Mathematics. — *"On a Group of Representations of the Linear Complex of Rays"*. By M. N. VAN DER BIJL. (Communicated by Prof. JAN DE VRIES).

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§ 1. We have in view all the representations of a linear complex L on the points of space *for which each ray contains its own image*. These have a number of properties in common of which the most important follow here. In the first place these two:

a. A plane pencil out of L is represented on a conic k^2 through the null point of the plane of the pencil which is touched at this point by the ray that has its image in the point.

b. The image of a net of rays out of L is a cubic surface O^3 through the directrices of the net (u and v). O^3 has the image of u as double point if u belongs to L.

Proof of *a*: the image curve is plane, has a single point in the null-point, and cuts any ray of the pencil, besides, in the image point of this ray.

Proof of *b*: A plane ν through u contains of the image surface the conic k^2 that corresponds to the pencil in ν, and u itself (each point of u corresponds as image to one ray of the net). The same holds good for v, the polar line of u. If $u \equiv v$ is a complex ray, the other part of the intersection, k^2, continues to pass through the image point of u if ν turns round u. This point is, therefore, a node of O^3.

§ 2. *The singular figures.*

If ν turns in the indicated way, k^2 degenerates 5 times into a pair of lines, according to a well known property of the cubic surfaces. The vertices of complex-plane pencils with degenerate image curves form, therefore, a surface of the fifth order; for the arbitrary line v contains 5 of these vertices. As the degeneration occurs only when the corresponding plane pencil contains a singular ray, v is cut by 5 such rays; in other words: there exists a scroll of the fifth order $R^5{}_s$ of singular rays (*singular scroll*). It coincides with the surface of the fifth order mentioned above.

The nodal curve δ of this surface is the locus of the singular points (*singular curve*), for a point on δ is among others the image of 2 singular rays and, inversely, through any singular point there must pass 2 singular rays, to wit the pair into which k^2 degenerates for this point.

Let S be a point of δ, s_1 and s_2 the 2 singular rays through S. A plane ν through s_1 cuts $R^5{}_s$ along a figure of the fifth order consisting of s_1 and a curve of the fourth order which cuts s_1 in the point of contact of ν and in 3 more points, including S, which belong to δ. Applying the same to s_2 we find in all 5 points (δ, ν). A point P of ν outside s_1 and s_2 cannot belong to δ, for then the complex ray SP would be singular so that the image of the plane pencil S would become a figure of the third order. In this way we have found:

There exists a singular scroll of the fifth order with a nodal quintic as singular curve (δ^5_s).

Considering these figures we may add to property I_a that k^2 is wholly defined by the 5 points of intersection of δ^5_s and the plane of the pencil; and to I_b that O^3 contains 5 singular rays, to wit the images of the points (u, R^5_s), and further the whole curve δ^5_s, because each of its points corresponds as image to the ray of the net passing through it.

§ 3. *a. Image of a point-range u.* Let v be again the associated polar line. A plane ν through u contains 2 generatrices of the image: the 2 rays that join the nullpoint (ν, v) to the points of intersection (k^2, u). Through any point of u there passes one generatrix: the image ray of that point. Accordingly we find a cubic scroll R^3_u, that has u as single, v as double directrix. To the points (R^5_s, u) there correspond 5 image rays, which are common generatrices of R^5_s and R^3_u. If u is a complex ray, R^3_u becomes a surface of CAYLEY, for in this case u is a directrix and at the same time a generatrix.

b. Image of a field of points V. The conic which represents the plane pencil through an arbitrary point or in an arbitrary plane, cuts V in 2 points. Hence the locus of the image rays is a congruence [2,2]. 16 plane pencils belong to this. In the first place the pencil in V; further 5 plane pencils with vertices in (V, δ^5_s); the remaining 10 lie in the planes in which k^2 has one of the 10 joins of these vertices as a non-singular component. Non-singular component of a degenerate k^2 can be all the chords k of δ^5_s in the plane through k and the singular ray through the point (k, R^5_s) outside δ^5_s. The congruence [2, 2] also contains the whole singular scroll; it is produced by the points of its intersection with V.

