The single directrix of Ψ is the line s which represents the system π that is defined by l and the pole of l relative to k^2.

If l is a tangent to k^2, k becomes a tangent to k^3 so that this directrix of Ψ is at the same time a generatrix. In this case Ψ is a scroll of CAYLEY.

A system Π has ∞^1 pairs of points in common with a system Λ, all the carriers of which pass through P and which have one point on l; the conjugated point always lies on the conic corresponding to l in the general inversion with center P and base curve k^2. This system is represented on the intersection different from k^3 of Φ and Ψ, i.e. a cubic (Φ, Ψ) which cuts k^3 in the images of the two coincidences of (Π, Λ) that lie in the points of intersection of l and k^2 and in the image point of the pair of points on the line BP. In the latter image point, the point L corresponding to P, the surface Φ touches the cone $\varkappa$ according to § 2; hence this is also the case for the curve (Φ, Ψ). The two former points of k^3 are the intersections of the chord k corresponding to l and k^3. These are cuspidal points of Ψ; the two corresponding torsal lines are the tangents to k^3 at these points, the images of the systems π of which the point P lies in one of the intersections of l and k^2. As in these cuspidal points of Ψ Φ and Ψ have only the tangent to k^3 as a common tangent, the curve (Φ, Ψ) must touch k^3 at this point.

A system (Π, Λ) is accordingly represented on a twisted cubic that touches $\varkappa$ once on k^3 and the latter curve twice.

If l is the polar line of P relative to k^2, (Π, Λ) consists of the two systems λ corresponding to the tangents to k^2 through P, and of the system π of P. In this case the image consists of two tangents of k^3, and the line touching $\varkappa$ on k^3 which cuts the two said tangents.

The system which is common to two systems Λ and Λ' which correspond to the lines l and l', consists of the system π of the point P that lies in the intersection of l and l', and of the system (Λ, Λ') of the ∞^1 pairs of points associated relative to k^2 of which one point belongs to l and the other to l'. The former system is represented on the line s belonging to Ψ and Ψ' which touches $\varkappa$ on k^3 and cuts k and k'. Besides s and k^3, along which curve Ψ and Ψ' touch each other, these surfaces have a conic (Ψ, Ψ') in common, the image of (Λ, Λ'). As the carriers of (Λ, Λ') envelop a conic, there are two pairs of points of this system the joins of which pass through B, so that the conic (Λ, Λ') cuts the curve k^3 in two points, where it touches $\varkappa$.

Consequently the system of the ∞^1 pairs of points that have a point on each of two given lines of α, is represented on a conic which touches $\varkappa$ twice on k^3.

A check of the latter two results is given by the fact that the class of the envelope of the carriers and the order of the curve of the points of the individuals of a system of ∞^1 pairs of points (C_1, C_2) are resp. equal to the numbers of points of intersection outside k^3 of the image curve with a surface Φ and with a surface Ψ.

§ 7. The pairs of points that are associated to each other relative to k^2 as well as relative to another conic k'^2 of α, form a quadratic involution I, which is represented on a surface X. The class of I, i.e. the number of pairs of I which belong to a straight line l of α, is one. For the points of the pair of I on l are the points of intersection of l with the conic described by C_2 if C_1 moves on l, or the double points of the involution which is defined by the pairs of points lying on k^2 and k'^2. As, accordingly, any line through B contains one pair of points of I, k^3 is a single curve of X. A chord k of k^3 cuts X outside this curve in one point, the image of the pair of points of I which lies on the line l of α corresponding to k. Consequently X is a cubic surface. As the involution I contains one pair of points, of which one point is given, a straight line touching $\varkappa$ on k^3 cuts the surface X only once outside k^3. Hence the surface X must touch k^3 on $\varkappa$.

