Mathematics. — "*On the expression of positive integers in the form* $ax^2 + by^2 + cz^2 + dt^2$ [1])". By H. D. KLOOSTERMAN. (Communicated by Prof. J. C. KLUYVER).

(Communicated at the meeting of October 31, 1925).

In my dissertation (Over het splitsen van geheele positieve getallen in een som van kwadraten, P. NOORDHOFF, Groningen 1924), I have proved an asymptotic formula for the number of representations $r(n)$ of the positive integer n in the form $a_1\, x_1^2 + a_2\, x_2^2 + \ldots . +$ $+ a_s\, x_s^2$ ($a_1, a_2, \ldots . a_s$ are positive integers) if $s \geqq 5$. The proof of this formula is merely a direct application of a method, due to HARDY and LITTLEWOOD. This method however does not give at once an asymptotic formula if $s = 4$, that is to say for the form $ax^2 + by^2 + cz^2 + dt^2$ (in order to avoid the indices I now write a, b, c, d, x, y, z, t instead of $a_1, a_2, a_3, a_4, x_1, x_2, x_3, x_4$). In this note I give the outline of a method, by means of which it is also possible to prove an asymptotic formula in this case. I write (r, q, ν are integers):

$$S_{r,q,\nu} = \sum_{j=0}^{q-1} exp\left(\frac{2\pi i r j^2}{q} + \frac{2\pi i \nu j}{q}\right)$$

To every integer p which is prime to q, a number p_1 can be determined by the conditions

$$p\,(p_1 + N) + 1 \equiv 0 \ (mod\ q) \quad , \quad 0 < p_1 \leqq q,$$

where N is a positive integer. Then the proof of the asymptotic formula depends on the following

Lemma: Let $n, N, a, b, c, d, \nu_1, \nu_2, \nu_3, \nu_4$ be integers, of which n, N, a, b, c, d are supposed to be > 0. Let μ be an integer such that $0 \leqq \mu \leqq q - 1$.

Then, if p runs through those positive numbers which are less than and prime to q and for which also $p_1 \leqq \mu$, then

$$\left| \sum_{p_1 \leqq \mu} S_{ap,q,\nu_1}\, S_{bp,q,\nu_2}\, S_{cp,q,\nu_3}\, S_{dp,q,\nu_4}\, exp\left(-\frac{2n\,\pi\, ip}{q}\right)\right| < K q^{2+\frac{7}{8}+\varepsilon}\, (n, q)^{\frac{1}{4}}, \qquad (1)$$

where K is a constant, depending on a, b, c, d only (and therefore not depending on $q, \mu, n, N, \nu_1, \nu_2, \nu_3, \nu_4$), (n, q) is the greatest common divisor of n and q and ε is an arbitrary small positive number.

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[1]) More detailed proofs will be published in the „Acta Mathematica"

The proof of this lemma is difficult and intricate. In the following pages I shall give an outline of the proof of the inequality

$$\left| \sum_{p_1 \leqq \mu} \exp\left(-\frac{2n\pi i p}{q}\right) \right| < K q^{\frac{7}{8}+\varepsilon} (n, q)^{\frac{1}{4}}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$ [1]

since the essential difficulties are the same as in the case of the more general inequality (1). For the sake of simplicity, I shall suppose $N=0$, so that

$$pp_1 + 1 \equiv 0 \pmod q.$$

In a ξ—η plane we shall consider the square $0 < \xi \leqq 1$, $0 \leqq \eta < 1$ and in this square we consider the point whose coordinates are $\xi = \frac{p_1}{q}$, $\eta = \frac{\{np\}}{q}$, where $\{np\}$ is the integer, which is $\equiv np \pmod q$ and for which $0 \leqq \{np\} \leqq q-1$. To every number p_1 lying between 0 and q a corresponding point can be found.

Therefore there are $\varphi(q)$ points. Let N_m be the number of these points, which are situated in the rectangular region

$$\frac{m}{M} \leqq \eta < \frac{m+1}{M} \quad , \quad 0 \leqq \xi \leqq \frac{\mu}{q},$$

where M is a positive integer and $m = 0, 1, 2 \ldots, M-1$.