§ 4. *a.* Image of a plane curve r^n (in the plane V with nullpoint N). This intersects V along r^n and along the $2n$ rays of the plane pencil (N, V) that are the images of the points (r^n, k^2_V). It is accordingly a scroll R^{3n} with r^n as directrix and with $5n$ singular rays among its generatrices owing to the $5n$ points of intersection (r^n, R^5_s). The nodal curve of R^{3n} passes through the $2n(n-1)$ points of intersection of r^n with the generatrices through N in so far as they do not lie on k^2_V.

b. Image of a twisted curve ϱ^n. The congruence (u, v) of L has as image an O^3. To the $3n$ points (ϱ^n, O^3) correspond $3n$ rays, that rest on u and have their images on ϱ^n. The image scroll is accordingly an R^{3n}. Among the generatrices there are $5n$ singular rays corresponding to the points $(\varrho^n, R^5_s.)$

c. Image of a scroll R^n. This is found through inversion of *b*. The image curve ϱ^x has as image an R^{3x} but also R^n completed by the null planes of the $5n$ singular points (R^n, δ^5_s) that lie on ϱ^x. Hence: $3\,x = n + 5\,n$, consequently $\varrho^x = \varrho^{2n}$.

Also directly in the following way: ϱ^x cuts a plane in as many points as there are rays common to R^n and the congruence [2,2] which represents the points of V. And for this number we find $2n$.

§ 5. *a*. Image of a surface O^n. As O^n cuts the conic k^2 of an arbitrary point in $2n$ points and also the k^2 of an arbitrary plane, the image congruence is a $[2n, 2n]$. Any generatrix of R^5_s has n points in common with O^n so that the singular scroll, counted n times, belongs to the $[2n, 2n]$. Further $5n$ plane pencils of the congruence correspond to the points (O^n, δ^5_s).

b. Image of a $[p, p]$ of L. Let this be a surface O^x. This passes through δ^5_s with p leaves because in any point of the singular curve p rays of the $[p, p]$ are represented. Now inversely the image of O^x is a $[2x, 2x]$, but also the $[p, p]$ completed by p times the congruence [5, 5] which consists of all the complex-plane pencils with nullpoints on δ^5_s. This leads to the equation $2\,x = p + 5\,p$, hence $O^x = O^{3p}$.

§ 6. A straight line u that contains 3 singular points, is a (singular) ray, as otherwise the scroll R^3_u which represents the points of u, would be of an order higher than 3. Inversely a singular ray u always contains 3 singular points for if this number were more or less the order of R^3_u would be too high or too low; in other words: u is a trisecant of δ^5_s. Hence R^5_s is *the scroll of the trisecants of* δ^5_s.

For this reason δ^5_s cannot be a rational curve. For this would have a surface of trisecants of the order 8. The singular curve is of the genus 1. This appears as follows. The nets of rays (u, v) and (w, x) out of L have as images cubic surfaces O^3_u and O^3_w, which cut each other along δ^5_s and along a ϱ^4 on which the scroll $R^2 = (u, v, w)$ is represented which is common to the two congruences [1, 1]. The generatrices of R^2 which belong to the nets, are unisecants of ϱ^4 for all the representations in question. Such a generatrix can only have its image point in common with ϱ^4 as it is not cut by any other straight line of the same kind and consequently cannot contain any other image. The other system of straight lines on R^2 consists, accordingly, of trisecants of ϱ^4 and this curve is, therefore, rational; the number of its apparent double points is 3.

Now the theory of the intersection of 2 algebraic surfaces O^m and O^n of which the intersection ϱ^{mn} consists of a ϱ^p and a ϱ^q, teaches that:

$$h_{pq} + 2h_q = q\,(m-1)\,(n-1).$$

h_{pq} is the number of straight lines through a point P which have one point in common wich ϱ^p and another with ϱ^q and h_q is the number of chords of ϱ^q through P.

We apply this to $\varrho^9 = (O^3_u, O^3_w) = \varrho^4 + \delta^5_s$. Now $h_{pq} = 10$, for although the cones (P, ϱ^4) and (P, δ^5_s) have 20 generatrices in common, 10 of them do not cut ϱ^4 and δ^5_s in different points for ϱ^4 contains the 10 points (R^2, δ^5_s). Further $q = 5$, $m = n = 3$, so that the above mentioned equation gives: $10 + 2h_q = 5(3-1)(3-1)$, consequently $h_q = 5$, hence δ^5_s is of the genus 1.

It is in accordance with this that we can pass ∞^4 cubic surfaces through δ^5_s. For the complex contains ∞^4 nets (u, v) and each of them has its own image O^3 through δ^5_s. The condition that O^3 must pass through δ^5_s is accordingly 15-fold.

