Mathematics: — "*On the foundations of combinatory Analysis Situs. I. Definitions and elementary theorems.*" By M. H. A. NEWMAN, Cambridge. (Communicated by Prof. L. E. J. BROUWER).

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In the Encyclopedia article of DEHN and HEEGAARD [1]), published in 1907, a set of definitions was given from which those parts of Analysis Situs which deal essentially with properties of finite collections of objects were to be developed without the use of the ideas and axioms of the Theory of Infinite Aggregates. Such a course is obviously desirable on logical grounds, but serious difficulties have been encountered in actually carrying out the programme indicated by the authors of the article: proofs have not been found, in the intervening years, for a number of theorems which stand at the very beginning of the subject. [2]) These difficulties appear to arise rather from the particular definitions adopted in the article than from the nature of the undertaking. In "combinatory" systems of Analysis Situs manifolds are regarded as sets of spherical "cells", and two manifolds are defined to be "topologically equivalent" or "homoeomorphic" if there is a third manifold which can be obtained from either by a series of modifications of certain prescribed types. The construction of a proof usually reduces, in the more fundamental theorems, to the consideration of the effect of combining, or re-arranging, a series of these steps. In DEHN and HEEGAARD's system the primitive modification adopted is the *subdivision* of a cell into two cells by any barrier drawn across it. The variety of configurations which can arise from a combination of even two steps of this kind is so great that no classification by mere enumeration can be undertaken; so that from the commencement of the theory recourse must be had to the inductive use of general theorems which would naturally present themselves for investigation at a much later stage, and of which the majority are, in fact, at present unproved.

In the present paper an alternative system of combinatory definitions is proposed. The fundamental transformations ("moves") are the *addition* and *removal*, instead of the *subdivision*, of cells. This allows the use of

[1]) Encykl. der Math. Wiss. III AB 3, *Analysis Situs.*

[2]) E.g. the "Theorem of Superposition": *If two manifolds Λ and M, are obtainable from the same manifold by subdivision of its cells, there is a manifold obtainable by subdivision from both Λ and M.* On the difficulties of proceeding from HEEGAARD and DEHN's definitions to a proof of this theorem cf. E. BILZ, Beiträge zur . . . Analysis Situs. (Math. Zschr. 18 (1923) p. l.).

simplexes as building units instead of spheres of general character; [1]) and the figure directly affected by a single transformation can only have one of a finite number of forms ("primitive clusters").

This paper contains the preliminary analysis of the structure of manifolds, and also some lemmas on primitive clusters, which find their application in the more general theorems of the paper which follows. [2])

§ 1. *Arrays of Simplexes.*

If n is a positive integer or zero, an *n-dimensional array*, or *n-array* is formed from a finite or enumerable set of objects by specifying which among the groups of $n+1$ objects contained in the set are the *units* of the array. The choice is unrestricted save that every object must belong to at least one unit. The objects are called the *vertices* of the n-array,[3]); and if $0 \leqslant k < n$ any $k+1$ vertices all belonging to the same unit form a k-component of the array. An $(n-1)$-component is called a *face*; units having a face in common are *adjacent.*

An *n-simplex* is an n-array with only one unit.

The sum, $\Gamma + \Delta + \ldots$, of a number of n-arrays, $\Gamma, \Delta, \ldots$, is the n-array whose units are all the units of all the arrays. If Γ contains [4]) Δ, $\Gamma - \Delta$ is the sum of the n-simplexes belonging to Γ but not to Δ.

If S and T are distinct [5]) m- and n-simplexes, ST is the $(m+n+1)$-simplex containing all the vertices of both. If U is ST, U/S is T (the component of U *opposite* S). If Γ and Δ are distinct m- and n-arrays, $\Gamma\Delta$ is the sum of all products ST, where S is a unit of Γ, and T of Δ.

(The number n is from this point supposed not less than 1).

An n-array is *regular* if each vertex belongs to only a finite number of units, and each face to only two. It is *connected* if every two units are the extreme members of a chain, i.e. of a finite sequence of units such that adjacent members of the sequence are adjacent units of the array.

The sum of the faces of an n-array, Δ, which belong each to only one unit is the *boundary* of Δ, written $\overline{\Delta}$.

If there are no boundary faces the array is *unbounded*, otherwise it is *bounded*. A component not belonging to the boundary is *internal*, a component belonging to only one unit is *free*.

(If Γ and Δ are unrestricted, an internal component of Γ is not necessarily interior to $\Gamma + \Delta$).

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[1]) Manifolds were first defined as sums of *simplexes*, (regarded as infinite aggregates of points), by L. E. J. BROUWER, "Ueber Abbildungen von Mannigfaltigkeiten", Math. Ann. 71 (1911).

[2]) These Proceedings, p. 627.

[3]) The name *points* is reserved for the complex entities which are considered in the theory of continuous manifolds.

[4]) The k-array Θ is *contained in*, or *belongs to*, the n-array Γ only if every unit of Θ is a unit or component of Γ.

[5]) i. e. having no vertex in common.

1. *If Γ and Δ have no common vertex and $d(\Gamma\Delta) > 0$,* [1] $\overline{\Gamma\Delta}$ *is* $\Gamma.\overline{\Delta} + \overline{\Gamma}.\Delta$.

(If S is a unit of Γ, T a unit of Δ, S' a face of S, and T' a face of T, typical faces of $\Gamma\Delta$ are $S'T$ and ST'; and the condition that, e.g. $S'T$ belongs to no unit of $\Gamma\Delta$ save ST is that S' belongs to no unit of Γ save S).

In order that this theorem may remain true when Γ or Δ is a vertex or is unbounded, it is agreed that when $d(\Phi) = 0$, $\overline{\Phi}$ is to be omitted from all terms containing it, and that when Φ is unbounded all terms containing $\overline{\Phi}$ are to be omitted.

If Γ and Δ are both unbounded, $\Gamma\Delta$ is unbounded.

2. *If Γ is a regular, unbounded n-array, there is no n-array, Δ, not contained in Γ, such that $\Gamma + \Delta$ is regular and connected.*

(Clear).

3. *If $n \geqslant 2$, the boundary of a regular, bounded n-array is an unbounded $(n-1)$-array.*

Let Γ be the bounded n-array, S_{n-2} an $(n-2)$-component of $\overline{\Gamma}$, S_{n-1} a boundary face containing it, and S_n the unit of Γ containing S_{n-1}.

