Mathematics. — *"A Representation of the Rays of Space on the Pairs of Points of a Plane"*. By Prof. JAN DE VRIES.

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§ **1.** Let l be an arbitrary straight line which cuts the planes α' and α'' in the points L' and L''. L' and L'' are projected on the plane α out of the fixed points A' and A''; the projections L_1 and L_2 form a *pair of points* which we consider as the *image* of the *ray* l. Apparently two arbitrary points L_1 and L_2 of α generally define *one* ray.

Let G be a point of the intersection g of the planes α' and α''; all the rays of the sheaf round G are represented in the pair of points G_1, G_2, which we call a *cardinal pair*. The cardinal pairs form two projective point-ranges on the lines g_1 and g_2 (the projections of g out of A' and A''); these meet in the point of intersection G_{12} of g and α. Accordingly the point-ranges (G_1) and (G_2) lie *perspectively*; the *center of perspectivety* A_{12} is the point of intersection of α and the line a which joins the centers A' and A''.

Any two points of g may be considered as intersections with α' and α''. Hence any pair consisting of an arbitrary point G_1 and an arbitrary point G_2 may be considered as the image of the *cardinal ray* g.

If l lies in α', L' is an arbitrary point of $l' \equiv l$; if G_2 is the projection of the point $g\,l$, the image of l consists of G_2 and an arbitrary point L_1 of the line l_1 (the projection of l out of A' as center).

The rays in α' and α'' are, therefore, *singular rays*.

§ **2.** The image of a straight line d cutting a consists of two points D_1, D_2, that are collinear with A_{12}. The pairs of points on a line d_{12} through A_{12} are the images of the rays in a plane through a.

The pairs of points L_1, L_2 on an arbitrary line are the images of the rays of a bilinear congruence of which the directrices lie in α' and α''.

The *image* of a *field of rays* consists of the ∞^2 pairs of points L_1, L_2, of which L_1 lies on a straight line f_1, L_2 on a straight line f_2. One of these pairs is formed by the images G_1, G_2 of the plane pencil of the rays resting on g.

The point of intersection D_{12} of f_1 and f_2 is the image of a ray d which cuts a. The *congruence* of the rays for which the images L_1 and L_2 coincide, has accordingly one ray in any plane.

If D_{12} describes a straight line d_{12}, the points D' and D'' describe two projective point-ranges in α' and α'' and d envelops a conic which touches a. Hence through a given point there pass two rays d of the congruence in question; this has accordingly the symbol [2, 1]. In the plane (G_{12}, a) the conic degenerates, for G_{12} is the image of a pencil in that plane.

Consequently *the rays with coinciding image points* are the *tangents* of a *quadratic cone* with *vertex* G_{12} which rest on the *fixed* tangent a [1]).

§ **3**. *Image of a plane pencil.* The image of a plane pencil is formed by *two projective point-ranges* on two straight lines f_1, f_2. One pair consists of the image G_1, G_2 of the ray resting on g, and another pair, D_1, D_2, is the image of the ray that cuts a.

If we choose the line f_1 at random and if G_1 is its intersection with g_1, G_2 the point which forms a cardinal pair with G_1, f_2 must pass through G_2. If we associate two arbitrary points P_2, Q_2 of f_2 to two arbitrary points P_1, Q_1 of f_1, and if G_2 corresponds to G_1, the point-ranges on f_1 and f_2, which in this way have become projective, are the image of a plane pencil.

The projective point-ranges on f_1 and f_2 define a conic δ^2 as the envelope of the lines $l_{12} \equiv L_1 L_2$. One of the tangents through A_{12} contains the cardinal pair G_1, G_2, the other the pair D_1, D_2. If we consider δ^2 as the image of the plane pencil, the ∞^5 *plane pencils* of space are *represented* on the ∞^5 *conics* of a plane.