§ 7. Also several properties regarding degeneration of image figures hold good for all the representations in question, e. g.: there exists a complex of the fifth order of lines u for which R^3_u degenerates into a scroll and a plane; this complex contains all the lines that rest on δ^5_s and has, therefore, this curve as locus of the cardinal points. A congruence [5, 10] belongs to it which consists of the chords of the singular curve which contains lines u for which R^3_u degenerates into a triple of planes one of which passes through u. To this congruence there belongs again the singular scroll for the generatrices of which R^3_u degenerates into 3 planes through u. As a locus of points R^5_s is the set of the vertices of all the plane pencils with degenerate image-conics k^2, the non-singular components of these pairs of lines form the above mentioned congruence [5, 10].

§ 8. The singular figures themselves can also degenerate. This will appear from a few examples which serve at the same time as a check on the above.

a. Suppose a fixed plane α and in it 2 projective pencils (F_1, α) and (F_2, α); further a fixed straight line a through F_2 outside α. A ray s of L cuts one ray t_1 of (F_1, α). Associated to this is t_2 of (F_2, α). Put $\mu = (a, t_2)$ and take $S = (\mu, s)$ as image of s. Inversely we can find the image ray of a point S through the following construction: $\mu = (a, S)$; $t_2 = (\mu, \alpha)$; t_2 gives the homologous ray t_1 of (F_1, α); $T = (\nu, t_1)$, where ν is the null-plane of S; $s = S\,T$.

b. If S is a point of a, μ is indefinite; also t_2 and, accordingly, t_1, so that T becomes any point of (ν, α) and s any ray of (S, ν). In other words: all the points of a are singular points.

The projective pencils in α produce a conic α^2 through F_1 and F_2. A point S on it defines a definite plane $\mu = (S, a)$, hence also definite lines t_2 and t_1, but $T = (\nu, t_1) \equiv S$, so that for s we may choose any ray of the plane pencil (S, ν) : a^2 is another part of the locus of the singular points.

Let t_1, t_2 and μ have the usual meaning. Let β be the nullplane of F_1; this passes through A, the nullpoint of α. β contains p_1 through A, the directrix of L associated to t_1, and $p_2 = (\mu, \beta)$. Consider $S = (p_1, p_2)$, which lies in μ. The null plane ν of S passes through t_1; the point of intersection (ν, t_1) is, therefore, indefinite; consequently S is a singular point. Owing to the correspondence (1,1) between the plane pencils (p_1) and (p_2), the locus of S is a conic b^2 in β through A and $B = (a, \beta)$. Let C be the second point of intersection of AF_1 with a^2 and choose $p_1 = AC$. Then $t_1 = AC$, because AC belongs to L. Further $t_2 = F_2C$ and $p_2 = BC$, hence $S = C$. Consequently the conics a^2 and b^2 cut each other in C.

Suppose that P outside a, a^2 and b^2 is a singular point. The cones (P, a^2) and (P, b^2) have 3 generatrices in common besides PC, which of course do not lie in one plane. But according to § 6 they must nevertheless be (singular) complex-rays. Hence P is not a singular point. The locus of the singular points is $a + a^2 + b^2$, a degenerate δ^5_s; it has 5 apparent double points. For the cones (P, a^2) and (P, b^2) have three generatrices in common which rest on a^2 and b^2 in two different points; and the plane (P, a) cuts each of these cones besides along PF_2, resp. PB, along one generatrix in 2 non-coinciding points.

c. A ray of (F_1, β) cuts all the rays t_1 (in F_1); t_1, t_2 and μ are therefore, indefinite, hence also $S = (\mu, s)$. The same holds good for the rays of (A, α), for these also cut all the t_1. The scroll which has a, a^2 and b^2 as directrices, is of the third order: R^3_s. Each of the generatrices of R^3_s contains 3 singular points and is, therefore, a singular ray.

Accordingly we have found a figure of the fifth order, $R^5_s = \alpha + \beta + R^3_s$, which consists of such rays. That this is the locus of these rays appears e.g. in the following way: let s be a singular ray which does not belong to R^5_s; its intersection S with α is the image of s and SA and would, therefore, be a singular point; but α does not contain any such a point outside a^2.

As it should be $a + a^2 + b^2$ appears to be the nodal curve of $\alpha + \beta + R^3_s$.

