Physics. — *"The determination of the potentials in the general theory of relativity, with some remarks about the measurement of lengths and intervals of time and about the theories of* WEYL *and* EDDINGTON." By Prof. H. A. LORENTZ.

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§ 1. It was remarked some years ago by KRETSCHMANN[1]) that from observations solely concerning the course of rays of light and the motion of material particles, the values of the potentials g_{ab} which characterize a gravitation-field can be so far deduced that only a constant factor remains undetermined. He showed, in fact, that, if two sets of values g_{ab} and $\bar{g}_{ab}$ are in agreement with these observations, the ratio $\bar{g}_{ab}/g_{ab}$ must be the same for all suffixes a and b, and independent of the coordinates.

It is easily seen in what manner this determination of the potentials can be effected. Let us imagine that a physicist explores a gravitation-field by attending to the motion of light-signals and material particles which he throws into the field in any way he likes, and let us suppose that he does so a great number of times and under varied conditions. His object will be to note and clearly to record the encounters between these projectiles.

To this effect he might tabulate the encounters in a register, after having numbered the projectiles, but a better picture of the phenomena may be obtained by means of a diagram drawn in a four-dimensional space R_4 in which each projectile has its „world-line". An encounter between two projectiles will be represented by an intersection of their world-lines and the lines have to be drawn in such a way that, along any one of them, the intersections with other lines follow each other in the order in which the successive encounters have taken place.

It is clear that the observer has a good deal of liberty in the construction of the diagram. A particular figure will continue to serve his purpose though it be subjected to an arbitrary deformation, provided only that the connexions between its parts remain unbroken. Even, as we are concerned with the intersections and their order only, all diagrams thus derivable from each other may, in a sense, be said to be the *same* figure. It ought also to be remarked that points in R_4 are defined exclusively by the intersection of world-lines, there being, according to the conceptions of the theory of relativity, no other means for defining the position of a point. Each point represents an „event".

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[1]) E. KRETSCHMANN, Ueber den physikalischen Sinn der Relativitätspostulate, Ann. d. Physik, (4) **53** (1917), p. 575.

We shall suppose the word-lines and the encounters to be so numerous that we may speak of a continuous succession of points along each line, and also of lines lying infinitely near each other.

After having drawn the world-diagram we can introduce coordinates, assigning to each point four numbers $x_1 \ldots . x_4$. In doing so we are limited only by the restriction of continuity and by the condition that, as we proceed along a world-line in the positive direction, corresponding to the succession of the encounters, the time-coordinate x_4 must constantly increase.

§ 2. EINSTEIN's theory postulates the possibility of associating with each point in R_4 ten numbers g_{ab} $(g_{ba} = g_{ba})$, such that, if one puts

$$ds^2 = \Sigma\,(ab)\, g_{ab}\, dx_a\, dx_b\,,$$

ds being the „line-element", the world-line of a flash of light satisfies the condition

$$ds = 0$$

and that the world-line of a material particle is a geodesic, i.e. such that, if its beginning and its end are kept fixed,

$$\delta \int ds = 0.$$

Admitting this, and remembering that the values of the coordinates can be directly read from the diagram, we may solve our problem as follows.

We consider in the first place the world-lines belonging to light-signals and passing through a definite point P. Let, on any one of these, Q be a point infinitely near P. We see at once the values of the four differentials dx_a, corresponding to the transition from P to Q, and the condition

$$\Sigma\,(ab)\, g_{ab}\, dx_a\, dx_b = 0$$

gives us a homogeneous linear relation between the potentials, with known coefficients $dx_a\, dx_b$. Proceeding in the same way with eight other lines of the same class, passing through P, we are led to nine equations from which the ratios between the potentials may be found. This can be done for any position of the point P and the result may be written in the form

$$g_{ab} = \omega\, \gamma_{ab}, \tag{1}$$

where the quantities γ_{ab} are known functions of the coordinates $(\gamma_{ab} = \gamma_{ab})$, whereas the function ω remains to be determined. If, as we shall suppose, the field is free from discontinuities, the quantities γ_{ab} may be chosen as continuous functions, and then ω will be of the same kind.

Any world-line of light passing through P and not included in the group selected will give a verification of the theory, because for it also the equation

$$\Sigma\,(a\,b)\, g_{ab}\, dx_a\, dx_b = 0$$

must hold.

§ 3. The world-lines for particles, the geodesics, may now serve for the determination of the function ω. Indeed, at any point of such a line, we have the four equations

$$\frac{d^2x_c}{ds^2} = - \Sigma\,(ab) \left\{ \begin{matrix} a\,b \\ c \end{matrix} \right\} \frac{dx_a}{ds}\,\frac{dx_b}{ds} \tag{2}$$

and these may be put in the form of differential equations for ω.

The symbol $\{\ \ \}$ in (2) is defined by

$$\left\{ \begin{matrix} a\,b \\ c \end{matrix} \right\} = \Sigma\,(e)\, g^{ce} \begin{bmatrix} a\,b \\ e \end{bmatrix} = \tfrac{1}{2}\, \Sigma\,(e)\, g^{ce}\,(g_{ae,b} + g_{be,a} - g_{ab,e}),$$

$g_{ac,b}$ being the differential coefficient of g_{ac} with respect to x_b and the set of quantities g^{ab} the "inverse" to the set g_{ab}. Similarly, we may introduce γ^{ab} and the derivatives $\gamma_{ab,c}$. All these quantities will have definite values because γ_{ab} is known.