If the face of S_n, other than S_{n-1}, which contains S_{n-2} is internal, it belongs to a second unit, T_n, of Γ. If the face of T_n, not belonging to S_n, which contains S_{n-2} is internal, it belongs to a third unit V_n. If any member of the series obtained by continuing in this way recurs, it can only be S_n, and its immediate predecessor contains S_{n-1}, contrary to the hypothesis that S_{n-1} is in the boundary. Hence the series does not recur, but terminates, and its last member contains another boundary face containing S_{n-2}.

§ 2. *Topological Equivalence.*

If a (1,1) correlation can be established between the vertices of two n-arrays, Γ and Δ, in such a way that the correlates of the vertices of each unit of Γ are the vertices of a unit of Δ, and *vice versa*, the two arrays are *congruent*.

(It is assumed from this point that all the n-arrays considered are *closed*, i. e. that they contain only a finite number of simplexes.) [2]

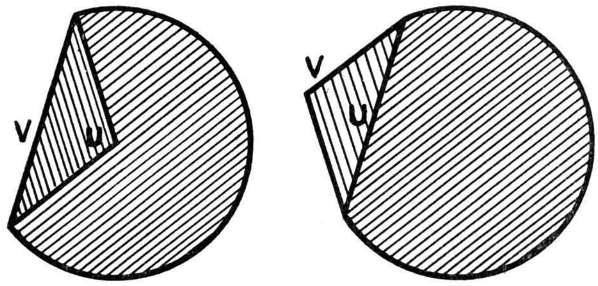
Let Γ be a bounded n-array and S an n-simplex, not belonging to Γ. S is said to have *regular contact with* Γ if it is the product of two components, U and V such that

$A\,(i)$ U belongs to $\overline{\Gamma}$
$A\,(ii)$ U is interior to $UV + \Gamma$
$A\,(iii)$ V does not belong to Γ

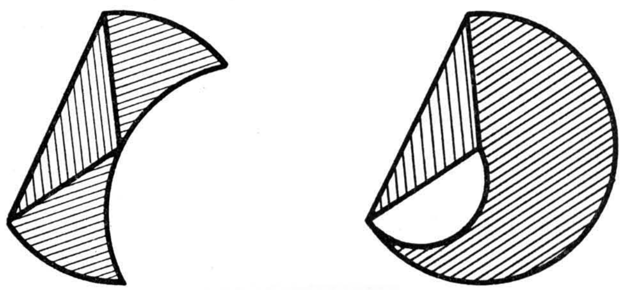
[1] $d(\Phi)$: "the dimension number of Φ".

[2] The definitions remain significant and most of the theorems remain valid, without this restriction. But it is obvious that to give an adequate account of the equivalence of open n-arrays infinite sequences of "moves" must be admitted; and the investigation of such sequences it is convenient to postpone.

Both U and V must contain at least one vertex.



Regular contact.



Irregular contact.
Fig. 1.

Clearly the faces $U.\overline{V}$ are interior to $\Gamma + UV$, the faces $V.\overline{U}$ are in its boundary.

4. *If Γ is regular, those of its boundary faces which contain U form the array $U.\overline{V}$.*

For a boundary face of Γ containing U but not belonging to $U.\overline{V}$ would be a boundary face of $\Gamma + UV$, contrary to $A\,(ii)$; if a unit of $U.\overline{V}$ did not belong to $\overline{\Gamma}$, the set of boundary faces of Γ at U would be a subset of the units of $U.\overline{V}$, and U would therefore be on its boundary. But the boundary of the regular array Γ is unbounded, and therefore U is interior to the set of units of $\overline{\Gamma}$ containing it.

Hence

5. *If Γ is regular the common faces of Γ and UV are boundary faces of Γ.*

(For they all belong to $U.\overline{V}$).

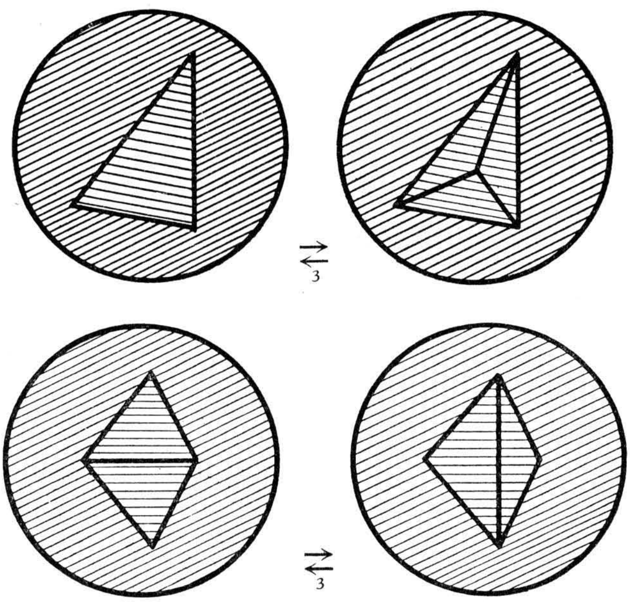
If Γ is a bounded n-array we define as a *move of type* 1 the operation of adding to Γ an n-simplex having regular contact with it; and as a *move of type* 2 the operation of removing from Γ an n-simplex, S, having regular contact with $\Gamma - S$.

If X and Y are distinct k- and $(n-k-1)$-simplexes, $X.\overline{Y}$ and $Y.\overline{X}$ are *complementary primitive n-clusters*. Their sum is seen to be $\overline{XY}$.

If Γ is any n-array, and $X.\overline{Y}$ a primitive n-cluster contained in Γ, it is a *move of type* 3 to replace $X . \overline{Y}$ by its complement, $Y . \overline{X}$, provided that

$B\,(i)$ X does not belong to $\Gamma — X . \overline{Y}$

$B\,(ii)$ Y does not belong to $\Gamma — X . \overline{Y}$



Moves of type 3.

Fig. 2.

If a *bounded* n-array, Γ, can be transformed by a finite succession of moves, of any or all of the types 1, 2, and 3, into an n-array which is congruent to the n-array Δ, Γ is said to be *topologically equivalent* to Δ, and we write: $\Gamma \rightarrow \Delta$ [1]). "Topologically equivalent" may usually be shortened to "equivalent."

