Mathematics. — *"On the Motion of a Plane Fixed System with Two Degrees of Freedom"*. By Prof. W. VAN DER WOUDE. (Communicated by Prof. J. C. KLUYVER).

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Literature.

I. FARID BOULAD: Sur la détermination du centre de courbure des trajectoires orthogonales d'une famille quelconque de courbes planes. (Bulletin des Sciences mathématiques, t. XL, 1916, p. 292—295).

II. G. DARBOUX: Remarque sur la note de M. FARID BOULAD. (Bulletin des Sciences mathématiques, t. XL, 1916, p. 295—296).

III. G. KOENIGS: Recherches sur les mouvements plans à deux paramètres. (Bulletin des Sciences mathématiques, t. XLI, 1917, pp. 120—127, 153—164, 181—196).

I. In the first of the above mentioned papers the author gives the theses which we reproduce here in a slightly altered form.

Let [C] be an arbitrary system of curves, depending on one parameter; [T] be the system of the orthogonal trajectories. Let a straight angle XOY move in such a way that the angular point O describes an arbitrary curve and OX and OY continually touch the curve (C), resp. (T), of the systems [C] and [T] which pass through O. In this case a plane system fixed to OXY has a motion with two degrees of freedom and any infinitesimal movement starting from an arbitrary initial position, defines a momentary pole of rotation I.

Now the theorems of FARID BOULAD read: "the locus of these poles I is a straight line d; if in the considered position (C) and (T) are the two curves of which O is the intersection, d joins the centers of curvature of (C) and (T). If further O describes a curve (Φ) which cuts the curves of [C], hence also the curves of [T], under a constant angle, — while, as we supposed, OX and OY continually touch curves of these systems —, I is at the same time the center of curvature of (Φ)".

II. In a note added to the above, the editor G. DARBOUX remarks that the first thesis of FARID BOULAD is a special case of the following one:

"If the motion of a figure in its plane depends on two parameters, to any infinitesimal movement of which the figure is capable, there corresponds a definite pole of rotation; the locus of these poles is a straight line".

If the motion depends on the parameters u and v, it is easily seen — as DARBOUX remarks — that the projections of the displacement of a point (x, y) of the movable plane on axes fixed to this plane, are given by the expressions:

$$\left.\begin{aligned} D_x &= (\xi_1 - \omega_1 y)\, du + (\xi_2 - \omega_2 y)\, dv \\ D_y &= (\eta_1 + \omega_1 x)\, du + (\eta_2 + \omega_2 x)\, dv \end{aligned}\right\} \text{[1]}$$

in which the "translations" and "rotations" ξ, η, ω do not depend on the coordinates x and y.

A point which can be the center of rotation of any of these movements, must satisfy the condition

$$D_y = D_x = 0,$$

hence:

$$\begin{vmatrix} \xi_1 - \omega_1 y & \xi_2 - \omega_2 y \\ \eta_1 + \omega_1 x & \eta_2 + \omega_2 x \end{vmatrix} = 0$$

or

$$\xi_1 \eta_2 - \eta_1 \xi_2 + (\xi_1 \omega_2 - \xi_2 \omega_1)\, x + (\eta_1 \omega_2 - \eta_2 \omega_1)\, y = 0.$$

Consequently the locus of these points is a straight line.

III. The above mentioned papers have led G. KOENIGS to publish his investigations on "les mouvements plans à deux paramètres". It would take too much space if, however briefly, we reproduced these extensive considerations; it is the less necessary as we shall refer to him repeatedly. We therefore only draw the attention here to the remarkable thesis of KOENIGS which we mentioned in § 8.

Our aim is to derive the results of FARID BOULAD and KOENIGS, of the latter at least the main points, and a few more, in an entirely different way, so that for instance it becomes possible to treat entirely according to the general method the cases which require a special discussion of KOENIGS. Moreover we believe that the geometrical meaning of the used formulas appears more clearly in our method.

§ 1. Perhaps it is not quite unnecessary to draw the attention to the fact that by the movements with two degrees of freedom considered by DARBOUX and KOENIGS they always understand "holonomous" movements.