Now

$$g^{ab} = \frac{1}{\omega}\,\gamma^{ab}, \qquad g_{ae,b} = \frac{\partial}{\partial x_b}\,(\omega\,\gamma_{ae}),$$

and therefore

$$\left\{ \begin{matrix} a\,b \\ c \end{matrix} \right\} = \tfrac{1}{2}\, \Sigma\,(e)\, \gamma^{ce}\,(\gamma_{ae,b} + \gamma_{be,a} - \gamma_{ab,e}) + \\ + \tfrac{1}{2}\, \Sigma\,(e)\, \gamma^{ce} \left(\gamma_{ae} \frac{\partial \log \omega}{\partial x_b} + \gamma_{be} \frac{\partial \log \omega}{\partial x_a} - \gamma_{ab} \frac{\partial \log \omega}{\partial x_e} \right). \tag{3}$$

If, further, we put

$$d\sigma^2 = \Sigma\,(ab)\, \gamma_{ab}\, dx_a\, dx_b\,, \tag{4}$$

we have

$$ds = \sqrt{\omega}\, d\sigma. \tag{5}$$

The differential $d\sigma$ may be considered as the line-element expressed in a new measure, and since by (4) it is known, we know also for any point of the geodesic under consideration the value of the integral $\sigma = \int d\sigma$, reckoned from some fixed point of the line. Thus, along the line, the coordinates become known functions of σ, and the same may be said of their first and second derivatives with respect to that variable.

Now, on account of (5),

$$\frac{dx_c}{ds} = \frac{1}{\sqrt{\omega}}\,\frac{dx_c}{d\sigma}\,, \text{ etc.}$$

$$\frac{d^2x_c}{ds^2} = \frac{1}{\sqrt{\omega}}\,\frac{d}{d\sigma}\left(\frac{1}{\sqrt{\omega}}\,\frac{dx_c}{d\sigma}\right) = \frac{1}{\omega}\,\frac{d^2x_c}{d\sigma^2} - \frac{1}{2\omega^2}\,\frac{dx_c}{d\sigma}\,\frac{d\omega}{d\sigma}$$

by which, after multiplication by ω, the equation of the geodesic line becomes

$$\frac{d^2x_c}{d\sigma^2} - \tfrac{1}{2}\,\frac{dx_c}{d\sigma}\,\frac{d \log \omega}{d\sigma} = - \Sigma\,(ab) \left\{ \begin{matrix} a\,b \\ c \end{matrix} \right\} \frac{dx_a}{d\sigma}\,\frac{dx_b}{d\sigma}\,. \tag{6}$$

If, finally, we substitute the expression (3) and the value

$$\frac{d\, log\, \omega}{d\sigma} = \Sigma\,(f)\,\frac{dx_f}{d\sigma}\,\frac{\partial\, log\, \omega}{\partial x_f},$$

we are led to four equations, linear in the unknown quantities

$$\frac{\partial\, log\, \omega}{\partial x_f}.$$

§ 4. Since we started from the assumption that EINSTEIN's theory is true, we need fear no contradiction between the different equations. But we must make sure that they are mutually independent and can give us definite values for the derivatives of log ω.

The equation which we deduce from (6) as just explained may be written in the form

$$-\,\Sigma\,(f)\,\frac{dx_c}{d\sigma}\,\frac{dx_f}{d\sigma}\,\frac{\partial\, log\, \omega}{\partial x_f}$$

$$+\,\Sigma\,(a\,b\,e)\,\gamma^{ce}\,\frac{dx_a}{d\sigma}\,\frac{dx_b}{d\sigma}\left(\gamma_{ae}\,\frac{\partial\, log\, \omega}{\partial x_b} + \gamma_{be}\,\frac{\partial\, log\, \omega}{\partial x_a} - \gamma_{ab}\,\frac{\partial\, log\, \omega}{\partial x_e}\right) = \ldots \qquad (7)$$

where the terms on the right-hand side are completely known. We have simply represented them by . . . . , their values being irrelevant to our purpose, which is merely to show the definiteness of the solution.

Multiplying (7) by γ_{ch} and adding the resulting equations [1]) with $c = 1,2,3,4$, we find four new equations ($h = 1, \ldots 4$)

$$\Sigma\,(a\,b)\left(\gamma_{bh}\,\frac{\partial\, log\, \omega}{\partial x_a} - \gamma_{ab}\,\frac{\partial\, log\, \omega}{\partial x_h}\right)\frac{dx_a}{d\sigma}\,\frac{dx_b}{d\sigma} = \ldots$$

or, again, if we add the equation that is obtained by interchanging the suffixes a and b,

$$\Sigma\,(a\,b)\,\Phi_{ab}\,\frac{dx_a}{d\sigma}\,\frac{dx_b}{d\sigma} = \ldots \qquad (8)$$

$$\Phi_{ab} = \gamma_{ah}\,\frac{\partial\, log\, \omega}{\partial x_b} + \gamma_{bh}\,\frac{\partial\, log\, \omega}{\partial x_a} - 2\,\gamma_{ab}\,\frac{\partial\, log\, \omega}{\partial x_h}.$$

Now, consider a definite point P and a definite suffix h. Since $\Phi_{ba} = \Phi_{ab}$ there are ten mutually independent values $\Phi_{11}, \ldots \Phi_{12}, \ldots$ and these are the same whatever be the direction of the geodesic line. The values of $\frac{dx_a}{d\sigma}$ and $\frac{dx_b}{d\sigma}$ being known, each line gives us an equation of the form (8) that must be satisfied by Φ_{ab}. Hence, if we apply (8) to ten geodesic lines passing through P, we obtain a sufficient number of equations for the determination of Φ_{ab}. There will be but one solution, if the determinant on the coefficients is not zero, a condition that will be fulfilled when the lines selected do not happen to lie on a cone of the second degree.

[1]) The relations $\Sigma\,(c)\,\gamma_{ch}\,\gamma^{ce} = \delta^e_h$ ($\delta^e_h = 1$ for $h = e$, and $= 0$ for $h \neq e$) may be used here.