But in this way any conic is the *image* of *two* plane pencils; for each of the two tangents through A_{12} may be considered as $g_{12} \equiv G_1, G_2$, the other containing the pair D_1, D_2. The tangents through G_1 and G_2 define the carriers f_1, f_2 of the projective point-ranges.

By means of the conics δ^2 we find accordingly an *involution* in the *plane pencils of space.*

If the vertex T of a plane pencil lies in a', δ^2 degenerates. For in this case the image consists of the pairs formed by T_1 and the points L_2 of a straight line f_2 and of the pairs formed by the point G_2 on f_2 and the points L_1 of f_1.

If the whole plane pencil lies in a', each ray has ∞^1 images consisting of a point G_2 and a point of a definite ray of the plane pencil round T_1.

If the plane of the pencil passes through G_{12}, its image consists of two perspective point-ranges and δ^2 degenerates into two plane pencils. If the plane passes through a, the image consists of two collocal projective point-ranges.

§ **4**. *Image of a quadratic scroll.* The image is formed by *two projective point-ranges* on two *conics* a_1^2 en a_2^2. The points of intersection of a_1^2 and g_1 form cardinal pairs with the points where a_2^2 is cut by g_2;

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[1]) A congruence [1,2] consists of the transversals of a conic and a fixed straight line cutting it. A [2,1] corresponds to it dually.

these pairs are the images of the rays that rest on g. The two rays resting on a have images L_1, L_2, for which l_{12} passes through A_{12}.

Let $\alpha_1{}^2$ be an arbitrary conic, G_1 and $G_1{}^*$ its points of intersection with g_1. If we pass a conic $\alpha_2{}^2$ through G_2 and $G_2{}^*$ and establish a projective correspondence between the point-ranges on $\alpha_1{}^2$ and $\alpha_2{}^2$ so that G_1 and $G_1{}^*$ are associated to G_2 and $G_2{}^*$, we have obtained the image of a quadratic scroll. If the scroll has a *directrix* f' in α' and a *directrix* f'' in α'', its image is formed by two projective point-ranges on the lines f_1 and f_2. In this case the points $f_1\, g_1$ and $f_2\, g_2$ do not form a cardinal pair. To the image there belong also the pairs of points which represent the rays of the scroll in α' and α''.

§ **5.** *Image of a sheaf.* The rays through the point S make the fields of points $[L']$ and $[L'']$ perspective. Accordingly the image of the sheaf is formed by the pairs L_1, L_2 of *two projective fields.*

The plane pencil of the rays resting on g, has its image in the *cardinal pairs.* To the sheaf there belong two rays of the congruence [2, 1], the rays of which are represented by points D_{12}. These two points and the point G_{12} are the *coincidences* of the two fields.

If the projective correspondence between the points of the fields $[L_1]$ and $[L_2]$ is such that the cardinal pairs consist of homologous points, we have the image of a sheaf. In order to see this we investigate what this image, **B**, has in common with the image **V** of a field of rays and with the image **S** of an arbitrary sheaf.

The image **V** consists of the ∞^2 pairs L_1, L_2 on two lines f_1, f_2. The straight line $f_2{}^*$ which is associated to f_1 in **B** cuts f_2 in the point G_2 which is associated to the point G_1 on f_1. Hence **V** and **B** have only this pair (G_1, G_2) in common. But a cardinal pair is the image of a sheaf (round G); accordingly the field of rays generally does not contain any ray of the congruence that has **B** as image.

Let L_1, L_2 be a pair of **B**, $L_1{}^*$ the point which through **S** is associated to L_2. In this case the points L_1 and $L_1{}^*$ are homologous in a projectivity that has all the points of g_1 as double points and is, therefore, a *homology.*

The center of the homology is a point L_1 to which **B** and **S** associate the same point L_2. But then **B** must be the image of a congruence [0,1], hence of a sheaf.

§ **6.** Let us suppose that a *homology* in α contains the cardinal points; these are the images of the rays of the congruence that is represented by the homology.