Then

$$\sum_{\frac{m}{M} \leqq \frac{\{np\}}{q} < \frac{m+1}{M};\, p_1 \leqq \mu} \exp\left(-\frac{2n\pi i p}{q}\right) = N_m\, e^{-\frac{2\pi i m}{M}} + O\left(\frac{N_m}{M}\right)$$

and therefore

$$\sum_{p_1 \leqq \mu} \exp\left(-\frac{2n\pi i p}{q}\right) = \sum_{m=0}^{M-1} N_m\, e^{-\frac{2\pi i m}{M}} + O\left(\frac{\mu}{M}\right). \quad . \quad . \quad . \quad (3)$$

It remains to calculate N_m. In order to do this, we consider the function $f(\xi, \eta)$ which can be defined as follows:

1. $\quad f(\xi, \eta) = 1$, if $\frac{m}{M} < \eta < \frac{m+1}{M}$, $\quad 0 < \xi < \frac{\mu}{q}$;
2. $f(\xi, \eta) = \frac{1}{2}$ on the boundary of this rectangle;
3. $f(\xi, \eta) = \frac{1}{4}$ in the summits of this rectangle;
4. $f(\xi, \eta) = 0$ in all other points of the square $0 < \xi \leqq 1$, $0 \leqq \eta < 1$;
5. $f(\xi, \eta)$ is periodic both in ξ and η, with period 1.

Then we have

$$N_m = \sum_{p} f\left(\frac{p_1}{q}, \frac{np}{q}\right) + O(1),$$

[1] In what follows the letter K denotes a constant, depending on a, b, c, d only. However it does not always denote the same constant.

where p runs through a system of residus $mod\, q$ and prime to q. The function $f(\xi, \eta)$ has an analytical expression in the form of a double FOURIER-series:

$$f(\xi, \eta) = \sum_{\alpha=-\infty}^{+\infty} \sum_{\beta=-\infty}^{+\infty} a_{\alpha,\beta} \sum_{p} exp\,(2\pi i \xi \alpha + 2\pi i \eta \beta), \quad . \quad . \quad . \quad (4)$$

where

$$a_{\alpha,\beta} = \int_0^1 \int_0^1 f(\xi, \eta)\, exp\,(2\pi i \xi \alpha + 2\pi i \eta \beta)\, d\xi\, d\eta.$$

This FOURIER-series is convergent for all values of ξ and η.

We now have

$$N_m = \sum_{\alpha=-\infty}^{+\infty} \sum_{\beta=-\infty}^{+\infty} a_{\alpha,\beta} \sum_{p} exp \left(\frac{2\pi i \alpha p_1}{q} + \frac{2\pi i \beta n p}{q} \right) + O\,(1) \quad . \quad (5)$$

Now we can prove

$$\sum_{p} exp \left(\frac{2\pi i \alpha p_1}{q} + \frac{2\pi i \beta n p}{q} \right) = O\,(q^{\frac{3}{4}}\,(\beta n, q)^{\frac{1}{4}}) \quad . \quad . \quad . \quad (6)$$

We can substitute this in (5). The non-uniform convergence of the FOURIER-series causes some difficulty, which, however, is not serious.

We find

$$N_m = \frac{\mu}{Mq}\, \varphi\,(q) + O\,(q^{\frac{3}{4}+\varepsilon}\,(n, q)^{\frac{1}{4}}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

for every positive ε. The inequality (2) now follows, if we substitute (7) into (3), taking $M = [q^{\frac{1}{8}}]$.

The essential idea of the method described, is that the calculation of a sum, where p does not run through a complete system of residus to *mod* q and prime to q (in fact we had $p_1 \leqslant \mu$), can be reduced to the calculation of sums, like (6), where p runs through a complete system of residus $mod\, q$ and prime to q. The proof of (6) is also difficult. It has been proved by LITTLEWOOD in the particular case, where $(\alpha, q) = (\beta n, q) = 1$. (This proof has not been published). I have succeeded in extending LITTLEWOOD's method for the most general case.

