Mathematics. — *"On transformations of projective spaces"*. By Prof. L. E. J. BROUWER.

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In my article: *"Ueber Abbildung von Mannigfaltigkeiten"* (Math. Ann. 71, p. 97—115) I have proved the following theorem:

A uniform continuous transformation without fixed point of an n-dimensional sphere into itself, has degree -1 *for even n and* $+1$ *for odd n.*

Since it appears that the specialisation of this theorem which gives an analogous property of uniform continuous transformations of n-dimensional projective spaces into themselves, and of which a particular case had already been published in these Proceedings XI, p. 798, is not to that degree, as I then thought permissible to assume, self evident to all readers, it shall be set forth here in full detail.

In the first place let there be given a uniform continuous transformation τ of a $(2n-1)$-dimensional projective space E into itself. We provide E with a positive indicatrix and with an elliptic metric. Let S be the $(2n-1)$-dimensional sphere provided with a positive indicatrix and a metric, obtained by duplication of E. Let P be a point of E, P_1 and P_2 the corresponding points of S, P' the image of P in E under τ, and P_1' and P_2' the points of S corresponding to P'. Let τ_1 be the uniform continuous transformation of S into itself which brings P_1 into P_1', and which becomes τ by the folding of S into E [1]). Then the volume of the image of S under τ_1 (measured by the volume of the image of S under the simplicial approximations of τ_1) is twice as much as the volume of the image of E under τ. Since, however, the volume of S is also twice that of E, the degree of τ_1 appears to be equal to that of τ. Further, since the absence of a fixed point for τ implies the absence of one for τ_1, *the transformation* τ, *if exhibiting no fixed point must be necessarily of degree* $+1$.

On the other hand there exist arbitrarily small *congruent* transformations of S into itself without fixed point. To these correspond arbitrarily

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[1]) The existence of τ_1 is due to the fact that to a circuit of S passing through P_1 corresponds a contractible circuit of E passing through P the image of which under τ is a contractible circuit of E passing through P' which corresponds to a circuit of S passing through P_1'.

The antipodal point-pairs of S become under τ_1 again antipodal point-pairs or simple points, according as uncontractible circuits of E become under τ again uncontractible or contractible.

small congruent transformations of E into itself without fixed point. *Thus there exist uniform continuous transformations of E into itself of degree $+1$ and without fixed point.*

In the second place let there be given a uniform continuous transformation τ of a $2n$-dimensional projective space E into itself. We provide E with an elliptic metric. Let S be the $2n$-dimensional sphere provided with a metric, obtained by duplication of E. We provide S with a positive indicatrix. Let P be a point of E, P_1 and P_2 the corresponding points of S, P' the image of P in E under τ, and P_1' and P_2' the points of S corresponding to P'. Let τ_1 and τ_2 be the uniform continuous transformations of S into itself which carry P_1 into P_1' and P_2' respectively, and which become τ by the folding of S into E. Then corresponding image simplexes of corresponding simplicial approximations of τ_1 and τ_2 have equal volumes of opposite signs; thus τ_1 and τ_2 have equal degrees of opposite signs, and thus either τ_1 or τ_2 has a fixed point. *Then, however, τ also must have a fixed point* [2]).

[2]) Dr. HOPF points out to me that a uniform continuous transformation of a $(2n-1)$-dimensional projective space E into itself possesses at least two invariant points, if its degree is odd and $\neq +1$. We can add that on the other hand a uniform continuous transformation of a $2n$-dimensional projective space E into itself has at least two invariant points, if its *absolute degree* (i.e. the absolute value of the degrees of the correspondent transformations of the sphere S duplicating E) is $\geqq 2$.