A ray of this congruence which cuts a, has for image a pair D_1, D_2. The line d_{12} through D_1 and D_2 contains also a cardinal pair and is, therefore, a double ray of the homology; accordingly this has A_{12} for center and its axis passes through G_{12}. The point-ranges on homologous

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lines are projective, hence images of a plane pencil that has a as a ray. Consequently the homology in question is the *image* of a *parabolic congruence* [1,1] that has a as *directrix*.

§ **7.** *Image of a bilinear congruence.* The rays of a congruence [1,1] define a *quadratic correspondence* in α. For the line f' of α', which is the projection of a line f_1 in α, defines, together with the directrices r and s of the congruence, a quadratic scroll, hence a conic φ'' in α'', consequently also a conic φ_2 in α. The ray t' of the [1,1] that lies in α', defines a point G^* on g. Hence φ_2 passes through G_2^* and through the images R_2 and S_2 of r and s. These three points are the *cardinal points* of the latter system. The *cardinal points* of the former system are R_1, S_1 and the image G_1^* of the ray in α''.

The plane pencil of the congruence that has R' as vertex, has for image the point-range on $S_2G_2^*$, apart from the image of t'. Together with the point-range on R_1S_1 the cardinal point G_2^* forms the image of the ray t'.

The plane pencils of the [1,1] that have their vertices on s, are represented in the point-ranges on lines f_1 and f_2 of which f_1 passes through R_1 and f_2 through R_2; the plane pencil (f_1) is projective with the plane pencil (f_2). Analogously S_1 and S_2 are the centers of two projective plane pencils, and any two homologous rays contain the image of a plane pencil that has its vertex on the directrix r.

A *parabolic* [1,1] consists of ∞^1 plane pencils which have a ray r in common while the vertices form a point-range on r which is projective with the pencil of their planes. The quadratic scroll which has a line of α' as directrix, contains a ray of the plane pencil of the [1,1] that has R'' as vertex. The plane of this pencil touches the carrier of the scroll at R''; accordingly the conics φ'' have a fixed tangent at R''. But then the conics φ_2 have also the same tangent at R_2. Consequently the *quadratic correspondence* has *two coinciding cardinal points* in R_2; this is also the case in R_1. This result could be foreseen because in this case the directrix s coincides with r.

§ **8.** We arrive at an *involution* in the *rays of space* by associating to a ray l with image L_1, L_2 the ray m of which the image consists of the points $M_1 \equiv L_2$ and $M_2 \equiv L_1$.

If l describes a plane pencil, L' and L'' describe projective point-ranges; hence L_1 and L_2 describe projective point-ranges on two lines l_1 and l_2. The points $G_1 \equiv g_1 l_1$ and $G_2 \equiv g_2 l_2$ form a cardinal pair. But this is not the case with the points $H_1 \equiv G_2$ and $H_2 \equiv G_1$; hence the point-ranges (M_1) on $m_1 \equiv l_2$ and (M_2) on $m_2 \equiv l_1$ form the representation of a quadratic scroll. Accordingly our *involution* transforms a plane pencil into a *quadratic scroll*.

The double rays of this involution form the congruence [2,1] found in § 2.

The rays in a plane through a are arranged in involutorial pairs.

To the ray g there corresponds the field of rays in the plane defined by $h_1 \equiv g_2$ and $h_2 \equiv g_1$.

§ **9**. As we pointed out, the conics δ^2 give rise to an *involution* in the *plane pencils of space*.

In this way each of the two systems of plane pencils of a congruence [1,1] is transformed in itself. Let us consider e.g. the pencils in the planes through the directrix r, which, therefore, have their vertices on the directrix s. Let δ^2 be the image of such a plane pencil; one of its tangents through A_{12}, G_{12}, contains a cardinal pair G_1, G_2, the other, d_{12}, contains the image (D_1, D_2) of the ray of the pencil that rests on a. The straight lines $f_1 \equiv R_1 G_1$ and $f_2 \equiv R_2 G_2$ carry the image points L_1, L_2 of the other rays of the pencil.