The involution I has four double points, the points of intersection $Q_1, \ldots, Q_4$ of k^2 and k'^2, which are resp. represented on the corresponding points $P_1, \ldots, P_4$ of k^3. As no line through a point Q_i on which there lies no other point Q_i, contains any pair of I besides the coincidence in Q_i, no chord of k^3 through P_i which does not contain any of the other points P, has any point in common with X outside P_i and k^3. The points $P_1, \ldots, P_4$ are necessarily conical points of X. The lines Q_i Q_k are singular for I, as all the pairs of points of Q_i Q_k that are associated relative to k^2, are also associated relative to k'^2 and belong, accordingly, to I. The lines P_i P_k belong, therefore, all to X.

Consequently a quadratic involution of pairs of points associated relative to k^2, is represented on a surface X that contains four conical points on k^3.

Such a surface may be represented on a plane χ by means of a cubical involution. Through this representation the six lines P_i P_k are transformed into the angular points R_{ik} of a complete quadrilateral. A twisted cubic through the points P_i is transformed into a straight line of χ. The diagonals of the said complete quadrilateral are the images of three lines of χ different from P_i P_k which cut k^3 once, touch the cone $\varkappa$ in the points of intersection with k^3, and form a triangle. These three lines s are the images of systems π of I of which the points P are singular points of I, to each of which in I the whole polar line relative to k^2 is associated. As each pair of these systems π has a pair of points in common, the points P and the associated straight lines are resp. the angular points and the sides of a polar triangle of k^2. The straight line of X that corresponds to the diagonal $R_{ik} R_{lm}$, cuts the lines $P_i P_k$ and $P_l P_m$ so that the associated system π of I has its point P in the intersection of Q_i Q_k and Q_l Q_m, as it contains one pair of points of each of these lines. The line associated to this point P in I, joins the other two diagonal-points of the complete quadrilateral $Q_1 Q_2 Q_3 Q_4$.

A cubic surface through k^3 which touches $\varkappa$ on k^3, is always the image of such a quadratic involution. For a surface of this kind has one point of intersection with a line s that is not singular so that to a point of α one point is associated in the corresponding involution. Further the surface cuts a conic which touches $\varkappa$ twice on k^3, in two points that are not singular for the representation, so that I contains two pairs of points that have a point on each of two given lines of α, and if a point describes a straight line of α the point associated to it in I describes a conic. I cannot be an inversion relative to k^2, as this is represented on a quadratic surface.

Accordingly a cubic surface through k^3 which touches a quadratic projecting cone of this curve in each point of k^3, has always four conical points on k^3.

If I has a coincidence in B, the quadratic cone $\varkappa$ splits off from X. Consequently I is represented on a plane.

Let us assume besides I an involution I' which is defined by k^2 and a third conic k''^2. I' is represented on a surface X'. The surfaces X and X' define a pencil of cubic surfaces that touch $\varkappa$ on k^3, the image of the pencil of quadratic involutions defined by I and I', of which each individual may be derived from a pencil of conics containing k^2 out of the net defined by k^2, k'^2 and k''^2. The individual of the said pencil of surfaces which passes through a point of $\varkappa$ outside k^3, must degenerate into $\varkappa$ and a plane, as a non-degenerate surface of the pencil cannot have any point in common with $\varkappa$ outside k^3, along which curve it touches $\varkappa$. The corresponding quadratic involution may be derived from the pencil with a base point in B of the net defined by k^2, k'^2 and k''^2.

Hence the intersection of X and X' consists of k^3, counted double, and a plane cubic (X, X') which touches $\varkappa$ in three points on k^3. This curve is the image of the system (I, I') of the pairs of points that are associated relative to k^2, k'^2 and k''^2. As (X, X') has three points that are not singular for the representation in common with a surface Φ as well as with a surface Ψ, the locus of the pairs of points of (I, I') is a cubic, and the carriers of the pairs of (I, I') envelop a curve of the third class. As (X, X') has one point in common with each of the lines of the surfaces of the pencil defined by X and X' which cut k^3 once, the locus of the pairs of (I, I') passes through the singular points of the involutions of the pencil defined by I an I'. Accordingly this locus is the Jacobian of the net defined by k^2, k'^2 and k''^2.