An application of the HARDY-LITTLEWOOD-method making use of the lemma mentioned above, now gives the following result:

$$r\,(n) = \frac{\pi^2}{\sqrt{abcd}}\, n\, S\,(n) + O\,(n^{\frac{17}{18}+\varepsilon}) \quad . \quad . \quad . \quad . \quad . \quad (8)$$

for every positive ε, where

$$S\,(n) = \sum A_q\,, \quad A_q = q^{-4} \sum_{q=1}^{\infty} S_{ap,q}\, S_{bp,q}\, S_{cp,q}\, S_{dp,q}\, exp \left(- \frac{2n\pi i p}{q} \right) \quad (9)$$

(where $S_{ap,q}$ etc. are GAUSSIAN sums and where p runs through all positive integers less than and prime to q), and $S(n)$ is the *singular series*.

Before we can conclude, that

$$r(n) \sim \frac{\pi^2}{\sqrt{abcd}}\, n\, S(n) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

a detailed examination of the singular series is necessary. For it may happen that $S(n)$ is small for large values of n.

We have

$$A_{qq'} = A_q\, A_{q'}, \quad \text{if} \quad (q, q') = 1.$$

Hence

$$S(n) = \Pi\, \chi_{\varpi},$$

where the right hand side is an infinite product, where ϖ runs through all prime numbers and where

$$\chi_{\varpi} = 1 + A_{\varpi} + A_{\varpi^2} + A_{\varpi^3} + \ldots$$

The factors χ_{ϖ} can be calculated from the second formula (9), by substituting the explicit values of the GAUSSIAN sums.

If we carry out the examination of the singular series in this way, we are led to complicated algebraical calculations, and a large number of special cases have got to be treated separately. Therefore I have carried out the examination for the special case only, where a, b, c and d are odd. Then it is possible to prove

$$S(n) > \frac{K}{\log \log n} > 0$$

if the following conditions are satisfied:

A. There is no prime number which divides three or four of the numbers a, b, c, d.

B. There is no odd prime ϖ, for which the following relations are true simultaneously

$$\varpi^2 \mid a, \quad b = \varpi\, b_1, \quad (b_1, \varpi) = (c, \varpi) = (d, \varpi) = 1, \quad \left(\frac{cd}{\varpi}\right)(-1)^{\frac{\varpi-1}{2}} = -1.$$

C. The same for the relations:

$$(c, \varpi) = (d, \varpi) = 1, \quad \varpi^2 \mid a, \quad \varpi^2 \mid b, \quad \left(\frac{cd}{\varpi}\right)(-1)^{\frac{\varpi-1}{2}} = -1.$$

D. The same for the relations:

$$a = \varpi\, a_1, \quad b = \varpi\, b_1, \quad (a_1, \varpi) = (b_1, \varpi) = (c, \varpi) = (d, \varpi) = 1,$$

$$\left(\frac{a_1\, b_1\, c\, d}{\varpi}\right) = 1, \quad \left(\frac{cd}{\varpi}\right)(-1)^{\frac{\varpi-1}{2}} = -1.$$

E. The same for the relations

$$a + b \equiv 2 \pmod 4, \quad a + c \equiv 2 \pmod 4,$$
$$a + d \equiv 2 \pmod 4, \quad a + b + c + d \equiv 4 \pmod 8.$$

F. The conditions, that are obtained from *B.—E.* by permutation of a, b, c, d.

If these conditions are satisfied the formula (10) is true, and therefore we have the following result:

If a, b, c and d are odd and if the conditions *A.—F.* are fulfilled, then there is only a finite number of integers which can not be written in the form $ax^2 + by^2 + cz^2 + dt^2$ and the asymptotic behaviour of the number $r(n)$ of representations of the number n is given by (10).

The Hague, October 1925.