The outcome of the calculations so far sketched is this, that for all combinations of the suffixes a, b, h, we know the expression

$$\gamma_{ah}\frac{\partial \log \omega}{\partial x_b}+\gamma_{bh}\frac{\partial \log \omega}{\partial x_a}-2\,\gamma_{ab}\frac{\partial \log \omega}{\partial x_h}=\ldots \tag{9}$$

Let us now multiply this by γ^{ab} and add the equations which we find by giving both to a and to b the values 1, 2, 3, 4. The first term gives $\frac{\partial \log \omega}{\partial x_h}$, the second leads to the same result and from the third we find

$$-8\frac{\partial \log \omega}{\partial x_h},$$

so that the derivative $\frac{\partial \log \omega}{\partial x_h}$ becomes known. Thus, since h may be 1, 2, 3 or 4, one finds at any point of the diagram the derivatives of $\log \omega$ with respect to the coordinates. By this, apart from a constant term, $\log \omega$ becomes known. Finally, one finds the function ω and, by virtue of (1), the potentials g_{ab}, only a constant factor being left undeterminate [1]).

Here again there would be opportunities for verifications of the theory. Indeed, we have used no more than ten of the geodesics passing through P, the number of the equations (9) is far greater than four, the number of the first derivatives of $\log \omega$, and finally, when these have been determined as functions of the coordinates, the relations of the form

$$\frac{\partial^2 \log \omega}{\partial x_b\,\partial x_a}=\frac{\partial^2 \log \omega}{\partial x_a\,\partial x_b}$$

may be put to the test.

§ 5. So far we used an arbitrarily chosen system of coordinates x_a. If, instead of these, we want to introduce new coordinates x'_a, certain functions of x_a, we can follow the same method for determining the corresponding potentials g'_{ab}. We may, however, just as well take for these the values that are derived from the potentials g_{ab} first determined by means of the transformation-formulae for covariant tensors. These formulae, when applied to g_{ab}, are equivalent to the statement that ds^2

[1]) According to WEYL (Raum, Zeit, Materie, 1st ed., p. 182) the world-lines of light-signals would suffice already for this determination of the potentials. I think this cannot be said. Suppose f.i., that, after having properly chosen the coordinates, one has been able to account for the course of these lines by assuming for g_{44} and g_{ab} $(a \neq 4, b \neq 4)$ certain values that are functions of the space-coordinates x_1, x_2, x_3 only, and by putting $g_{a4}=0$ for $a \neq 4$ (so that the field is a stationary one). Then, one may multiply all potentials by one and the same arbitrarily chosen function of x_1, x_2, x_3, without altering the velocities of propagation, which are determined by $ds^2=0$, and therefore without modifying the course of rays of light which follows from the velocities by means of HUYGENS' construction.

is invariant, and it is clear that, if $ds' = ds$, the new conditions $ds' = 0$, and $\delta \int ds' = 0$ for the world-lines are equivalent to the original ones $ds = 0$ and $\delta \int ds = 0$.

As to the constant factor in g_{ab} or g'_{ab}, we shall suppose it to have been chosen for the first system and to have such a value for the second that g_{ab} and g'_{ab} are related to each other in the way just mentioned.

When the potentials g_{ab} have been determined, the geometry of the extension R_4 may be completely developed on the assumption that ds represents the line-element. It will be easy f. i. to define the angle between two directions and to find the differential equation for geodesic lines. We need not speak of all this, but perhaps the following remarks will not be out of place.

1. Two line-elements (PQ and PQ') ds and $d's$ with the components dx_a and $d'x_a$ are said to be at right angles to each other when

$$\Sigma\,(ab)\, g_{ab}\, dx_a\, d'x_b = 0.$$

2. It may be inferred from this that, if the line-element QQ' is denoted by $d''s$,

$$d''s^2 = ds^2 + d's^2.$$

3. A line-element is a contravariant vector, whose direction-constants ξ^a are given by the differential coefficients $\dfrac{dx_a}{ds}$.

4. If a vector is displaced parallel to itself (the word "parallel" being used in the sense that was given it by LEVI CIVITA [1])), its starting point

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1) In order to state what is the meaning of a parallel displacement of a vector we may remark in the first place that, when, at any point P — we have two directions at right angles to each other and determined by the constants ξ^a and ξ'^a, the four quantities

$$\xi''^a = \xi^a \cos\varphi + \xi'^a \sin\varphi$$

will also satisfy the condition that is fulfilled by direction-constants, (viz. the condition

$$\Sigma\,(a\,b)\, g_{ab}\, \xi^a\, \xi^b = 1).$$

The direction which they determine is said to lie in the plane of the two given directions and to make an angle φ with the former of these.

Let P be a point in R_4 and L a geodesic line starting from it. We shall now define a parallel displacement of a vector PA, the starting point P of which moves along L.

1. If, in the first place, at the point P the vector PA is directed along L, it shall constantly be directed along that line.

2. Similarly, if originally the vector is perpendicular to the line, it shall remain at right angles to it. This, however, does not completely determine the direction of the vector when a point Q has been reached, and we therefore complete the definition as follows: Draw from the point P a second geodesic line L' that makes an infinitely small angle with the line L and whose direction at the point P lies in the plane containing the initial direction of L and that of the vector PA. Take equal infinitely small segments PQ and PQ' on L and L'. Then the line-element QQ' will give us the direction of the vector PA after its displacement to the point Q.

moving along a line-element dx_a, the changes of its direction-constants are given by

$$d\xi^a = -\Sigma\,(b\,c)\left\{ {b\,c \atop a} \right\} \xi^c\, dx_b\,. \tag{10}$$

§ 6. Let us now imagine that not only the values of the coordinates but also those of the potentials g_{ab} are inscribed in the diagram R_4.

A physicist who wants to study phenomena as affected by the gravitational field will then be enabled, using the numbers which he sees in the diagram, to assign definite values, independent of the choice of coordinates, to lengths of lines and to intervals of time; he can express them in what may be called "invariant" measure. A first instance of this kind is the distance between two neighbouring points in the diagram, the invariant measure for which will simply be the value of ds. As a second example we may take the length of an infinitely short rod. Let L and L' be the world-lines of its extremities, A a point of the first line corresponding to the instant x_4 for which we want to evaluate the length, and B a point of L' determined by the condition that $A\,B$ is perpendicular to L'. Then we shall measure the length l of the rod by [1])

$$l^2 = -AB^2, \tag{11}$$

calculating $A\,B^2$ by means of the formula for ds^2.

