Mathematics. — *"Projective and conformal invariants of half-symmetrical connections".* By J. A. SCHOUTEN. (Communicated by Prof. JAN DE VRIES).

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1. *The quantity of projective curvature.*

We consider a linear connection [1]) leaving transvections invariant:

$$\begin{aligned} \delta v^{\nu} &= dv^{\nu} + \Gamma^{\nu}_{\lambda\mu}\, v^{\lambda}\, dx^{\mu} \\ \delta w_{\lambda} &= dw_{\lambda} - \Gamma^{\nu}_{\lambda\mu}\, w_{\nu}\, dx^{\mu} \end{aligned} \quad \ldots \ldots \ldots \quad (1)$$

If

$$S_{\lambda\mu}^{\cdot\cdot\,\nu} = {}^{1}/_{2}\,(\Gamma^{\nu}_{\lambda\mu} - \Gamma^{\nu}_{\mu\lambda}) \quad \ldots \ldots \ldots \quad (2)$$

has the form

$$S_{\lambda\mu}^{\cdot\cdot\,\nu} = S_{[\lambda}\, A^{\nu}_{\lambda]}, \quad \ldots \ldots \ldots \quad (3)$$

the connection is called a halfsymmetrical one [2]).

First we take $S_{\lambda\mu}^{\cdot\cdot\,\nu}$ quite general and using the quantity of curvature $R_{\omega\mu\lambda}^{\cdot\cdot\cdot\,\nu}$ we form a quantity $P_{\omega\mu\lambda}^{\cdot\cdot\cdot\,\nu}$ in the same way as this is done in an affine connection [3])

$$P_{\omega\mu\lambda}^{\cdot\cdot\cdot\,\nu} = R_{\omega\mu\lambda}^{\cdot\cdot\cdot\,\nu} - 2\, P_{[\omega\mu]}\, A^{\nu}_{\lambda} + 2\, A^{\nu}_{[\omega}\, P_{\mu]\lambda} \quad \ldots \quad (4)$$

$$P_{\mu\lambda} = -\frac{1}{n^2-1}\,(n\, R_{\mu\lambda} + R_{\lambda\mu}) \quad \ldots \ldots \quad (5)$$

$$R_{\mu\lambda} = R_{\nu\mu\lambda}^{\cdot\cdot\cdot\,\nu} \quad \ldots \ldots \ldots \quad (6)$$

Then the following theorems hold:

I. *In a halfsymmetrical connection* $P_{\omega\mu\lambda}^{\cdot\cdot\cdot\,\nu}$ *is invariant with all geodesical transformations of the* $\Gamma^{\nu}_{\lambda\mu}$, *which leave* S_{ν} *invariant.*

II. *If* $P_{\omega\mu\lambda}^{\cdot\cdot\cdot\,\nu}$ *is invariant with all transformations wich leave* $S_{\lambda\mu}^{\cdot\cdot\,\nu}$ *invariant, the connection is halfsymmetrical.*

III. A *halfsymmetrical connection can then and only then be transformed by a geodesical transformation of the* $\Gamma^{\nu}_{\lambda\mu}$, *which leaves* S_{λ} *invariant, into a displacement with a zero quantity of curvature, if* $P_{\omega\mu\lambda}^{\cdot\cdot\cdot\,\nu}$ *is zero.*

We have some remarkable identities. For a halfsymmetrical connection holds:

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[1]) Der Ricci Kalkül, SPRINGER 1924, p. 67.

[2]) R. K. p. 69.

[3]) R. K. p. 131.

$$\left.\begin{aligned}\nabla_{\nu} P_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = & -2(n-2)\nabla_{[\omega} P_{\mu]\lambda} + \\ & + \frac{12}{n+1} S_{[\omega} R_{\mu\lambda]} - \frac{12}{n+1} S_{[\lambda} \nabla_{\omega} S_{\mu]} - 4 S_{[\omega} R_{\mu]\lambda} + 2 S_{\nu} R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}\end{aligned}\right\} . \quad (7)$$

For an affine connection this identity passes into:

$$\nabla_{\nu} P_{\omega\lambda\mu}^{\cdot\cdot\cdot\nu} = -2(n-2)\nabla_{[\omega} P_{\mu]\lambda} \quad \ldots \ldots \quad (8)$$

If $P_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = 0$ the conditions

$$\nabla_{[\omega} S_{\mu]} = 0 \quad \ldots \ldots \ldots \ldots \quad (9)$$

$$\nabla_{[\omega} P_{\mu]\lambda} = 2 S_{[\omega} P_{\mu]\lambda} \quad \ldots \ldots \ldots \quad (10)$$

hold for a halfsymmetrical connection.