If Γ and Δ are two *unbounded* n-arrays, and units S of Γ, and T of Δ exist, such that $\Gamma — S$ is bounded and topologically equivalent to $\Delta — T$, Γ is said to be *topologically equivalent* to Δ. (It will be shewn later, (24), that in the case of manifolds the property is independent of the choice of S and T).

Equivalence is both *transitive* and *symmetrical*. For if $\Gamma \underset{1}{\rightarrow} \Delta$, $\Delta \underset{2}{\rightarrow} \Gamma$; and if $\Gamma \underset{3}{\rightarrow} \Delta$, $\Delta \underset{3}{\rightarrow} \Gamma$.

6. *If Γ is a regular, connected, bounded n-array, and $\Gamma \rightarrow \Delta$, Δ is a regular, connected bounded n-array.*

The case $n = 1$ is trivial. We therefore suppose $n \geqq 2$.

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[1]) "$\Gamma \underset{pq}{\rightarrow} \Delta$" denotes, when Γ and Δ are bounded, that only moves of types p and q need occur in the transformation of Γ into Δ. It is to allow this notation that the unsymmetrical sign $\rightarrow$ is used for equivalence.

It is sufficient to consider the effect of a single move. *Regularity* is clearly not destroyed if the move is of type 2; nor, in virtue of 5, if it is of type 1; nor, by the conditions B, if it is of type 3. The *boundary* is unaffected by a move of type 3, and since it is itself unbounded, only a part of it is affected by a move of type 1 or 2. *Connection* could be destroyed only by removing a unit, S_n, which was the only means of connecting in Γ two units, P_n and Q_n, both adjacent to it. But if the removal of S_n is a move of type 2 the common $(n-2)$-component, S_{n-2}, of P_n and Q_n, being interior to the set of common faces of S_n and $\Gamma - S_n$, is interior to Γ, and therefore a chain of simplexes starting from P_n, all containing S_{n-2}, constructed in the manner described in 3, cannot terminate, and so forms a cycle, whose units include P_n and Q_n. This is incompatible with the hypothesis that P_n and Q_n can be connected only through S_n.

7. *If Γ is a regular, connected, unbounded n-array and Δ an equivalent unbounded n-array, then Δ is regular and connected.*

We may again suppose that $n \geqslant 2$. It may be shewn, by an argument similar to that of 6, that the removal of any unit from Γ gives a regular, connected, bounded n-array. There is, therefore, a unit, T, of Δ such that $\Delta - T$ is a regular, connected, bounded n-array. Since Δ is unbounded, all the boundary faces of $\Delta - T$ must be among those of T. But the boundary of the regular array $\Delta - T$ is unbounded. It is therefore identical with $\overline{T}$, and $(\Delta - T) + T$ is regular, connected and unbounded.

§ 3. *Element, Sphere, Cluster.*

An *n-element* is an n-array which is equivalent to an n-simplex. If q is the smallest number of moves in which the element can be transformed into a simplex, the element is said to be *of order q*.

An *n-sphere* is an n-array congruent to the boundary of an $(n+1)$-element [1]).

8. *E_n is a regular, connected, bounded n-array.*

(Follows from 6).

9. $\Sigma_n \,.\!\underset{3}{\rightarrow}\!.\, \overline{S}_{n+1}$. [2]) For since an $(n+1)$-element is a regular $(n+1)$-array, the addition or removal of a simplex with regular contact replaces a primitive n-cluster of the boundary by its complement (5). Conditions $B(i)$ and $B(ii)$ are satisfied in virtue of 4 and $A(iii)$, or $A(iii)$ and 4

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[1]) The case $n=0$ is included: a 0-element is a vertex, a 0-sphere a pair of vertices. The assumption $n \geqq 1$ is, however, still retained in the text which follows. S and T always denote simplexes, E an element and Σ a sphere. If there is a lower index, (as in E_{n-k-1}) it denotes the dimension-number; upper indices are merely distinguishing marks.

[2]) If Γ and Δ are unbounded $\Gamma \,.\!\underset{3}{\rightarrow}\!.\, \Delta$ means that Δ can be obtained from Γ by a succession of moves of type 3. The dots are to recall the fact that $\Gamma \rightarrow \Delta$ cannot at present be inferred.

according to whether the alteration to the $(n+1)$-element is of type 1 or type 2. A move of type 3 does not alter the boundary.

10. *Σ_n is a regular, connected, unbounded n-array.*

(Cf. the proof of 6).

The smallest number of moves of type 3 required for the transformation of an n-sphere into the boundary of an n-simplex is called the *order* of the n-sphere.

If S_k and Σ_{n-k-1} have no common vertex, $(0 \leqslant k \leqslant n-1)$, $S_k\, \Sigma_{n-k-1}$ is a *complete* $(n:k)$*-cluster*; and if S_k and E_{n-k-1} have no common vertex, $S_k\, E_{n-k-1}$ is an *incomplete* $(n:k)$*-cluster*.

In both clusters S_k is the *core*; and Σ_{n-k-1} is the *shell* of $S_k\, \Sigma_{n-k-1}$ and E_{n-k-1} of $S_k\, E_{n-k-1}$.

If $k=0$ the clusters are called *complete* and *incomplete n-stars.*

Clearly, a "primitive cluster" is a complete n-cluster.

11. *The core of a complete cluster is an internal component; the core of incomplete cluster is in its boundary.*

For the face which is the product of the core and an $(n-k-2)$-component of the shell is interior to, or in the boundary of, the cluster as the $(n-k-2)$-component is interior to, or in the boundary of the shell.

12. *An n-star is an n-element.*

a. If the n-star is complete, let a be its core and Σ its shell.

The theorem is true if the shell is the boundary of a simplex. Suppose it true if the shell is of order $q-1$. Let Σ be obtained by a move of type 3 from $\Sigma^\star$, of order $q-1$. Let Σ be $\Gamma + U.\overline{V}$, $\Sigma^\star$ be $\Gamma + V.\overline{U}$. (If V is a single vertex it may be supposed not to be a).

By hypothesis Γ does not contain U or V. The sum of the units of $a\Sigma^\star + UV$ which contain V is therefore $a\,V.\overline{U} + UV$, i.e., $V.\overline{aU}$, to which V is interior; and U does not belong to $a\Sigma^\star$. Hence UV has regular contact with $a\Sigma^\star$. Again, since $a\Gamma$ contains neither aU nor V it is a move of type 3 to change $a\Sigma^\star + UV$, which is $a\Gamma + V.\overline{aU}$, into $a(\Gamma + U.\overline{V})$, which is $a\Sigma$.