§ 9. Another possible degeneration of δ^5_s is: 2 crossing straight lines (a and a_1) with a transversal (a_2) and a conic (b^2) which cuts the former two lines. This happens in the representation through 2 projective plane pencils if we choose them in perspective correspondence. In this case a^2 degenerates into $a_2 = F_1F_2$ and the axis of perspectivity a_1, which continues to have a point C in common with F_1A through which b^2 also passes. The singular surface (R^5_s) consists of the planes α, β and (a, a_2) and the scroll that has a, a_1 and b^2 as directrices.

We can also establish a perspective correspondence between the plane pencils (A, β) and (B, β). For this it is only necessary that we associate the directrix of L corresponding to AB as homologous ray to the ray AF_2 of the plane pencil (F_2, α).

Finally we can transform both projectivities into perspective correspondences, through which δ_s^5 is transformed into a skew pentagon. Instead of b^2 we find $b_1 = AB$ and a straight line b_2 cutting b_1 and cutting a_1 in C. R_s^5 has degenerated into 5 planes (a, b_1), (b_1, b_2), (b_2, a_1), (a_1, a_2) and (a_2, a).

It is easily seen that for the latter degenerations the number of apparent double points is again 5 and that R_s^5 has the curve δ_s^5 as nodal curve.

A very special case is $F \equiv F_1 \equiv F_2$. In this case we have 2 collocal projective plane pencils; we may also use an involution of rays. Geometrically it is easily seen that again the fixed straight line a and a conic b^2 in the null plane β of F (quite analogous to the homonymous plane of § 8 *b*), belong to the singular points, and that $a + b^2$ is completed to a δ_s^5 by the rays of coincidence c_1 and c_2. It appears as above that the complex pencils in β and in $\alpha = (c_1, c_2)$ consist of singular rays.

The cubic scroll R_s^3 splits up into the planes (a, c_1), (a, c_2) and β.

Accordingly the null plane β of F must be considered as a double plane in the locus of singular rays and, therefore, an arbitrary point of $\delta_s^5 = a + b^2 + c_1 + c_2$ is again the point of issue for 2 singular rays. As b^2 appears to pass through the null points of α, β, (a, c_1) and (a, c_2), all these rays are again trisecants of δ_s^5.

§ 10. Also each of the other forms of degeneration of the singular curve has its own singular scroll, degenerate or not, which may always be derived from it through the relation: $R_s^5 \equiv$ *surface of trisecants of* δ_s^5.

a. $\delta_s^5 =$ *rational* $\delta^4 +$ *chord k.*

The singular scroll consists of the scroll of trisecants of δ^4 and the scroll formed by the chords of δ^4 through the points of k; it is easily seen that this surface is of the third order and has k as a nodal line.

b. $\delta_s^5 =$ *non-rational* $\delta^4 +$ *unisecant k.*

In this case the surface R_s^5 is not degenerate: it consists of the chords of δ^4 that rest on k. Besides the line k, counted double, the intersection with a plane V through k contains 3 more chords of δ^4 through 3 singular points, namely the joints of the 3 points (V, δ^4) outside k; accordingly this intersection is indeed of the fifth order.

c. $\delta_s^5 = \delta^3 + \delta^2$; δ^3 *and* δ^2 *have* 2 *points in common.*

R_s^5 consists of the pencil in the plane of δ^2 with vertex in the point where this plane is cut by δ^3 outside δ^2, and of a scroll of the fourth order consisting of the chords of δ^3 that rest on δ^2.

This is the case with a few representations of L found by professor JAN DE VRIES, e. g.:

A ray s cuts the fixed plane α in P; let p be the polar line of P relative to a given conic a^2 in α, ϱ the plane through p and a fixed point C; $S = (s, \varrho)$ is chosen as the image of s. Inversely the image ray of a point S is found by choosing that ray of the null plane of S which rests on the polar line q of Q relative to a^2, if Q is the intersection of CS and α.

It is at once clear that the complex plane pencil (A, α) consists of singular rays and a^2 of singular points. We find further that at any point of a^2 one tangent to the cone (C, a^2) may be drawn which belongs to L and does not yield any definite point S because ϱ passes through it. Closer examination shows that this kind of singular rays forms a surface of the fourth order with nodal curve δ^3 which passes through A and through the points where a^2 is touched by the tangents through A. In this way the aforesaid is justified.