2. *The quantity of conformal curvature.*

We consider a metrical connection leaving transvections invariant, and we suppose $S_{\lambda\mu}^{\cdot\cdot\nu} \neq 0$ [1]). The $\Gamma_{\lambda\mu}^{\nu}$ are given by the equation [2]):

$$\Gamma_{\lambda\mu}^{\nu} = \left\{ {\lambda\mu \atop \nu} \right\} + S_{\lambda\mu}^{\cdot\cdot\nu} - 2 S^{\nu}_{\cdot(\lambda\mu)} \quad \ldots \ldots \quad (11)$$

Using the quantity of curvature, which we will call $K_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$, we form a quantity $C_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ in a perfectly analogous manner as this is done in a RIEMANNIAN connection:

$$C_{\omega\mu\lambda\nu} = K_{\omega\mu\lambda\nu} - \frac{4}{n-2} g_{[\omega[\lambda} L_{\mu]\nu]} \quad \ldots \ldots \quad (12)$$

$$L_{\mu\lambda} = -K_{\mu\lambda} + \frac{1}{2(n-1)} K g_{\mu\lambda} \quad \ldots \ldots \quad (13)$$

$$K_{\mu\lambda} = K_{\nu\mu\lambda}^{\cdot\cdot\cdot\nu} \quad ; \quad K = K_{\mu}^{\cdot\mu} \quad \ldots \ldots \quad (14)$$

Then the following theorems hold:

I. *In a halfsymmetrical connection* $C_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ *is invariant with all conformal transformations of* $g_{\lambda\mu}$.

II. *If* $C_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ *is invariant with all conformal transformations of* $g_{\lambda\mu}$, *the connection is halfsymmetrical.*

III. *A halfsymmetrical connection can then and only then be transformed by a conformal transformation of* $g_{\lambda\mu}$, *which leaves* S_{λ} *invariant, in a connection with a zero quantity of curvature, if* $C_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ *is zero.*

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[1]) R. K. p. 72.

[2]) R. K. p. 73, equation (53*b*).

[3]) R. K. p. 170.

Here also some remarkable identities exist.
For a halfsymmetrical connection we have

$$\left.\begin{aligned}\nabla_\nu C_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = -2\frac{n-3}{n-2}\nabla_{[\omega} L_{\mu]\lambda} - S_{[\omega} R_{\mu]\lambda} + \\ + 2S_\nu R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} + \frac{4}{n-2} g_{\lambda[\omega} G_{\mu]\nu} S^\nu\end{aligned}\right\} . \quad (15)$$

$$G_{\mu\nu} = -L_{\mu\lambda} + L g_{\mu\lambda} \quad ; \quad L = L_\mu^{\cdot\mu} \; . \; . \; . \; . \; . \quad (16)$$

$$\nabla^\alpha G_{\mu\alpha} = 2 S^\alpha G_{\alpha\mu} \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (17)$$

For a RIEMANNIAN connection this identity passes into:

$$\nabla_\nu C_{\omega\lambda\mu}^{\cdot\cdot\cdot\nu} = -2\frac{n-3}{n-2}\nabla_{[\omega} L_{\mu]\lambda} \; . \; . \; . \; . \; . \; . \quad (18)$$

If $C_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = 0$, the relations

$$L_{[\mu\lambda]} = -(n-2)\nabla_{[\mu} S_{\lambda]} \; . \; . \; . \; . \; . \; . \; . \; . \quad (19)$$

$$\nabla_{[\omega} L_{\mu]\nu} = 2 S_{[\omega} L_{\mu]\nu} . \; . \; . \; . \; . \; . \; . \; , \; . \quad (20)$$

hold for a halfsymmetrical connection.