The involution of the pairs of points on the Jacobian of a net containing k^2 which are associated relative to all the conics of the net, is represented on a plane cubic which touches $\varkappa$ three times on k^3.

Besides k^3 three surfaces X, X' and X'' have three points in common on a straight line. This line is the intersection g of the three planes containing the intersections of each pair of the chosen surfaces. The surfaces X, X' and X'' define a net; the degenerate individuals consist of the planes through g and $\varkappa$. A plane through g cuts this net along a pencil of cubics of which three base points lie in the three isolated base points of the net and the remaining six coincide in pairs in the three points of intersection with k^3. A surface Ψ through two of the isolated base points must also pass through the third. For Ψ cuts the plane through g along a cubic which passes through eight base points of the said pencil that lie independently, and which must, therefore, also contain the ninth base point [1]).

[1]) The independence appears from the fact that through any seven of the base points we can pass a cubic which does not contain the eighth base point. In the first place we can choose the intersection with $\varkappa$ and a straight line through an isolated base point of the net, further the conic through the base points and the line which joins two base points lying on k^3.

Hence the three involutions I, I' and I'' have three pairs of points in common, so that the line which joins a point of one of these pairs to a point of the second, also contains a point of the third. Consequently the common pairs of points of I, I' and I'' form the pairs of opposite angular points of a complete quadrilateral. If I, I' and I'' are defined, besides by k^2, resp. by k'^2, k''^2 and k'''^2, these pairs of points are associated relative to all the conics of the linear system defined by $k^2, \ldots, k'''^2$.

In this way we have found a proof of the theorem that the pairs of points which are associated relative to all the conics of a complex, are the pairs of opposite angular points of a complete quadrilateral.

§ 8. A plane V of points S is the image of a system of ∞^2 pairs of points (C_1, C_2). As V has resp. one, one and two points in common with a line s, a chord k and a conic which touches $\varkappa$ twice on k^3, we may conclude that to V there corresponds a quadratic involution of pairs of points associated relative to k^2. This involution has a coincidence in B, because V cuts all the generatrices of $\varkappa$. It may be derived from a pencil of conics containing k^2 which has a base point in B.

A plane is the image of a quadratic involution of pairs of points associated relative to k^2 which has a coincidence in B.

The six singular lines of the involution are represented on the points of intersection of V and k^3 and on the three chords of this curve in V. The three systems π each of which is defined by a singular point of the involution and the polar line associated to it, correspond to the three lines in V which touch $\varkappa$ on k^3. A surface Ψ cuts V along a cubic which touches $\varkappa$ in the three points of intersection with k^3, the image of the correspondence between the points of a straight line and the conic associated to it through the involution, which, as the said cubic cuts any line s in V once outside k^3, passes through the singular points of the involution.

To a plane through A as well as to an arbitrary tangent plane to k^3 there corresponds an involution with two coinciding coincidences, etc.

As a special case let us choose for V a tangent plane to $\varkappa$. We can consider V as consisting of lines s which form a plane pencil the vertex of which lies in the point where V touches the curve k^3. Accordingly the associated system is formed by the systems π that have their point P on a straight line of α through B; it is, therefore, a system Λ. A tangent plane to $\varkappa$ forms with $\varkappa$ a degeneration of a surface Ψ.

Consequently the system of the pairs of points of which one point lies on a straight line through B, is represented on a tangent plane to $\varkappa$.

In particular the plane of osculation to k^3 at A is the image of the system of the pairs of which one point lies on the tangent b to k^2 at B.

The points S of a line g in space are the images of the pairs (C_1, C_2) of a system γ of ∞^1 pairs of points that are associated relative to k^2.