Thus $a\Sigma^\star \underset{1}{\rightarrow} a\Sigma^\star + UV \underset{3}{\rightarrow} a\Sigma$, and $a\Sigma$ is an n-element.

b. If the n-star is incomplete, let a be its core and E its shell.

The theorem is true if E is a simplex. Suppose it true when E is of order $q-1$, and let E be obtainable in a single move from $E^\star$, of order $q-1$.

If the move changing $E^\star$ into E is of type 3 the arguments of the preceding paragraph are valid. If E is $E^\star + XY$, X being interior to E and Y free, aE is $aE^\star + aXY$, aX being interior to aE and Y free. Hence $aE^\star \underset{1}{\rightarrow} aE$ and aE is an n-element. If E is $E^\star - XY$, the removal of aXY from $aE^\star$ is a move of type 2, and aE is again an element.

13. *If an n-simplex and an n-element,* (*not a simplex*) *have the same boundary, their sum is an n-sphere.*

Let E be the element, S the simplex, and a a new [1]) vertex. a lies

[1]) i. e. a vertex not belonging to any of the arrays already under consideration.

in the boundary of the $(n+1)$-element aE; it is clearly interior to $aE + aS$; and S does not belong to aE. Hence $a(E+S)$ is an $(n+1)$-element, and its boundary, $E+S$, an n-sphere.

14. *If* $\Gamma_n \;.\underset{3}{\rightarrow}.\; \Sigma_n$, Γ_n *is an n-sphere.* We may suppose only one move is required—say the substitution of $U.\overline{V}$ for $\overline{U}.V$. If then Γ_n is $\Delta_n + \overline{U}.V$ and a is a new vertex

$$\begin{aligned} a\Gamma_n &\underset{1}{\rightarrow} a\Gamma_n + UV, \\ &\text{is } a\Delta_n + a\overline{U}.V \\ &\underset{3}{\rightarrow} a\Delta_n + aU.\overline{V} \\ &\text{is } a\Sigma_n. \end{aligned}$$

But $a\Sigma_n$ is an $(n+1)$-element, and so, therefore, is $a\Gamma_n$; and its boundary, Γ_n, is a sphere.

15. *Every* $(n:k)$*-cluster is an incomplete* $(n-k-1)$*-cluster.*

Let the cluster be $S\Pi$, were Π may be an $(n-k-1)$-sphere, or $(n-k-1)$-element, let a be a vertex of S and S' the opposite face. $a\,\Pi$ is a complete or incomplete $(n-k)$-star, and therefore an $n-k$-element. Hence $S\Pi$, which is $S'a\Pi$, is an incomplete $(n:k-1)$-cluster.

16. *Every* $(n:k)$*-cluster is an n-element.*

(Follows from 15 and 12).

§ 4. *Manifolds.*

An *n-array* is an *n-manifold* if (1) it is connected, and (2) the sum of the units at each vertex is a complete or incomplete n-star [1]). If at each vertex the simplexes form a complete n-star the manifold is *unbounded*; if not it is *bounded.*

It will be shewn that bounded and unbounded n-manifolds are in fact bounded and unbounded n-arrays in the sense already defined. For the present "bounded-manifold" "unbounded-manifold" must be regarded as indivisible phrases.

17. *An n-array which can be obtained from an unbounded manifold by a succession of moves of type* 3 *is an unbounded manifold.*

It is sufficient to examine the effect of a single move of type 3. Let M be the manifold, let the substitution of $V.\overline{U}$ for $U.\overline{V}$ be the given move, and let Γ be $M - U.\overline{V}$, Δ be $\Gamma + V.\overline{U}$. Γ, then, contains neither U nor V.

Let a be a vertex of Δ. If a does not belong to M it must be identical with V, and the sum of the simplexes at a is $V.\overline{U}$, a complete n-star. If, on the other hand, all the units of Δ containing a belong to M, the units at a in Δ are identical with the units at a in M, which form a complete n-star. Suppose then that a belongs to $V.\overline{U}$ but is not V.

(a). Let V be aV'. Of the units of $U.\overline{V}$ in M, the subset $a\,U.\overline{V'}$ con-

[1]) The name "manifolds" is usually given to the arrays here called "unbounded manifolds", but as bounded manifolds appear to play the more important part in the present theory, this unsymmetrical nomenclature is not adopted.

tains a. The set of simplexes at a in M has therefore the form $a\,(\varDelta + U\,.\,\overline{V'})$, where $\varDelta + U.\overline{V'}$ is an $(n-1)$-sphere. Neither V' nor U belongs to $\varDelta$, or V or U would belong to $a\varDelta$, and therefore to Γ. Hence $\varDelta + U.\overline{V'}.\underset{3}{\rightarrow}.\varDelta + V'.\overline{U}$ which is therefore also a sphere. The array $a\,\varDelta + U.\overline{V}$ in M, which includes all units of M containing a, is replaced in $\varLambda$ by $a\,\varDelta + V.\overline{U}$ which is $a\,(\varDelta + V'.\,\overline{U})$. This is therefore the set of simplexes at a in $\varLambda$, and, its boundary being a sphere, is a complete star.

(*b*). If U is $a\,U'$, all the units of $U\,.\overline{V}$ belong to the n-star at a in M, which is of the form $a\,(\varDelta + U'.\,\overline{V})$. This is replaced in $\varLambda$ by $a\,\varDelta + V.\overline{U}$ and the sum of the units at a in $\varLambda$ is therefore $a\,(\varDelta + V.\,\overline{U'})$. It follows as before that this is an n-star.

Hence

18. *Every n-sphere is an unbounded n-manifold.*

19. *The n-simplexes at a k-component of an unbounded n-manifold form a complete $(n:k)$-cluster.* [1])

The theorem is true when $k=0$. Suppose it true when $k \leqq j-1$, where $1 \leqq j \leqq n-1$. Let S be a j-component of the manifold, a a vertex of S, and S' the opposite face.

The n-simplexes at S' form a certain $(n:j-1)$-cluster, say $S'\Sigma$. The n-simplexes at S are those units of $S'\Sigma$ which contain a. Since a does not belong to S', it belongs to Σ, and (18) the units of Σ containing it form an $(n-j)$-star, $a\Sigma'$. Hence the simplexes at S form the array $S'.\,a\Sigma'$, which is a complete $(n:j)$-cluster.