It would be easy to give examples which fall outside these. Let us assume for instance a straight angle with a system fixed to it which moves over a fixed plane; the path of the point O is entirely arbitrary whereas OX continually touches the path of O. Now O may coincide with any point A_1 of the fixed plane; but the position of OX does not only depend on A_1 but also on the path described by O.

[1] These formulas will be derived in § 5.

We shall also exclude such non-holonomous movements; accordingly the position of the system of axes OXY is entirely defined by the point in the fixed plane which is reached by O.

§ 2. We begin by remarking that DARBOUX gives a proof of FARID BOULAD's starting point to which we have no objection, but that he undervalues the significance of this theorem; FARID BOULAD's theorem is as general as the one proved by DARBOUX.

Let Π_f be the fixed plane over which the plane Π_m moves to which the system of axes OXY is fixed; the position of Π_m or of OXY relative to a system of axes in Π_f depends on two parameters. We consider the line element (O, OX) of Π_m defined by the point O and the direction of the X-axis. If Π_m moves over Π_f, O will coincide with any point of Π_f and everywhere the line element (O, OX) will define a direction. In Π_f a singly infinite system of curves $[C]$ is defined in each point A_1 of which the tangent coincides with the direction defined in that point. Let $[T]$ be the system of the orthogonal trajectories. Now the movement of Π_m over Π_f is entirely defined by the condition that OX must continually coincide with the tangent to a curve of $[C]$, hence OY with the tangent to a curve of $[T]$. *Accordingly any plane movement with two degrees of freedom may be defined by fixing to the movable system a system of axes which moves in the way indicated by* FARID BOULAD.

§ 3. Now in the plane Π_f the systems of curves $[C]$ and $[T]$ are given; the movable system of axes has a definite position in which OX touches a curve (C), OY a curve (T).

We begin by accepting the proof given by DARBOUX of the first thesis of FARID BOULAD: the locus of the possible poles of rotation is a straight line d. These possible movements contain:

1. a displacement of O along (C) where OX continually touches (C).
2. a displacement of O along (T) where OY continually touches (T).

It is known that in the former case the pole of rotation coincides with the center of curvature of (C), in the latter case with that of (T). Consequently, as also FARID BOULAD and DARBOUX remark, the line d is the join of these points.

To this we shall add a few remarks. We have chosen OXY arbitrarily in the plane Π_m; if we replace OXY by OX^1Y^1 where $\angle XOX^1 = \varphi$, also the line element (O, OX^1) describes the whole plane Π_f; in the possible movements, the system of curves $[C]$ is replaced by another system $[C^1]$, formed by isogonal trajectories (φ-trajectories) of $[C]$.

Whereas accordingly the two-dimensional system of possible movements which in the future we shall indicate by M_2, is quite defined by the system of curves $[C]$, the reverse, that M_2 quite defines this system of curves, is not true. If we replace $[C]$ by $[C^1]$, M_2 does not change;

but then d, the locus of the poles of rotation for an arbitrary position, has not changed either.

Consequently any point I of d is not only a possible pole of rotation, but also the center of curvature of one of the isogonal trajectories of (C) in O.

This is the last thesis of FARID BOULAD. We may also formulate its geometrical contents in the following way:

The centers of curvature of (C) and (T) form with O the angular points of a right-angled triangle; the hypotenuse also contains the center of curvature of (C^1). If R_1, R_2, R_φ are the radii of curvature of these three curves in O, we have:

$$\frac{1}{R_\varphi}=\frac{\cos\varphi}{R_1}+\frac{\sin\varphi}{R_2} \text{ [1]} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(o, R_1) and $(-R_2, o)$ are the centers of curvature of (C) and (T).

§ 4. *This gives us at once the solution to the problem: to produce two mutually orthogonal systems of curves so that the ratio of the radii of curvature of two curves which cut each other in a given point, is constant, i.e. independent from the chosen point.*

Let m be this constant and $cotg\,\varphi = m$. We consider an arbitrary singly-infinite system of straight lines together with the systems of their orthogonal trajectories — φ- and $(\frac{1}{2}\pi+\varphi)$-trajectories. In any point the following relations between the radii of curvature exist:

$$\frac{1}{R_\varphi}=\frac{\sin\varphi}{R_{1/2\,\pi}}\;;\quad \frac{1}{R_{(1/2\pi+\varphi)}}=\frac{\cos\varphi}{R_{1/2\,\pi}};$$

$$\frac{R_\varphi}{R_{(1/2\pi+\varphi)}}=cotg\,\varphi=m.$$

Accordingly the φ- and the $(\frac{1}{2}\pi+\varphi)$-trajectories give the solution of the problem.