The necessary calculation can be performed in the following manner. Let A' be the point of L' corresponding to the same time x_4 as A, and let B correspond to $x_4 + \tau$. Then the infinitely short time τ is determined by the condition that $A\,B$ is at right angles to L' (we may just as well say, at right angles to L) and having found τ one knows $A'B^2$ and $A\,B^2 = A\,A'^2 - A'B^2$. The result is

$$l^2 = -\Sigma\,(a\,b)\, g_{ab}\,(x'_a - x_a)\,(x'_b - x_b) + \frac{\{\Sigma\,(a\,b)\, g_{ab}\,\dot{x}_a\,(x'_b - x_b)\}^2}{\Sigma\,(a\,b)\, g_{ab}\,\dot{x}_a\,\dot{x}_b} \tag{12}$$

Repeating this construction, one can displace the vector parallel to itself over any finite part of the geodesic L,

3. If finally the vector PA in its initial position has a direction neither along the line L nor perpendicular to it, we decompose it into two components having these directions. Displacing each of them parallel to itself along the line, say to a point R, and keeping their magnitudes constant, we shall find two vectors at the point R. Compounding these we obtain a definite resulting vector and this will give us the direction of PA after a parallel displacement to the point R.

This definition of a parallel displacement along a geodesic implies the definition of such a displacement along a given infinitely short line, for such a line may always be considered as the first element of a geodesic. Proceeding by infinitely small steps, we may now also displace a vector parallel to itself along any length of an arbitrarily chosen line that is no geodesic.

Working out what has been said here, one is led to eq. (10).

[1]) If the potentials have the values that are often ascribed to them (f.i. $g_{11} = g_{22} = g_{33} = -1$, $g_{44} = c^2$, $g_{ab} = 0$ for $a \neq b$, or values little different from these) AB^2 becomes negative. In order to find a real value for l (11) has been written with the negative sign.

Here the coordinates of A are denoted by x_a and those of A' by x'_a (so that $x'_4 = x_4$). The symbols $\dot{x}_1$, $\dot{x}_2$, $\dot{x}_3$ represent the components of the velocity of the first extremity of the rod, ($\dot{x}_4 = 1$).

Owing to the way in which it has been found, the above expression (12) is invariant. It may be remarked that, if instead of the length of AB we had taken that of AA', which depends on the "simultaneous" positions of the two ends, the result would have depended on the choice of coordinates.

That l, as defined by (12), may appropriately be termed the "length" of the rod, will be clear if one remarks that, if all circumstances remain the same, l is proportional to the differences of corresponding coordinates $x'_a - x_a$, and that for a rod at rest, placed in a field characterized by

$$g_{11} = g_{22} = g_{33} = -1, \quad g_{44} = c^2, \quad g_{ab} = 0 \quad \text{for} \quad a \neq b,$$

the formula becomes

$$l^2 = (x_1' - x_1)^2 + (x_2' - x_2)^2 + (x_3' - x_3)^2.$$

§ 7. In what precedes nothing has been said of the phenomena presented by rods placed in a gravitational field; we have only adopted a rule for measuring their length. If a physicist, adhering to the theory of relativity, were able to observe all the minute effects required by this theory and wanted to account for them, he certainly would follow this rule, because it would enable him to discuss all his observations, f.i. those about the influence of temperature and of external forces, in terms that are independent of the choice of coordinates. An "ideal rod of unvariable length" would mean to him a rod whose "invariant" length l would be the same under all circumstances.

Take f.i. the case of a field, which, when x_1, x_2, x_3 are rectangular cartesian coordinates, is characterized by the potentials specified at the end of § 6, that is a field in which there are no forces of gravitation. Let a rod be placed in this field in the direction x_1 and let it move with the velocity v in that direction. Then $\dot{x}_1 = v$, $\dot{x}_2 = \dot{x}_3 = 0$, $\dot{x}_4 = 1$, so that (12) becomes

$$l^2 = (x_1' - x_1)^2 + \frac{v^2 (x_1' - x_1)^2}{c^2 - v^2} = \frac{c^2}{c^2 - v^2} (x_1' - x_1)^2 .$$

Now, it would be very natural to measure the length of the rod by the difference of the simultaneous values of the coordinates x_1 and x'_1. If l is the same under all circumstances, this new length l_e (say the "euclidian" length) will change with the velocity v, according to the formula

$$l_e = \sqrt{1 - \frac{v^2}{c^2}}\, l,$$

in which one recognizes the well known contraction that is brought about by a motion of translation.

§ 8. The influence of a gravitation-field on the motion of a clock can be treated in a similar way. Let the clock be so small that we may speak of its world-line L; on this line the successive ticks will mark a series of points P, Q, R, ..., which we shall suppose to be infinitely near each other. The statement that a clock is "perfect" will have a meaning independent of the choice of coordinates, if we understand by it that the distances PQ, QR, ..., when expressed in invariant measure, will be equal for the particular clock considered, whatever be the circumstances. If the length of this distance is τ and if dx_4 is the interval of time between successive ticks, we have

$$\tau^2 = \Sigma\,(a\,b)\, g_{ab}\, dx_b\, dx_b\,,$$

or

$$\tau^2 = (g_{11}\,\dot{x}_1^2 + \ldots + 2\,g_{12}\,\dot{x}_1\,\dot{x}_2 + \ldots + 2\,g_{14}\,\dot{x}_1 + \ldots g_{44})\, dx_4^2,$$

a relation from which the value of dx_4 in different cases can be deduced.