In particular, the n-simplexes containing a *face* of the manifold form a complete $(n:n-1)$-cluster i.e. a pair of simplexes. The "unbounded-manifold" is therefore in fact a regular unbounded n-array.

20. *If the units of a primitive cluster contained in an unbounded n-manifold are removed, one by one, in any order, all the contacts after the first are regular.*

It is sufficient to shew that if M is an unbounded manifold containing $U\,.\,\overline{a\,V}$, $U\,V$ has regular contact with $M - U\,.\,\overline{a\,V}$.

We first notice that all units of M containing U belong to $U\,.\,\overline{aV}$. For if not, $\overline{aV}$ would be drawn on [2]) the shell of the complete cluster at U; and this is impossible, for one sphere cannot be drawn on another. (Cf. 2).

Consider the contact of UV with $M - U\,.\,\overline{aV}$. The boundary of $M - U\,.\,\overline{aV}$ contains $\overline{aV}$, and therefore V; U does not belong to $M - U\,.\,\overline{aV}$; and V, which is interior to M and does not belong to $aU\,.\overline{V}$, is interior to $M - aU\,.\overline{V}$, which is $M - U\,.\,\overline{aV} + UV$. The contact is regular and $M - U\,.\,\overline{aV} + UV \underset{2}{\rightarrow} M - U\,.\,\overline{aV}$.

[1]) Cf. H. KNESER, Ein Topologischer Zerlegungssatz. (These Proceedings **27**, (1924) p. 603).

[2]) "Γ is drawn on Δ" means that the units of Γ form a subset of the units of Λ. It implies that $d(\Gamma) = d(\Delta)$ and that Γ is not Δ.

21. *If M is unbounded and $M. \underset{3}{\rightarrow}. \Lambda$, then $M \rightarrow \Lambda$.*

It is sufficient to examine the effect of a single move, say the substitution of $V.\overline{U}$ for $U.\overline{V}$. Let S be a unit of $U.\overline{V}$, T a unit of $\overline{U}.V$. If the units of $U.\overline{V}—S$ are removed from $M—S$ in any order, the contacts are all regular: i.e. $M—S \underset{2}{\rightarrow} M—U.\overline{V}$. Λ is also an n-manifold, and therefore $\Lambda—T \underset{2}{\rightarrow} \Lambda—V.\overline{U}$; i.e. $\Lambda—V.\overline{U} \underset{1}{\rightarrow} \Lambda—T$. Since $M—U.\overline{V}$ is $\Lambda—V.\overline{U}$ this is the required result [1]).

22. *If an n-simplex is removed from an n-sphere the remainder is an n-element.*

Sketch of the proof. In extending the range of this theorem from spheres of order $q—1$ to the sphere Σ, of order q, the crux is the case when S, the simplex to be removed, belongs to the new cluster, $V.\overline{U}$, in Σ. If then T is any unit of $U.\overline{V}$ it can be shewn, as in 21, that $\Sigma—S$ can be obtained by moves of types 1 and 2 from the n-element $\Sigma^*—T$, where Σ^* is $\Sigma—V.\overline{U}+U.\overline{V}$.

24. *If S and T are any two units of an unbounded n-manifold, M, $M—S \rightarrow M—T$.*

It is sufficient to examine the case in which S and T have a common face. But in that case $S+T$ is a primitive cluster and therefore, (Cf. 20), $M—S \rightarrow M—(S+T) \rightarrow M—T$.

25. *An n-dimensional bounded manifold is a bounded n-array whose boundary is one or more unbounded $(n—1)$-manifolds.*

It follows easily from 11 that the set of boundary vertices is identical with the set of vertices at which the simplexes form an incomplete star. The boundary faces at any boundary vertex, α, are those boundary faces of the incomplete n-star at α which contain α. They are obtained by joining α to the boundary of the shell of the n-star and therefore form a complete $(n—1)$-star.

26. *If Γ is a bounded manifold, conditions $A(i)$ and $A(ii)$ governing regular contact may be replaced by: $A(iia)$ $U.\overline{V}$ belongs to $\overline{\Gamma}$.*

If, condition $A(iia)$ being satisfied, U were not interior to $\Gamma+UV$, there would be boundary faces of Γ containing U, but not belonging to UV and therefore not contained in $U.\overline{V}$; i.e. the complete $(n—1:k)$-cluster $U.\overline{V}$ would be contained in another complete $(n—1:k)$-cluster, which is impossible.

27. *If $\Gamma+UV$ is a bounded manifold, the conditions $A(i)$ and $A(iii)$ governing regular contact may be replaced by: $A(iiia)$, $V.\overline{U}$ belongs to $\overline{\Gamma+UV}$.* (Cf. 26).

28. *If all the units of an n-element have a common vertex, the element is an n-star.*

If the common vertex, ξ, is interior to the element, the truth of the theorem is obvious. If it is in the boundary, the array is formed by

[1]) We may now drop the dots and write $M \underset{3}{\rightarrow} \Lambda$. The symbols $M \underset{1}{\rightarrow} \Lambda$, $M \underset{1,2}{\rightarrow} \Lambda$ etc., are not used when Λ and M are unbounded.

joining ξ to those boundary faces which do not contain it. It is therefore sufficient to shew that *if an* $(n-1)$*-star is removed from an* $(n-1)$*-sphere the remainder is an* $(n-1)$*-element.* In this form the theorem is clearly true of the boundary of an n-simplex. The proof that it can be extended to spheres of order q involves considerations similar to those which arose in proving 12 and 17, and may be omitted.

29. *An n-array which is equivalent to a bounded n-manifold is a bounded n-manifold.*

The theorem is true when $n = 1$. Suppose it true when $n \leqslant m - 1$, where $m \geqslant 2$. From this hypothesis there follow:

30. $(n \leqslant m - 1)$. *An n-element is a bounded n-manifold.*

31. $(n \leqslant m)$. *The n-simplexes at a k-component of a bounded n-manifold form an* $(n:k)$*-cluster, complete or incomplete, as the k-component is, or not, internal.*

This is true when $k = 0$. Suppose it is known to follow from the inductive hypothesis if $n < m$, and also if $n = m$ and $k \leqslant j - 1$, where $1 \leqslant j \leqslant n - 1$.

a. Let S be a boundary j-component, S' a face of S, a the opposite vertex.