§ 5. We shall now give a geometrical meaning to the quantities which appear in the note of DARBOUX.

If on an arbitrary surface we choose orthogonal parameter curves $v=$ constant and $u=$ constant, and if we make a system of axes $OXYZ$ move in such a way that O describes this surface, OX and OY touch the u- and v-curves and OZ coincides with the normal to the surface,

[1]) Cf. the more general theorem (LIOUVILLE, 1850): if on a surface ρ_u and ρ_v are the geodetical curvatures of the orthogonal systems of curves u and v, we have for a curve which cuts the curves of the u-system (i.e. $v=$ constant) under an angle φ:

$$\frac{1}{\varrho_g}=\frac{d\varphi}{ds}+\frac{\cos\varphi}{\varrho_u}+\frac{\sin\varphi}{\varrho_v}.$$

the movements of the trihedron $OXYZ$, hence also the infinitesimal displacement of a point (x, y, z) fixed to this trihedron, may be expressed by geometrical quantities that are related to the parameter curves. (By the u-curves we always understand the curves $v =$ constant).

If the parameter curves are at the same time lines of curvature, these expressions assume a more simple form.

Let us apply this by assuming the systems $[C]$ and $[T]$ as u- and v-curves; they may be considered as lines of curvature of Π_f. In this case the possible infinitesimal displacements of a point (x, y) are expressed by [1]):

$$\left.\begin{aligned} D_x &= \sqrt{E}\left(1-\frac{y}{R_1}\right)du - \sqrt{G}\,\frac{y}{R_2}\,dv \\ D_y &= \sqrt{E}\,\frac{x}{R_1}\,du + \sqrt{G}\left(1+\frac{x}{R_2}\right)dv \end{aligned}\right\} \quad \ldots \ldots \quad (2)$$

Here du and dv are the increments of the coordinates of the point O in the plane Π_f, D_x and D_y are the projections of the infinitesimal displacement of a point (x, y) on OX and OY; (o, R_1) is the center of curvature of the u-curve ($v =$ constant), $(-R_2, o)$ that of the v-curve in the point O [2]), E and G are the values which the coefficients in the expression for the line element of:

$$ds^2 = E du^2 + G dv^2$$

assume in the point O.

We remark that between E and G there exists a relation, the formula of GAUSZ, which indicates that the total curvature of Π_f is zero:

$$\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right)=0; \quad \ldots \ldots \quad (3)$$

further that R_1 and R_2 are expressed in E and G by

$$\frac{1}{R_1} = -\frac{1}{\sqrt{EG}}\frac{\partial\sqrt{E}}{\partial v};\ \frac{1}{R_2} = \frac{1}{\sqrt{EG}}\frac{\partial\sqrt{G}}{\partial u} \quad \ldots \ldots \quad (4)$$

This enables us to write the formula of GAUSZ in the following form, which will appear to be more suitable for us:

$$\sqrt{G}\frac{\partial}{\partial u}\frac{1}{R_2} - \sqrt{E}\frac{\partial}{\partial v}\frac{1}{R_1} + \sqrt{EG}\left(\frac{1}{R_1^2}+\frac{1}{R_2^2}\right) = 0 \quad \ldots \quad (3^a)$$

From (2) we find as the equation of the line d, the locus of the poles of rotation:

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[1]) G. DARBOUX: Théorie des Surfaces, t. II, p. 398; GAUTHIER-VILLARS, Paris; 1915.