As g cuts a surface Φ and a surface Ψ resp. twice and three times, the carriers of γ envelop a conic and the locus of the pairs of γ is a cubic. This curve has a node in B. For γ contains two coincidences lying in B, the carriers of which are the lines through B which cut the generatrices of $\varkappa$ meeting g, on k^2. The system γ consists of the common pairs of points of the pencil of quadratic involutions which is represented on the pencil of planes of which g is the axis, hence of the pairs of points that are associated relative to all the conics of a net that has a base point in B. From the curve of the third class which is enveloped by the carriers of a general net, there splits off the plane pencil that has B for vertex.

Accordingly a line g is the image of the involution defined by a net of conics containing k^2 that has its base point in B, on its Jacobian.

The lines k corresponding to the carriers of γ, are the chords of k^3 which cut g. They form ∞^1 triangles inscribed in k^3, to which there correspond triangles inscribed in k^2 the sides of which are carriers of γ.

Consequently the conic enveloped by the carriers of γ, is inscribed in k^2, hence also in all the conics of the net from which γ is derived.

The lines g of a plane V are the images of systems γ of the quadratic involution I corresponding to V. The loci of the pairs of points of these systems γ are cubics which are *invariant* for I. They have a node in B, pass through the three singular points of I, and cut the singular lines of I which does not pass through B, in a pair of points that are associated relative to k^2. There are *four* systems of such cubics that are invariant for I.

A line g that cuts k^3 once, has resp. one and two points which are not singular for the representation, in common with a surface Φ and a surface Ψ. Accordingly the locus of the pairs of points of the corresponding system γ is a conic and the carriers of γ form a plane pencil. From the system γ corresponding to an arbitrary line g, also the involution has split off which is defined through k^2 on a straight line through B, and the system of pairs of points of which the carriers form the plane pencil that has for vertex the projection of the point of intersection of g and k^3.

To a line cutting k^3 once, there correspond, accordingly, a conic and a plane pencil. The conic passes through B, as γ contains a coincidence in this point, which is represented on the point of intersection outside k^3 of g and $\varkappa$. The vertex of the plane pencil is the center of an inversion which is represented on the quadratic surface Φ of the chords of k^3 cutting g, and through which the conic is transformed into itself.

From this it appears that the generatrices of a quadratic surface Φ corresponding to a point P of α that cut k^3 once, are associated to the conics through B which are invariant for an inversion with center P and base curve k^2.

As γ has two coincidences, one in B and one in the point of k^2 that corresponds to the point of intersection of g and k^3 [1]), we find:

The Jacobian of a net of conics containing k^2 which has two base points one of which lies in B, degenerates into the line joining these base points and a conic through the base points. The involution defined on the conic by the net, is represented on a line that cuts k^3 once.

The three plane pencils in a plane V of lines which cut k^3 once, give three systems of conics that are invariant for the involution I corresponding to V. Such a system consists of the conics passing through B and another double point of I as well as through the two singular points which do not lie on the side of the two said double points and which contain a pair of points associated relative to k^2, on the side through the angular points of which the conics do not pass.

In all, six systems of this kind correspond to I. The other three are represented on conics which pass through the points of intersection of k^3 and V and which, besides, touch $\varkappa$ in the point of intersection of k^3 and V corresponding to the double point of I through which the conics of such a system do not pass [2]).

A line g through A is the image of the system of the ∞^1 pairs of points associated relative to k^2, of the two straight lines through B,BK_1 and BK_2, if K_1 and K_2 are the points of contact of the tangents to k^2 through the point of intersection C of g and α. From the system associated to an arbitrary line g there splits off besides the parabolic involution which is defined by k^2 on the tangent b at B. The corresponding net of conics has two base points which coincide in B.

The Jacobian of a net with two base points that coincide in B, degenerates into the common tangent of the conics of this net and into two straight lines through B. The system of the pairs of points associated relative to the net of which each of the latter two lines contains one point, is represented on *a straight line through A.*

If g is a chord k of k^3, the corresponding net has three base points one of which lies in B. The Jacobian degenerates into the line l of α which corresponds to k that joins the base points different from B, and into the two lines which may be drawn through B and the other two base points, which lines are associated to the points of intersection of k and k^3.