The simplexes at S are those units of the $(n:j-1)$-cluster (say $S'E$) at S', which contain a. a belongs to E and is on its boundary; for if it were interior to E, aS' would be interior to ES' and therefore to the manifold. Since the element E, (of less than m dimensions) is a bounded manifold, those of its units which contain a form an incomplete $(n-j)$-star, aE'. The n-simplexes at S therefore form the incomplete $(n:j)$-cluster $S'a\,.\,E'$.

b. Let S be an internal j-component, and S' and a a face and opposite vertex.

If S' is internal, the arguments used in 19 apply. If not, and the $(n:j-1)$-cluster at S' is $S'E$,a is now interior to E. For if not, S would be in the boundary of $S'E$, and therefore, since this array includes all units of the manifold containing S, in the boundary of the manifold. The units of E containing a therefore form a complete $(n-j)$-star $a\Sigma'$, and the simplexes at S the complete $(n:j)$-cluster, $S\Sigma'$.

In particular the array at an internal *face* is a pair of simplexes. A bounded manifold is therefore a regular array.

The sum of the units of a manifold M which contain a component S will be called "*the S-cluster in M.*"

32. $(n \leqslant m)$. *An incomplete n-star is a bounded manifold.*

If the star is aE, and β is a vertex of E, the units of E at β form a complete or incomplete star, $\beta\Pi$. Hence the units of aE at β form the set $a\,.\,\beta\Pi$, or $\beta\,.\,a\Pi$, and $a\Pi$ is an $(n-1)$-element.

33. $(n \leqslant m)$. *If M is an n-manifold, a component of M interior to a subset, Γ, of M is interior to M and does not belong to $M-\Gamma$.*

The component, S, is interior to Δ, the set of units of Γ which contain it. Δ is contained in the S-cluster in M, and therefore can only have S as an internal component, if the S-cluster in M is a complete cluster and coincides with Δ.

Proof of 29. It must be shewn that an alteration of any of the three types to a bounded m-manifold leaves it a bounded m-manifold.

The effect of a move of type 3 has been discussed in the proof of 17.

Consider the effect of *adding* to the manifold, M, a simplex UV, U being interior to $M + UV$ and V free.

If U is a vertex, the boundary of the shell of the U-star in M coincides with $\overline{V}$. The boundary of the array at U in $M + UV$ is therefore (13) a sphere, and the array itself an m-star.

If V is a vertex the V-cluster is $U\,V$.

If α is a vertex of U but is not identical with it, let U be $\alpha\,U'$. The contact of $U\,V$ with αE, the α-star in M, is regular. For all the units of $\underline{U}\,.\,\overline{V}$ contain α, and therefore belong to $\alpha\,E$. Since they belong (by hypothesis) to $\overline{M}$, they belong (33) to $\overline{\alpha\,E}$; V does not belong to M, much less to $\alpha\,E$; and $\alpha\,E$ is a bounded manifold (32). Hence (26) $\alpha\,E + U\,V$ is an m-element and therefore an incomplete m-star.

Finally, consider a vertex, β, of V. Let V be $\beta\,V'$. If V' belonged to $\beta\,E^\star$, the β-star in M, it would belong to $E^\star$ and therefore V would belong to $\beta E^\star$, which is contrary to hypothesis. $U\,.\,\overline{V}$ is $U\beta\,.\,\overline{V'} + UV'$, and therefore the array $U\beta\,.\,\overline{V'}$ belongs to $\overline{\beta E^\star}$. Hence the contact of UV with $\beta E^\star$ is regular, and the set at β in $M + UV$ is an incomplete m-star.

Thus at all vertices of $M + UV$ the simplexes form a star: the sum is a manifold.

The case of the *removal* of a simplex is similar.

34. *The removal of a unit from an unbounded n-manifold leaves a bounded n-manifold.*

(Follows from 22).

35. *An n-array which is equivalent to an unbounded n-manifold is an unbounded n-manifold.*

(Follows from 34 and 13).

§ 5. *Some lemmas on Primitive Clusters.*

If Γ and Δ are two n-arrays, we write

$$\Gamma \underset{p,q}{\rightarrow} \Delta, \qquad [\text{not } P, Q, \ldots]$$

for the statement: "If Φ is any n-array, containing no unit of Γ or Δ, then $\Phi + \Gamma \rightarrow \Phi + \Delta$ provided none of $P, Q, \ldots$ is a component of Φ".

36. *If P, Q, R are three simplexes with no common vertex*

$$P\,.\,\overline{Q}\,.\,\overline{R} \underset{3}{\rightarrow} Q\,.\,\overline{P}\,.\,\overline{R}, \qquad [\text{not } P, Q]$$

If R is a vertex the theorem is reduced to $P\,.\,\overline{Q} \underset{3}{\rightarrow} Q\,.\,\overline{P}$ [not P, Q], which is true. Suppose its truth is known if $d(R) < q$. Let R be $R'\xi$, R' having $q-1$ dimensions.

$$
\begin{aligned}
P.\overline{Q}.\overline{R} \quad &\text{is} \quad P\xi.\overline{Q}.\overline{R'} + PR'.\overline{Q}, \\
&\underset{3}{\rightarrow} \; Q.\overline{P\xi}.\overline{R'} + PR'.\overline{Q}, && [\text{not } Q,\ P\xi], \\
&\text{is} \quad P.\overline{QR'} + \xi Q.\overline{P}.\overline{R'}, \\
&\underset{3}{\rightarrow} \; QR'.\overline{P} + \xi Q.\overline{P}.\overline{R'}, && [\text{not } QR',\ P], \\
&\text{is} \quad Q.\overline{P}.\overline{R}.
\end{aligned}
$$

The four conditions, [not P, $P\xi$, Q, QR'] are clearly contained in [not P, Q].

Noticeable special cases are:

$$P.\overline{R} \underset{3}{\rightarrow} a.\overline{P}.\overline{R}, \qquad [\text{not } P,\ a], \quad \ldots\ldots \quad (1)$$

$$\beta.\overline{R} \underset{3}{\rightarrow} a.\overline{R}, \qquad [\text{not } \beta,\ a]. \quad \ldots\ldots \quad (2)$$

The value of the last result is that it shews that the transformation need not proceed by way of $\beta.\overline{R} \rightarrow R$, which would require [not R].