[2]) R_2 *may* therefore differ in sign from the radius of curvature of $u =$ constant, if we wish to consider this radius of curvature as a positive quantity.

$$\left(1-\frac{y}{R_1}\right)\left(1+\frac{x}{R_2}\right)+\frac{xy}{R_1R_2}=0$$

or

$$-\frac{x}{R_2}+\frac{y}{R_1}-1=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

§ 6. We now investigate whether the possible movements always contain an infinitesimal translation. A necessary and sufficient condition is that for a definite ratio of du and dv the expressions for D_x and D_y in (2) must be independent from x and y. This is the case for

$$\frac{\sqrt{E}}{R_1}du+\frac{\sqrt{G}}{R_2}dv=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where the displacement of an arbitrary point is defined by

$$\left.\begin{array}{l} D_x=\sqrt{E}\,du \\ D_y=\sqrt{G}\,dv \end{array}\right\};$$

consequently according to (6)

$$\frac{D_x}{D_y}=-\frac{R_2}{R_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Accordingly in any position the possible movements contain one translation, defined by (6); the pole of rotation is the point at infinity of d; the direction of translation is therefore at right angles to d, which appears at once from (7).

We may give a slightly different form to the equations (6) and (7). If φ is the angle between the tangent to the path of O in Π_f, here at the same time the direction of translation, and the u-curve, and $(\frac{1}{2}\pi-\varphi)$ the angle between this tangent and the v-curve, we have:

$$tg\,\varphi=\frac{\sqrt{G}}{\sqrt{E}};$$

hence we may replace (6) en (7) by:

$$tg\,\varphi=-\frac{R_2}{R_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6^a)$$

$$\frac{D_y}{D_x}=tg\,\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7^a)$$

§ 7. Are there among the movements of the system M_2 *finite* translations along a straight line?

It is known and moreover it is easily seen that the direction of the translation makes a constant angle not only with axes fixed to Π_f but also with the axes OXY fixed to Π_m; the latter, however, coincide with the tangents to the u- and v-curves. Hence for a translation along a straight line φ is constant. From (6^a) it ensues besides that in the points of the path described by O, R_1 and R_2 have a constant ratio.

Inversely: if O moves along a straight line in Π_f which cuts the u-curves under a constant angle φ, the angle between a line in Π_f and OX in Π_m is constant so that the movement is a translation. According to (6^a) in this case the ratio of R_1 and R_2 is at the same time constant.

Whereas a system M_2 generally does not contain any finite translations along a straight line, from what preceeds we may deduce the systems which contain a singly infinite number of these translations.

We assume a system of straight lines depending on one parameter t, and a system of u-curves defined by the condition that any line is to be cut under an angle $\varphi(t)$ which only depends on t and not on the chosen curve. These curves define M_2 in the way indicated before (§ 2); the system of straight lines defines the translations in M_2. In the points of the same straight line the radii of curvature of the u-curve and of the v-curve have a constant ratio.

A special case is that where φ is constant. In this case the u- and the v-curves are the φ- and the $(\frac{1}{2}\pi + \varphi)$-trajectories of the system of straight lines.

§ 8. If O describes the plane Π_f, the line d of which the equation in Π_m is:

$$-\frac{x}{R_2}+\frac{y}{R_1}-1=0$$

moves in Π_f as well as in Π_m. As R_1 and R_2 are as a rule independent from each other, its equation in Π_m depends on 2 parameters; but it envelops a curve if there exists a relation

$$\varphi(R_1, R_2)=0.$$

For this case KOENIGS has proved the following fundamental theorem:

"If the position of d in Π_m depends on one parameter only, this is also the case with its position in Π_f". The two movements, to wit the one where d remains at rest in Π_m and the one where it remains at rest in Π_f, are independent from each other.

In order to prove this theorem we shall first answer the following question.

We consider an arbitrary plane system Π_m which moves over the fixed plane Π_f; the movement depends on one parameter t. A line d moves relative to Π_m; its equation relative to the system of axes OXY fixed to Π_m, is

$$\alpha x+\beta y-1=0 \quad \ldots \ldots \ldots \quad (8)$$

Which conditions must be satisfied by α and β if d is to be at rest in Π_f?

§ 9. If a point $P(x, y)$, that is a point with coordinates (x, y) relative to OXY, is *fixed* — i.e. if it is at rest relative to Π_f —, these coordinates satisfy the conditions:

$$\left.\begin{aligned} V_{a,x} &\equiv \xi - \omega y + \frac{dx}{dt} = 0 \\ V_{x,y} &\equiv \eta + \omega x + \frac{dy}{dt} = 0 \end{aligned}\right\} \quad \ldots \ldots \quad (9)$$

Here, as always, ξ and η are the velocity-components of O and ω is the rotation of the system of axes OXY.