§ 9. A rod may be conceived to have different lengths l according to the circumstances under which it is placed. A short discussion, under certain simplifying assumptions, of changes of this kind (in the case of an infinitely short rod) will be of interest with a view to a theory that has been proposed by WEYL [1]) and according to which there is a close and fundamental connexion between gravitational and electromagnetic phenomena.

The length of the rod might change with the time x_4, with the position of one of the extremities, determined by its coordinates x_1, x_2, x_3, and with the direction in which the rod is placed in the space R_3. We shall however discard this latter possibility, so that we are only concerned with variations of $x_1, \ldots x_4$. If these are infinitely small, the change produced in l may be assumed to be a homogeneous linear function of them, and if, further, we suppose it to be proportional to l itself, we may write

$$d \log l = \Sigma\,(a)\, P_a\, dx_a \tag{13}$$

with coefficients P_a solely depending on the coordinates. P_a will be a covariant vector, because, according to the fundamental idea of EINSTEIN's theory, for a given displacement in R_4, $d \log l$ must be independent of the choice of coordinates.

Eq. (13) may be applied to any part of a world-line, say between the points C and D, which, of course, means that during a certain interval of time the position of the rod in R_3 undergoes some definite change. We shall suppose the dimensions of the line CD to be very small and we shall calculate the change of l accurately up to quantities of the second order with respect to these dimensions.

Then, if for any point E of the path we put

$$x_a = x_{a,C} + \mathbf{x}_a\,,$$

[1]) Berl. Sitz. Ber., 1918, p. 465.

we must, in (13), replace P_a by

$$P_a + \Sigma\,(b)\,\frac{\partial P_a}{\partial x_b}\,\mathbf{x}_b\,,$$

where we have to take, both for P_a and for its differential coefficients, their values at the point C.

The total change of $log\, l$ now becomes

$$\triangle\, log\, l = \Sigma\,(a)\,P_a \int d\mathbf{x}_a + \Sigma\,(a\,b)\,\frac{\partial P_a}{\partial x_b}\int \mathbf{x}_b\, d\mathbf{x}_a\,.$$

The last term in this expression depends on the path along which the transition from C to D has taken place, and if $\triangle'\, log l$ is the change corresponding to a second path $CE'D$, we have

$$\triangle\, log\, l - \triangle'\, log\, l = \Sigma\,(a\,b)\,\frac{\partial P_a}{\partial x_b}\int \mathbf{x}_b\, d\mathbf{x}_a\,, \tag{14}$$

the integral being taken along the closed path $CEDE'C$ in the direction indicated by the order of the symbols.

The integral vanishes for $a = b$ and we have $\int \mathbf{x}_a\, d\mathbf{x}_b = -\int \mathbf{x}_b\, d\mathbf{x}_a$, the sum of the two integrals being $\int d\,(\mathbf{x}_a\,\mathbf{x}_b) = 0$.

Thus (14) may be replaced by

$$\triangle\, log\, l - \triangle'\, log\, l = \tfrac{1}{2}\,\Sigma\,(a\,b)\left(\frac{\partial P_a}{\partial x_b} - \frac{\partial P_b}{\partial x_a}\right)\int \mathbf{x}_b\, d\mathbf{x}_a\,. \tag{15}$$

This again shows that in general the final length of the rod will be different, according to the path along which the transition from C to D has been made. If there is to be no such difference, the components of the vector P_a must depend on a potential φ, so that

$$P_a = -\frac{\partial \varphi}{\partial x_a}\,.$$

§ 10. There is a certain formal similarity between the expressions to which we have now been led and the relations which exist in an electromagnetic field.

Indeed, it is well known that the state of things in such a field can be described by means of a fourfold vector $\overline{\overline{P}}_a$, the components of electric and magnetic force being given by the expressions

$$\frac{\partial \overline{P}_a}{\partial x_b} - \frac{\partial \overline{P}_b}{\partial x_a}\,, \tag{16}$$

which, taken together, form an antisymmetric covariant tensor of the second rank.

This analogy would have a deeper meaning if the two vectors P_a and $\overline{P}_a$ could be assimilated to each other, so that with a constant numerical coefficient λ,

$$P_a = \lambda\,\overline{P}_a\,. \tag{17}$$

This would mean, and it would certainly be very important, that the changes of length considered in the preceding § are the indications of an electromagnetic field and that, conversely, any electromagnetic field gives rise to changes of that kind. In particular, the electric and the magnetic force would be made responsable for the fact that the length of a rod depends on the path in R_4 that has been followed. So long, however, as these effects of an electromagnetic field have not been observed or, at all events, have not been made probable by other arguments (for one can always account for their apparent absence by a too low value of the coefficient λ), I think we had better not admit the connexion in question, confining ourselves to the introduction in electromagnetic theory of the fourfold potential and not ascribing to it any other physical meaning.

Two remarks more may be made. In the first place, if the fourfold vectors P_a and $\overline{P_a}$ really were indissolubly connected, this would amount to an action of an electromagnetic field widely different from anything that could reasonably be expected. This may be seen by taking the case of a constant electric field. In this we are concerned with one only of the components $\overline{P_a}$, namely with $\overline{P}_4$, the ordinary electrostatic potential, and the expression (13) would reduce to $P_4 dx_4$, showing that the length of a rod would, in course of time, continually and indefinitely increase or diminish. These changes might be detected by the following experiment. Of two equal rods, first juxtaposed in a region 1, one is left there, while the other is removed to a region 2 where the potential has a different value. After some time it is brought back to its original position and again compared with the first rod. The effect of these manipulations would be a difference in the two lengths that might be increased at will, simply by keeping the second rod for a longer time in the region 2.

In the second place, from electromagnetic phenomena one can deduce differences or changes only of potentials, the absolute values remaining undetermined. On the contrary, the numbers inscribed in the diagram R_4 enable us to determine in invariant measure the lengths of rods. Attending to their changes and applying eq. (13) one could obtain a knowledge of the potentials themselves.