If we also draw the tangents f_1^* and f_2^* through R_1 and R_2, these contain the images of the rays of another plane pencil belonging to the system; now $d_{12} \equiv g_{12}^*$ contains the cardinal pair G_1^*, G_2^*, $g_{12} \equiv d_{12}^*$ the pair (D_1^*, D_2^*).

Any tangent l_{12} to δ^2 contains the image (L_1, L_2) of a ray of the former plane pencil and the image (L_1^*, L_2^*) of a ray belonging to the latter. The line l_{12} contains the images of the rays of a congruence [1,1] that has as directrices a straight line of α' and one of α''. This congruence has two lines in common with the given [1,1]; each of them belongs to one of the plane pencils in question.

An arbitrary line of α is, therefore, touched by one δ^2; accordingly the conics of our system form a *scroll*. The two conics meeting in A_{12}, are the images of the plane pencils lying in the double planes of the involution round r.

The *pairs of points* of the scroll are (S_1, G_2^*), (S_2^*, G_1) and the pair to which G_{12} belongs.

If the congruence [1,1] is *parabolic* with the directrix r, the conics of the system have the line $R_1 R_2$ as common tangent.

§ **10.** Let us now consider the system Σ of the *plane pencils* with *common ray* r that have their *vertex* in the point S (and which belong, therefore, to the sheaf round S).

All the conics δ^2 touch the straight line $r_{12} \equiv R_1 R_2$ and the intersection d_{12} of their planes and the plane (Sa); for each plane pencil has one ray in the latter plane and the images of these rays form two projective point-ranges on d_{12}.

Now to any plane pencil there corresponds the plane pencil of the rays that have their images on the tangents f_1^*, f_2^* through the points G_1^*, G_2^* on d_{12}^*. The new pencils form a congruence Σ^*; their planes pass through the point G^*. Evidently the plane pencils (f_1^*) and (f_2^*) are projective; hence the lines f' and f'' (in α' and α'') describe two projective plane pencils round G^* and the *planes* of the pencils of Σ^* envelop a *quadratic cone*. Consequently an arbitrary point lies on *two* rays of the pencils and an arbitrary plane contains *one* ray.

The system Σ is transformed by the involution into a *congruence* [2,1].

§ **11.** Let Σ be the system of the *pencils* in a *plane* ϱ that have one *ray* r in common. If the field of rays in ϱ is represented by the pairs of points on f_1 and f_2, the plane pencils of Σ have as images the conics which touch f_1, f_2, r_{12}, and g_{12}, which form accordingly a *scroll*.

Any line d_{12} defines one δ^2; the tangents f_1^* and f_2^* through the points G_1^* and G_2^* on d_{12} contain the images of the rays of the associated plane pencil. Any plane contains one ray of the system Σ^*.

Let L_1 be a point of a, G_1 a point of g_1, δ^2 the conic which touches $L_1 G_1$. The tangent d_{12} cuts g_1 in a point G_1^*, which we associate to G_1. If we choose G_1^* arbitrarily on g_1, $A_{12} G_1^*$ defines a δ^2 of which two tangents meet in L_1. Hence g_1 contains *three* points G_1^* for which $A_{12} G_1^*$ and $L_1 G_1^*$ touch the same δ^2. Each of the three lines $f_1^* \equiv L_1 G_1^*$ corresponds to a plane pencil of Σ^*. Accordingly this system is a *congruence* [3,1].

It consists of the *pencils* in the *planes of osculation* of a *twisted cubic* of which the *vertices* lie on the intersection of two planes of osculation [1]).

[1]) This congruence corresponds dually to the [1,3] which has a twisted cubic and one of its bisecants as directrices. The other [1,3] which consists of the bisecants of a curve ϱ^3, does not contain any plane pencils.