The following generalization of 36 is easily proved.

37. *If $P, Q, R, \ldots, S$ are any number of simplexes, no two having a common vertex, then*

$$P.\overline{Q}.\overline{R}\ldots\overline{S} \underset{3}{\rightarrow} Q.\overline{P}.\overline{R}\ldots\overline{S}, \qquad [\text{not } P,\ Q].$$

It is convenient to prove together the following four theorems:

38. *If the common part*[1]) *of an n-manifold, M, and a primitive n-cluster, $S.\overline{T}$, is an $(n-1)$-element belonging to the boundaries of both, then $M \underset{1}{\rightarrow} M + S.\overline{T}$.*

(In 39—44 U and V are opposite components of an $(n+2)$-simplex and M a bounded n-manifold drawn on $\overline{U}.\overline{V}$).

39. *If U' is a face of U and Γ the set of units of $U'.\overline{V}$ which do not belong to M, then $M \underset{1}{\rightarrow} M + \Gamma$, provided $M + \Gamma$ is not $\overline{U}.\overline{V}$.*

40. *If no face of U is interior to M at least one face of V does not belong to M $(d(V) > 0)$.*

41. *If an h-component, X, but no $(h-1)$-component of U is interior to M, there is a face, V', of V such that $(U/X).V'$ does not belong to M. $(d(V) > 0)$.*

The four theorems are clearly true if $n = 1$. Suppose them true if $n \leqslant m-1$. From this assumption there follow:

42. $(n \leqslant m-1)$. *M is an n-element.*

For to each vertex, ξ, of U corresponds an array Γ_ξ, (which may in some cases have no units), consisting of the units of $(U/\xi).\overline{V}$ which do not belong to M. The arrays Γ_ξ can, by 39 Hyp., be added to M in

[1]) *Θ is the common part of Γ and Δ* means that the common units, (if any), and common components of Γ and Δ are the units and components of the array Θ.

succession, by moves of type 1, until only one of those which actually have units remains. This remaining array, Γ_η, being part of $(U/\eta) . \overline{V}$ is a primitive cluster, and the array that has been obtained from M, being the remainder of $\overline{U} . \overline{V}$, is an n-element. M itself is therefore also an n-element.

43. $(n \leqslant m - 1)$. *If an n-element is removed from* $\overline{U} . \overline{V}$ *the remainder is an n-element.*

(During the process described in 42 the simplexes added, by moves of type 1, to M are removed, by moves of type 2, from $\overline{U} . \overline{V} - M$ and the final result — one of the arrays Γ_ξ — is an n-element).

44. $(n \leqslant m-1)$. *If* U' *is any face of* U, *and* $\varDelta$ *the set of units of* $U'\overline{V}$ *belonging to* M, $M \underset{2}{\rightarrow} M-\varDelta$, *provided* M *is not* $\varDelta$.

Proof of 38. Suppose n equal to m and let $d(S) = k$. If $k = m$ the theorem is reduced to 26. Suppose then that the theorem is true if $n = m$ and $k \geqslant j + 1$, where $j \leqslant m-1$, and that in the given case $n = m$ and $k = j$. Let E be the common boundary element.

If no component of S is interior to E there is a face, T', of T which does not belong to E, (40); and the array $S.\overline{T} - S.\overline{T'}$, (i.e., if ξ is T/T', the array $\xi S . \overline{T'}$, which is a primitive $(m : j + 1)$-cluster), has also the array E in common with M. Its units may therefore be added to M by moves of type 1. The remaining simplex, ST', has regular contact with $M + \xi S . \overline{T'}$. For S is interior to $S . \overline{T}$ and T' belongs neither to $\xi S . \overline{T'}$ nor, by hypothesis, to M.

If an h-component, U, but no $h-1$-component of S is interior to E, let S be UV. There is then a face, T' of T, such that VT' does not belong to E. *If* ξ *is* T/T', *the common part of* M *and* $\xi S\overline{T'}$ *is an* $(m-1)$*-element.* The common *faces*, being those units of E which do not belong to $T' . \overline{S}$, certainly form an $(m-1)$-element (44) .Suppose then that M and $\xi S . \overline{T'}$ have a common component, W, which does not belong to a common face; from the conditions governing the contact of M and $S . \overline{T}$, W must belong to a common boundary face of M and ST'. $\overline{ST'}$ is $S . \overline{T'} + VT' . \overline{U} + U\,T' . \overline{V}$; the units of $S . \overline{T'}$ are interior to $S . \overline{T}$, those of $VT' . \overline{U}$ all contain VT' which we have supposed not to belong to E, and therefore $UT' . \overline{V}$ contains all units of E which belong to ST'. Also the part of the boundary of $S.\overline{T}$ belonging to $\xi S.\overline{T'}$ is $S.\overline{T'}$, i.e. $UV . \overline{T'}$. Thus W must be a component of both $UT' . \overline{V}$ and $UV . \overline{T'}$ and therefore of their set of common faces, $U . \overline{T'} . \overline{V}$, which belongs to the U-cluster in $\overline{S} . \overline{T}$. But since U is, by its definition, interior to E, the whole of the U-cluster in $\overline{S} . \overline{T}$ belongs to E and therefore those units of the boundary of $\xi S . \overline{T'}$ which contain $U . \overline{T'} . \overline{V}$ belong to M. The hypothesis that W does not belong to a common face of M and $\xi S . \overline{T'}$ is untenable.

Thus the primitive cluster $\xi\, S . \overline{T'}$ satisfies the conditions of the theorem; its units can by the inductive hypothesis be added to M by moves of of type 1; and ST' has regular contact with $M + \xi S . \overline{T'}$: for VT' does not belong to M, and U is interior to $M + S . \overline{T}$.