§ 11. I shall conclude with some remarks on a generalisation of WEYL's theory that has been proposed by EDDINGTON [1]). His considerations are the more interesting because they can be developed to a certain extent without it being necessary to introduce the potentials g_{ab}.

EDDINGTON's aim is to arrive at the anti-symmetric covariant tensor F_{ab} of electric and magnetic force, at the fourfold potential on which these forces depend, and at the gravitation-potentials, making all these quantities flow from one common source. For this purpose, he begins by assigning to each point of the diagram R_4, in which coordinates, but

[1]) Proc. Roy. Soc. A. 99, p. 104 (1921); The mathematical theory of relativity, p. 213.

no values g_{ab} have been inscribed, 40 numbers. These are regarded as continuous functions of the four coordinates and are then subjected to certain mathematical operations.

The *fundamental numbers* as we may call them are represented by the symbol Γ^c_{ab}, with the relation

$$\Gamma^c_{ba} = \Gamma^c_{ab}\,, \tag{18}$$

by which the number of mutually independent quantities that otherwise would be 64, is reduced to 40.

Since no potentials g_{ab} have been introduced, we can speak neither of the length of a line-element, nor of the magnitude of a vector A (as we may call any line-element); there can be question only of the components dx_a or A^a. There is nothing, however, that prevents us from imagining that, when the starting point of a vector moves along a line-element dx_b, the components of the vector change in some specified manner. Making a definite assumption concerning these changes, EDDINGTON defines what he calls a parallel displacement; I shall rather say the "selected" displacement in order to keep in mind that, so long as there is no ds, there can be no question of direction-constants and of angles, nor of a parallel displacement in the sense in which the term was understood in § 5.

The fundamental numbers serve precisely for the definition of the changes in the components of a vector A^a, which accompany its displacement along a line-element dx_b,. EDDINGTON's formula being

$$dA^a = -\,(b\,c)\,\Gamma^a_{bc}\,A^c\,dx_b\,. \tag{19}$$

Line-elements and vectors in R_4 may be conceived to remain the *same* whatever be the coordinates which one uses for the evaluation of their components, and in the definition contained in (19) it is to be understood that the element dx_b along which the displacement is effected and the vector A^a, both before and after its displacement, are always the same in this sense. Hence, equations of the form (19), but with other fundamental numbers Γ^a_{bc} will hold after a change of coordinates. It is not difficult to find the relations between the original and the new fundamental numbers, but these transformation-formulae are found to have a form different from the one that is characteristic of tensors. In other terms, Γ^a_{bc} is not a tensor [1]).

[1]) The transformation-formula for Γ^a_{bc} is

$$\left(\frac{\partial x'_a}{\partial x_b} = \pi_{ba}\quad,\quad \frac{\partial x_a}{\partial x'_b} = p_{ab}\right)$$

$$\Gamma'^a_{bc} = -\,\Sigma\,(kl)\,p_{kc}\,p_{lb}\,\frac{\partial \pi_{ka}}{\partial x_l} + \Sigma\,(k\,l\,m)\,p_{lb}\,p_{mc}\,\pi_{ka}\,\Gamma^k_{lm}\,. \tag{20}$$

If here, on the right-hand side we had the last term only, Γ^a_{bc} would be a tensor.

Using the relation $\dfrac{\partial \pi_{ka}}{\partial x_l} = \dfrac{\partial \pi_{la}}{\partial x_k}$, one can deduce from (20) that Γ'^a_{bc} has, like Γ^a_{bc}, the symmetry expressed in (18).

§ 12. We shall now use eq. (19) for calculating the changes $\triangle A^a$ which occur in the components of a vector, when in a succession of infinitely small selected displacements, its starting point is made to move in a closed line drawn in R_4. The dimensions of this line are supposed to be infinitely small and we shall limit ourselves to quantities that are of the second order with respect to them.

Let the motion begin at the point P, and let for any other point Q of the cycle

$$x_a - x_{aP} = \mathbf{x}_a \, .$$

Then we have to calculate

$$\triangle A^a = - \Sigma (b\, c) \int \Gamma^{a}_{bc} A^c \, d\mathbf{x}_b \, . \qquad (21)$$

We must here keep in mind that the first factor under the sign of integration is the value of Γ^{a}_{bc} at the point Q and that for the second factor we must take the component A^c such as it has become when that point has been reached. It is preferable, however, to understand by these symbols the values corresponding to the fixed point P. Doing so, we must replace the first factor by

$$\Gamma^{a}_{bc} + \Sigma (s)\, \Gamma^{a}_{bc,s}\, \mathbf{x}_s \, ,$$

where $\Gamma^{a}_{bc,s}$ is the value at P of the differential coefficient of Γ^{a}_{bc} with respect to x_s. As to the change of A^c during the motion from P to Q, it will be sufficient to calculate it up to terms of the first order and we can therefore directly deduce it from (19), replacing the differentials dx_b by $x_{bQ} - x_{bP} = \mathbf{x}_b$ and understanding by A_c the initial values at the point P. Thus the second factor in the integral has to be replaced by

$$A^c - \Sigma (h\, i)\, \Gamma^{c}_{hi} A^i \, \mathbf{x}_h \, .$$

After substitution (21) becomes $\left(\int d\mathbf{x}_b = 0, \text{and we may omit the product } \mathbf{x}_s \mathbf{x}^h \right)$

$$\triangle A^a = - \Sigma (b\, c\, s)\, \Gamma^{a}_{bc,s} A^c \int \mathbf{x}_s \, d\mathbf{x}_b + \Sigma (b\, c\, h\, i)\, \Gamma^{a}_{bc} \Gamma^{c}_{hi} A^i \int \mathbf{x}_h \, d\mathbf{x}_b \, .$$