To a straight line g that touches $\varkappa$ on k^3, there corresponds a net with two base points for which the Jacobian has degenerated into the join of these base points, counted double, and into the line p of the system π which is represented on the said straight line. The net associated to a tangent of k^3, has a base point in B and two coinciding base points outside B. The Jacobian degenerates into the common tangent and into the line joining the point of contact and B, counted double.

[1]) Cf. § 9, par. 2.

[2]) Cf. § 9.

§ 9. A curve k^n is the image of a system γ of ∞^1 pairs of points that are associated relative to k^2. We suppose that k^n touches $\varkappa\, p_1$ times on k^3 outside A, and cuts k^3 besides p_2 times outside A.

If a system γ has a pair of points on a straight line through B, the image curve cuts k^3. If the image S on the image curve approaches the point of intersection with k^3, the chord of k^3 through S approaches the line joining the said point of intersection to A. In this case the image curve must touch $\varkappa$ on k^3. A point where the image curve cuts k^3 but does not touch $\varkappa$, is consequently the image of a coincidence of γ, and to a point where the image curve touches k^3, there corresponds a double coincidence of γ, which, however, only forms a single pair of this system.

The class of the system γ which is represented on k^n, i. e. the class of the envelope of the carriers of γ, is equal to the number of points of intersection of k^n and a surface Φ that are not singular for the representation, hence: $2n-p_1-p_2$, and the order, i. e. the order of the locus of the pairs of points of γ, is equal to the number of points of intersection of k^n and a surface Ψ that are not singular for the representation, hence to $3n-2p_1-p_2$. As k^n cuts the cone $\varkappa$ outside k^3 in $2n-2p_1-p_2$ points, γ has $2n-2p_1-p_2$ coincidences in B, and accordingly the locus of the pairs of points of γ has a $2n-2p_1-p_2$-fold point in B.

A curve k^n of the order n which touches $\varkappa\, p_1$ times on k^3 outside A, and which besides cuts k^3 p_1 times outside A, is the image of a system γ of ∞^1 pairs of points associated relative to k^2 of the class $2n-p_1-p_2$ and of the order $3n-2p_1-p_2$ which has $2n-2p_1-p_2$ coincidences in B.

The number of the coincidences of γ outside B is p_2.

Let us now investigate the representation of a system γ of ∞^1 pairs of points associated relative to k^2 of the class μ and the order ν that does not have any coincidence in B and does not contain either any pair associated relative to k^2 which belongs to the tangent b at B to k^2. The latter two conditions can be satisfied by a system γ of any kind.

As the image curve of γ does not pass through A, we find by solving n, p_1, and p_2 out of the equations

$$2n - p_1 - p_2 = \mu$$

$$3n - 2p_1 - p_2 = \nu$$

$$2n - 2p_1 - p_2 = 0:$$

The image of a system γ of ∞^1 pairs of points associated relative to k^2 of the class μ and of the order ν that does not contain any pair of which one point lies in B, is a curve of the order ν which touches $\varkappa\,\mu$ times on k^3 outside A and which besides cuts k^3 $2(\nu-\mu)$ times outside A.

Hence the order of a system of ∞^1 pairs of points associated relative to k^2, is always greater than or equal to the class.

Further:

A system of ∞^1 *pairs of points associated relative to* k^2 *of the order* ν *and the class* μ *contains* $2(\nu-\mu)$ *coincidences.*

A surface Ω of the order r which contains s leaves through k^3 touching $\varkappa$ on k^3 and of which t more leaves pass through k^3, is the image of a system Γ of ∞^2 pairs of points associated relative to k^2. We suppose that A as well as any point of k^3 is an $(s+t)$-fold point of Ω.