Proof of 39. Let $n = m$. In view of 38 and 42 it is sufficient to shew that the common part of M and Γ is a bounded $(m - 1)$-manifold. It is convenient to prove the slightly more general result that if $M^{\star}$ and $\Gamma^{\star}$ are as M and Γ in 39 save that $M^{\star} + \Gamma^{\star}$ may be $\overline{U} . \overline{V}$, the common part of $M^{\star}$ and $\Gamma^{\star}$ is an $(m-1)$-manifold. Let this be assumed true when $\overline{U}.\overline{V}$ is replaced by the boundary of a primitive cluster of lower dimensions. Let Q be any common component of $\Gamma^{\star}$ and $M^{\star}$, let U' be $P_1 Q_1$, [1]) V be $P_2 Q_2$, Q be $Q_1 Q_2$. Then $U'.\overline{V}$ is $P_1 Q . \overline{P_2} + P_1 Q_1 P_2 . \overline{Q_2}$ so that the Q-cluster in $\Gamma^{\star}$ is contained in $P_1 Q . \overline{P_2}$. Again, if ξ is U/U', the Q-cluster in $M^{\star}$ is contained in $Q. \overline{\xi P_1} . \overline{P_2}$, the Q-cluster in $\overline{U} . \overline{V}$. The shell, Γ', of the Q-cluster in $\Gamma^{\star}$ is then an element drawn on $\xi \overline{P_1} . \overline{P_2}$, consisting of all those units of $P_1 . \overline{P_2}$ which do not belong to M', the shell of the Q-cluster in $M^{\star}$. Hence by the inductive hypothesis the common components of M' and Γ' form an $(h-1)$-manifold, Π_{h-1} (where $h = d(\Gamma') = d(M')$). Since $h < m$, Π_{h-1}, which is drawn on, or identical with, $\overline{\Gamma'}$, is a sphere or an element (42). The common components of $M^{\star}$ and $\Gamma^{\star}$ containing Q form the array $Q\,\Pi_{h-1}$, a complete or incomplete cluster. This establishes the subsidiary theorem, concerning $M^{\star}$ and $\Gamma^{\star}$, and as it is clear that the common array of M and Γ is a proper part of $\overline{\Gamma}$ it must be a bounded $(m-1)$-manifold, and the truth of 39 follows.

Proof of 40. Let $n = m$. Using the theorems which have now been proved, 40 can be enunciated in the symmetrical form:

If $\overline{U} . \overline{V}$ is the sum of two elements, E_1 and E_2, either a face of U is interior to E_1, or a face of V is interior to E_2.

It is therefore sufficient to consider the case in which $d(V) \leqslant d(U)$. The theorem is clearly true if $d(U) = 0$. We therefore suppose U has at least two vertices.

If there were a vertex, ξ, of U which did not belong to E_1, E_1 would be contained in $(U/\xi) . \overline{V}$; and if, in this case, U/ξ were not interior to E_1, there would be a face, V^1, of V, such that $(U/\xi) . V^1$, and therefore V^1 itself, did not belong to E_1. We may therefore assume that if ξ is any vertex of U, ξ belongs to E_1 but is not interior to it.

Let U be ξU^1, and let ξe_1 be the ξ-star in E_1. Since $\xi . \overline{U^1} . \overline{V}$ is the ξ-star in $\overline{U} . \overline{V}$, e_1 is an element drawn on $\overline{U^1} . \overline{V}$. No face of U^1 is interior to e_1; for if it were, a face of U would be interior to ξe_1, i. e. to E_1. Hence by the inductive hypothesis there is at least one face of V, say V^1, which does not belong to e_1. $U^1 V^1$ is the only unit of $\overline{U} . \overline{V}$ which contains V^1 but not ξ, and therefore is the only unit of E_1 containing V^1. Thus if every face of V belonged to E_1, there would correspond to every vertex, ξ, of U, a vertex, η, of V such that $(UV)/(\xi\eta)$ is the only

[1]) On this page suffixes are used (for typographical reasons) as distinguishing marks, not dimension numbers.

unit of E, containing V/η. Clearly, in this correlation, the same vertex of V cannot correspond to two different vertices of U and therefore if $d(V) < d(U)$ the hypothesis that every face of V belongs to E is untenable. If $d(U) = d(V)$ the only possibility to be considered is that E_1 is $\sum\left(\frac{UV}{\xi\eta}\right)$, each vertex of U and V appearing in one term only of the sum. But this array is clearly not an element, for no two units have a common face. The hypothesis that every face of V belongs to E_1 must therefore be abandoned.

Proof of 41. Let U be XY. The trivial or degenerate cases in which X is a vertex, or Y does not belong to M, may be excluded. Y cannot be interior to M, or M would contain the whole of $\overline{U}.\overline{V}$. The Y-cluster in M is then Ye, where e is an element drawn on $\overline{X}.\overline{V}$. If a face, X', of X were interior to e, the $Y.X'$-cluster in M would be the whole of $X'.Y.\overline{V}$. Since, (X being interior to M) the X-cluster in M is $X.\overline{Y}.\overline{V}$ the X'-cluster in M would in this case contain the sum of these two arrays, viz., $X'.\overline{\alpha Y}.\overline{V}$ where α is X/X'. But this is the whole of the X'-cluster in $\overline{U}.\overline{V}$, and cannot, owing to the conditions imposed on X, belong to M. There is therefore no face of X interior to e; by the inductive hypothesis at least one unit of $\overline{V}$ does not belong to e, and therefore at least one unit of $Y.\overline{V}$ does not belong to Ye, the Y-cluster in M.

45. *If $S.\overline{T}$ is drawn on an n-manifold, M, and the components of $S.\overline{T}$ belonging to $\overline{M}$ form a bounded $(n-1)$-manifold, then $M \underset{2}{\rightarrow} M - S.\overline{T}$.* (The proof is similar to that of 38).

From 38 and 45 we obtain the following generalisation of the rules governing regular contact with manifolds, (cf. 26 and 27).

46. *If*

$$C(iia)\quad U.\overline{V}.\overline{W} \text{ is drawn on } \overline{M},$$

$$C(iii)\quad V \text{ does not belong to } M,$$

then

$$M \underset{1}{\rightarrow} M + UV.\overline{W}.$$

Every component of $UV.\overline{W}$ which does not belong to $U.\overline{V}.\overline{W}$ contains or is contained in V, and therefore the components common to M and $UV.\overline{W}$ form the array $U.\overline{V}.\overline{W}$ itself, which is an $(n-1)$-element.

47. *If $UV.\overline{W}$ is drawn on M and*

$$D(ii)\quad U \text{ is interior to } M,$$

$$D(iiia)\quad V.\overline{U}.\overline{W} \text{ belongs to } \overline{M},$$

then

$$M \underset{2}{\rightarrow} M - UV.\overline{W}.$$

(From 45).