We may now, in the first term, write i and h instead of c and s and then obtain a new form of the same expression by interchanging in both terms the suffixes h and b. Finally, taking half the sum of the two forms and remembering that $\int \mathbf{x}_b \, d\mathbf{x}_h = - \int \mathbf{x}_h \, d\, \mathbf{x}_b$, we find

$$\triangle A^a = \tfrac{1}{2} \Sigma (b\, i\, h)\, B^{a}_{ihb} \, A^i \int \mathbf{x}_h \, d\mathbf{x}_b \, , \qquad (22)$$

where

$$B^{a}_{ihb} = \Gamma^{a}_{hi,b} - \Gamma^{a}_{bi,h} + \Sigma (c) \left[\Gamma^{a}_{bc} \Gamma^{c}_{hi} - \Gamma^{a}_{hc} \Gamma^{c}_{bi} \right] . \qquad (23)$$

Our conclusion is therefore that in general the components of a vector will have changed when it has been carried round along a closed path and that the changes are determined by the expressions (23). In what follows we shall be concerned with the quantities B^a_{ihb} only or rather with a tensor G_{ih} of the second rank that may be derived from them, and we shall scarcely have to think any more of the foregoing considerations, which were intended to deduce B^a_{ihb} from the fundamental numbers and to point out its geometrical meaning.

It is easily shown that B^a_{ihb} is a tensor, covariant as to the suffixes i, h, b and contravariant as to a [1]).

The covariant tensor G_{ih} that was mentioned just now is deduced from B^a_{ihb} by the operations indicated in the formula [2])

$$G_{ih} = \Sigma\,(a)\, B^a_{iha}. \qquad (24)$$

If the fundamental numbers are arbitrarily chosen, the tensor G_{ih} will be neither symmetric, nor antisymmetric. It may however be decomposed into a symmetric and an antisymmetric part, namely

$$R_{ih} = \tfrac{1}{2}\,(G_{ih} + G_{hi}) \quad , \quad F_{ih} = \tfrac{1}{2}\,(G_{ih} - G_{hi}).$$

It is easily seen that these are both covariant tensors and that

$$R_{hi} = R_{ih} \quad , \quad F_{hi} = -F_{ih}.$$

Performing the operations leading from the tensor (23) to F_{ih} and taking into account the relations (18), one finds

$$F_{ih} = \tfrac{1}{2}\,\Sigma\,(a)\left(\Gamma^a_{ah,i} - \Gamma^a_{ai,h}\right),$$

which shows that the tensor F depends on a fourfold potential A_k. If we put

$$A_k = \tfrac{1}{2}\,\Sigma\,(a)\,\Gamma^a_{ak}, \qquad (25)$$

1) In eq. (22) we may apply to A^l the transformation-formula for line-elements. We may proceed in the same way with $\mathbf{x}_h$, because this quantity is treated as infinitely small, and with $\triangle A^a$ because it is the difference of two vectors beginning at the same point. In all these cases the quantities p_{ab} and π_{ab} which occur in the transformation-formulae may be taken such as they are at the point P. We may do the same in transforming $d\mathbf{x}_b$. It is true that this element lies at a certain distance from P, but the influence which this has on p_{ab} would lead to terms of an order higher than needs be considered.

Thus:

$$\triangle A'^k = \Sigma\,(a)\,\pi_{ak}\,\triangle A^a = \tfrac{1}{2}\,\Sigma\,(a\,b\,i\,h)\,\pi_{ak}\,B^a_{ihb}\,A^i \int \mathbf{x}_h\,d\mathbf{x}_b$$

$$= \tfrac{1}{2}\,\Sigma\,(a\,b\,i\,h\,l\,m\,n)\,\pi_{ak}\,p_{il}\,p_{hm}\,p_{bn}\,B^a_{ihb}\,A'^l \int \mathbf{x}'_m\,d\mathbf{x}'_n\,,$$

an expression that has the same form as (22) if we put

$$B'^k_{lmn} = \Sigma\,(a\,b\,i\,h)\,\pi_{ak}\,p_{il}\,p_{hm}\,p_{bn}\,B^a_{ihb}.$$

2) Proof that G_{ih} is a covariant tensor. It suffices to express in the components B the quantities B' occurring in

$$G'_{ih} = \Sigma\,(a)\,B'^a_{iha}$$

and to use the relations

$$\Sigma\,(a)\,\pi_{ba}\,p_{ca} = \delta^c_b.$$

we may write

$$F_{ih}=\frac{\partial A_h}{\partial x_i}-\frac{\partial A_i}{\partial x_h}. \tag{26}$$

Following EDDINGTON we may now identify the quantities F_{ih} with the components of electric and magnetic force, or the components A_k with those of the fourfold electromagnetic potential.

It must be remarked, however, that the quantities A_a as defined by (25), do not constitute a tensor.

If we require them to do so, we must replace (25) by [1])

$$A_k=\tfrac{1}{2}\,\Sigma\,(a)\,\Gamma^a_{ak}+\frac{\partial\psi}{\partial x_k},$$

ψ being a function of the coordinates for which the transformation-formula is

$$\psi'=\psi-\tfrac{1}{2}\log p \tag{27}$$

(p is the functional determinant of the original coordinates with respect to the new ones). [1])

By the addition of the terms depending on ψ (26) is not changed.

§ 13. If h, i, j are all different, we have according to (26)

$$\frac{\partial F_{hi}}{\partial x_j}+\frac{\partial F_{ij}}{\partial x_h}+\frac{\partial F_{jh}}{\partial x_i}=0.$$

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[1]) Using (20) we may write

$$A'_r=\tfrac{1}{2}\,\Sigma\,(a)\,\Gamma'^a_{ar}+\frac{\partial\psi'}{\partial x'_r}=-\tfrac{1}{2}\,\Sigma\,(a\,k\,l)\,p_{kr}\,p_{la}\,\frac{\partial\pi_{ka}}{\partial x_l}+$$
$$+\tfrac{1}{2}\,\Sigma\,(a\,k\,l\,m)\,p_{la}\,p_{mr}\,\pi_{ka}\,\Gamma^k_{lm}+\frac{\partial\psi}{\partial x'_r}-\tfrac{1}{2}\,\frac{\partial\log p}{\partial x'_r}.$$