The order of Γ, i.e. the number of pairs of this system of which one point lies in a given point of α, hence the number of points of intersection of Ω with a straight line s which touches $\varkappa$ on k^3, that are not singular for the representation, is $r-2s-t$ and the class, i.e. the number of pairs of Γ that belong to a given straight line of α, hence the number of non-singular points of intersection of Ω and a chord k of k^3, is equal to $r-2s-2t$. As each generatrix of $\varkappa$ cuts the surface Ω in $r-3s-2t$ non-singular points, Γ has an $r-3s-2t$-fold full coincidence in B, i.e. a coincidence of which the carrier is indefinite and which forms an $r-3s-2t$-fold pair of points of Γ.

A surface Ω *of the order* r *containing* s *leaves through* k^3 *which touch* $\varkappa$ *on* k^3, *and which has* t *more leaves through* k^3 *and an* $(s+t)$*-fold point in* A, *is the image of a system of* ∞^2 *pairs of points associated relative to* k^2 *of the order* $r-2s-t$ *and of the class* $r-2s-2t$ *that has an* $(r-3s-2t)$*-fold full coincidence in* C.

The conic k^2 is a t-fold curve of coincidence for Γ.

The pairs of Γ define an involutorial correspondence $(r-2s-t,\ r-2s-t)$. If a point describes a straight line of α, the points corresponding to it in this involutorial correspondence describe a curve of which the order is equal to the number of non-singular points of intersection of Ω with the image of the system of the ∞^1 pairs of points associated relative to k^2 which have a point on each of two given straight lines of α, i.e. according to § 6 or according to the second theorem of this § a conic which touches $\varkappa$ twice on k^3. For the order of the curve associated to a straight line, we find therefore $2r-4s-2t$.

Inversely we investigate the representation of a system Γ of the order ϱ and of the class σ that has no full coincidence in B. The latter condition may again be satisfied by a system Γ of any kind. A straight line of α through B contains as many pairs of points non-singular for the representation and different from the coincidence in B, as an arbitrary line of α. A chord of k^3 through A must cut Ω outside A in as many points as a chord through an arbitrary point P of k^3 has in common with Ω outside P, so that A and P are points of the same multiplicity for Ω.

We may, therefore, find the characteristic numbers r, s and t of Ω by solving them out of the equations:

$$\begin{aligned} r-2s-t &= \varrho \\ r-2s-2t &= \sigma \\ r-3s-2t &= 0. \end{aligned}$$

We find:

A system Γ of the order ϱ and of the class σ that has no full coincidence in B, is represented on a surface Ω of the order $2\varrho + \sigma$ which contains σ leaves through k^3 touching $\varkappa$ on k^3, which has, besides, $\varrho - \sigma$ leaves through k^3, and which contains a ϱ-fold point in A.

Accordingly the order of a system Γ is always greater than the class.

For a system of ∞^2 pairs of points associated relative to k^2 of the order ϱ and of the class σ, k^2 is a $(\varrho - \sigma)$-fold curve of coincidence. For $2\varrho - \varrho - \sigma = \varrho - \sigma$ of the points of intersection outside k^3 of an arbitrary straight line s touching $\varkappa$ on k^3 lie on k^3 if we choose for s a tangent to this curve.

The order of the curve which in the involution defined by Γ is associated to a straight line, is equal to the number of non-singular points of intersection of Ω and a conic which touches $\varkappa$ twice on k^3; hence it is equal to 2ϱ.

§ 10. A system γ of ∞^1 and a system Γ of ∞^2 pairs of points associated relative to k^2, have as many pairs in common as the image curve of γ and the image surface of Γ have points of intersection that are not singular for the representation. If the class and the order of γ are resp. μ and ν, the class and the order of Γ resp. ϱ and σ, we find for the number of the points of intersection in question:

$$\nu(2\varrho + \sigma) - \mu(2\sigma + \varrho - \sigma) - 2(\nu - \mu)\varrho = \mu\varrho - \mu\sigma + \nu\sigma.$$

A system of ∞^1 pairs of points associated relative to k^2 of the class μ and the order ν has

$$\mu(\varrho - \sigma) + \nu\sigma$$

pairs in common with a system of ∞^2 similar pairs of the order ϱ and the class σ.

The intersection (Ω, Ω') different from k^3 of the surfaces Ω and Ω' which represent two systems $\Gamma(\varrho,\sigma)$ and $\Gamma'(\varrho',\sigma')$, is of the order:

$$(2\varrho + \sigma)(2\varrho' + \sigma') - 3\varrho\varrho' - 3\sigma\sigma' = \varrho\varrho' + 2\varrho\sigma' + 2\sigma\varrho' - 2\sigma\sigma'.$$

The curve which Ω has in common with a surface Φ, cuts each chord of k^3 on this surface in σ points, each straight line of Φ cutting k^3 once, in $\varrho + \sigma$ points, and it is accordingly a curve $(\sigma, \varrho + \sigma)$ of this quadratic surface. According to a well known theorem this curve has

$$\sigma(\varrho' + \sigma') + \sigma'(\varrho + \sigma) = \varrho\sigma' + \sigma\varrho' + 2\sigma\sigma'$$

points in common with the curve $(\sigma', \varrho + \sigma')$ along which Ω' cuts Φ outside k^3.

According to § 2 Φ touches the cone $\varkappa$ in one point of k^3, hence also σ leaves of Ω and σ' leaves of Ω'. The curves of intersection different from k^3 of Ω and Φ and of Ω' and Φ have resp. a σ- and a σ'-fold point in this point of k^3; they cut each other, therefore, outside k^3 in

$$\varrho\sigma' + \sigma\varrho' + \sigma\sigma'$$

points.

These are the points of intersection outside k^3 of the curve (Ω, Ω') and Φ. Hence in all (Ω, Ω') cuts the curve k^3 in

$$2\,(\varrho\varrho' + 2\varrho\sigma' + 2\sigma\varrho' - 2\sigma\sigma') - \varrho\sigma' - \sigma\varrho' - \sigma\sigma' = 2\,\varrho\varrho' + 3\,\varrho\sigma' + 3\,\sigma\varrho' - 5\sigma\sigma'$$

points.

If we choose Γ and Γ' so that the system common to these systems does not contain any pair of which one point lies in B, the curve (Ω, Ω') does not pass through A, nor does it cut the cone $\varkappa$ outside k^3. The points of intersection outside k^3 of (Ω, Ω') and an arbitrary quadratic surface through k^3, must lie on k^3 if we choose the cone $\varkappa$ for this surface.

Consequently (Ω, Ω') touches the cone $\varkappa$ in $\varrho\sigma' + \sigma\varrho' + \sigma\sigma'$ points on k^3, and cuts this curve besides in $2\,\varrho\varrho' + 2\,\varrho\sigma' + 2\,\sigma\varrho' - 6\,\sigma\sigma'$ points.

We find therefore by the aid of the first theorem of § 9:

Two systems Γ (ϱ, σ) and Γ' (ϱ', σ') have in common a system of ∞^1 *pairs of points associated relative to k^2 of the class $\varrho\sigma' + \sigma\varrho' + \sigma\sigma'$ and of the order $\varrho\varrho' + 2\varrho\sigma' + 2\sigma\varrho' - 2\sigma\sigma'$.*

If we determine the number of non-singular points of intersection of (Ω, Ω') with the image surface of a third system Γ'' (ϱ'', σ''), we find:

Three systems Γ (ϱ, σ), $\Gamma'(\varrho' \sigma)$ and Γ'' (ϱ'', σ'') have

$$\varrho\varrho'\sigma'' + \varrho\sigma'\varrho'' + \varrho\sigma'\sigma'' + \sigma\varrho'\varrho'' + \sigma\varrho'\sigma'' + \sigma\sigma'\varrho'' - 3\sigma\sigma'\sigma''$$

pairs of points in common.