But $\Sigma\,(a)\,p_{la}\,\pi_{ka}=\delta^l_k$ and consequently

$$\Sigma\,(a)\,\frac{\partial}{\partial x_l}\,(p_{la}\,\pi_{ka})=0.$$

By this the first term of the expression for A'_r becomes

$$\tfrac{1}{2}\,\Sigma\,(l)\,\frac{\partial p_{lr}}{\partial x_l}=\tfrac{1}{2}\,\Sigma\,(l\,u)\,\pi_{lu}\,\frac{\partial p_{lr}}{\partial x'_u}=\tfrac{1}{2}\,\Sigma\,(l\,u)\,\pi_{lu}\,\frac{\partial p_{lu}}{\partial x'_r}=$$
$$=\tfrac{1}{2}\,\Sigma\,(l\,u)\,\frac{1}{p}\,\frac{\partial p}{\partial p_{lu}}\,\frac{\partial p_{lu}}{\partial x'_r}=\tfrac{1}{2}\,\frac{\partial\log p}{\partial x'_r},$$

and if, in the third term, we write

$$\frac{\partial}{\partial x'_r}=\Sigma\,(m)\,p_{mr}\,\frac{\partial}{\partial x_m}$$

the formula becomes

$$A'_r=\Sigma\,(m)\,p_{mr}\,A_m,$$

showing that A_k is a covariant tensor.

It may also be noted that the transformations defined by (27) form a group.

There are four formulae of this kind and these form one group of MAXWELL's equations.

As to the other group, in which the density of electric charge and the components of the convection-current occur, these equations must necessarily contain a contravariant tensor connected with F_{ih}. A tensor F^{ab} of this kind can only be defined if we have introduced beforehand the components g_{ab}, the relation between the two tensors being expressed by the formula

$$F^{ab} = \Sigma\,(i\,h)\, g^{ai}\, g^{bh}\, F_{ih}.$$

Moreover the equations in question contain the factor $\sqrt{-g}$.

EDDINGTON has remarked, however, that the gravitation-potentials g_{ab} and all quantities that depend on them, may also be considered as derived from the fundamental numbers that have given us the components of electric and magnetic force.

Indeed, we have so far used only the antisymmetric part F_{ih} of the tensor G_{ih} and we may now have recourse to its symmetric part R_{ih}. Since g_{ih} must also be symmetric we may put

$$g_{ih} = \frac{1}{\lambda} R_{ih},$$

λ being a constant.

The effect of this will be that all quantities involved in the phenomena of gravitation and electromagnetism, namely the potentials g_{ab}, the electric and magnetic forces F_{ab} and the corresponding contravariant tensor F^{ab} have been derived from the fundamental numbers. [1])

§ 14. All that has been said in §§ 11—13 amounts to the establishment of certain rules for the mathematical operations by means of which the components of electric and magnetic force and, if so desired, the gravitation-potentials can be derived from the fundamental numbers.

Now it must be remarked that the variety of these numbers is considerably greater than that of the quantities which we want to deduce from them. Indeed, there are four components of the electromagnetic potentential and ten values g_{ab}, whereas there are no less than fourty fundamental numbers. It may well be asked whether after all it would not be preferable simply to introduce the functions that are necessary for characterizing the electromagnetic and gravitational fields, without encumbering the theory with so great a number of superfluous quantities. The introduction of these could be justified, and would, of course, become very important, only if we had good grounds for thinking that some-

[1]) The only quantity occurring in the above formulae of which this cannot, as yet, be said, is the function ψ which appears in our definition of A_k. This function is to a certain extent undeterminate, the only condition being that it must transform according to eq. (27). If we put $\psi = -\frac{1}{2} \log \sqrt{-g}$, an assumption that agrees with (27), ψ also will have its origin in the fundamental numbers.

thing that might sooner or later be observed lies behind their wide diversity.

It may also be remarked that, in any particular case, the fundamental numbers must be such that they lead to the really existing values of electric and magnetic force and of the gravitation-potentials. However, I have found it by no means easy to account f. i. for the values

$$g_{11} = g_{22} = g_{33} = -1, \quad g_{44} = c^2$$

by suitable assumptions concerning EDDINGTON's Γ^c_{ab}.

§ 15. Some words remain to be said about the analogy between eq. (10) and (19). The direction-constants in the former equation may be considered as the components of a vector of unit magnitude, so that, like (19), (10) expresses a rule for a certain selected displacement of a vector. In so far it is a special case of (19), Γ^a_{bc} being replaced by $\left\{ {bc \atop a} \right\}$ (by which condition (18) is satisfied).

By this the tensors $B^{\ a}_{ihb}$ and G_{ih} defined by (23) and (24) become the well known tensors connected with the curvature of R_4, and the latter of them becomes symmetric, a simplification that is important for the theory of gravitation.

In order to prove it, one has to show that the antisymmetric part (26) vanishes. This becomes clear if one takes into account that, according to (25),

$$2\,A_k = \Sigma\,(a) \left\{ {a\,k \atop a} \right\} = \Sigma\,(a\,b)\, g^{ab} \begin{bmatrix} a\,k \\ b \end{bmatrix}$$

$$= \tfrac{1}{2}\,\Sigma\,(a\,b)\, g^{ab} \left(\begin{bmatrix} a\,k \\ b \end{bmatrix} + \begin{bmatrix} b\,k \\ a \end{bmatrix} \right) = \tfrac{1}{2}\,\Sigma\,(a\,b)\, g^{ab}\, g_{ab,\,k}$$

$$= \frac{1}{2g}\,\Sigma\,(a\,b)\, \frac{\partial g}{\partial g_{ab}}\, \frac{\partial g_{ab}}{\partial x_k} = \frac{1}{2g}\, \frac{\partial g}{\partial x_k} = \frac{\partial \log \sqrt{-g}}{\partial x_k}.$$