If the line (8) is to be at rest relative to Π_f, there must be an infinite number of pairs of values x and y which satisfy (8) and (9), hence also the equation which is found from (8) through differentiation:

$$x\frac{da}{dt} + y\frac{d\beta}{dt} + a\frac{dx}{dt} + \beta\frac{dy}{dt} = 0 \quad \ldots \ldots \quad (10)$$

By eliminating $\frac{dx}{dt}$ and $\frac{dy}{dt}$ out of this equation and (9), we find:

$$\left(\frac{da}{dt} - \beta\omega\right) x + \left(\frac{d\beta}{dt} + a\omega\right) y - a\xi - \beta\eta = 0 \quad \ldots \quad (11)$$

Now an infinite number of values x and y must satisfy (8) and (11); for this it is necessary and sufficient that:

$$\left\| \begin{matrix} a & \beta & 1 \\ \dfrac{da}{dt} - \beta\omega & \dfrac{d\beta}{dt} + a\omega & a\xi + \beta\eta \end{matrix} \right\| = 0 \quad \ldots \quad (12)$$

This expresses the conditions in question.

We shall now return to the considered system M_2; we choose a movement out of it which depends on one parameter, by considering v as a function of u. If we put:

$$\frac{dv}{du} = \lambda,$$

the conditions that in this movement the line

$$\frac{x}{-R_2} + \frac{y}{R_1} - 1 = 0$$

is at rest relative to Π_f, are expressed by (12) on condition that we apply the following substitutions:

$$\left.\begin{gathered} a = -\frac{1}{R_2} \quad , \quad \beta = \frac{1}{R_1} \\ \xi = \sqrt{E} \quad , \quad \eta = \lambda\sqrt{G} \quad , \quad \omega = \frac{\sqrt{E}}{R_1} + \lambda\frac{\sqrt{G}}{R_2} \\ \frac{d}{dt} = \frac{\partial}{\partial u} + \lambda\frac{\partial}{\partial v} \end{gathered}\right\} \quad \ldots \quad (13)$$

The conditions found above assume the following form:

$$\left.\begin{aligned}\frac{\partial}{\partial u}\frac{1}{R_2}+\lambda\frac{\partial}{\partial v}\frac{1}{R_2}+\sqrt{E}\left(\frac{1}{R_1^2}+\frac{1}{R_2^2}\right)=0\\ \frac{\partial}{\partial u}\frac{1}{R_1}+\lambda\frac{\partial}{\partial v}\frac{1}{R_1}-\lambda\sqrt{G}\left(\frac{1}{R_1^2}+\frac{1}{R_2^2}\right)=0\end{aligned}\right\}$$

By means of (3^a) they may be reduced to

$$\left.\begin{aligned}\lambda\sqrt{G}\frac{\partial}{\partial v}\frac{1}{R_2}+\sqrt{E}\frac{\partial}{\partial v}\frac{1}{R_1}=0\\ \lambda\sqrt{G}\frac{\partial}{\partial u}\frac{1}{R_2}+\sqrt{E}\frac{\partial}{\partial u}\frac{1}{R_1}=0\end{aligned}\right\}\quad\ldots\ldots\quad(14)$$

If always the same value of λ is to satisfy them, in other words if in any position the system M_2 is to contain a movement which lets d be at rest in Π_f, it is necessary and sufficient that:

$$\begin{vmatrix}\frac{\partial}{\partial u}\frac{1}{R_1} & \frac{\partial}{\partial v}\frac{1}{R_1}\\ \frac{\partial}{\partial u}\frac{1}{R_2} & \frac{\partial}{\partial v}\frac{1}{R_2}\end{vmatrix}=0\quad\ldots\ldots,\quad(15)$$

As we found above the condition which must be satisfied if M_2 is always to contain a movement in which d does not move in Π_m, is

$$\varphi(R_1, R_2)=0.\quad\ldots\ldots\ldots\quad(16)$$

But the two conditions (15) and (16) have the same meaning. The first part of the theorem of KOENIGS is therefore proved.

The movement where d is at rest in Π_m, is given by:

$$\frac{1}{R_1}=\text{constant},$$

or

$$\frac{\partial}{\partial u}\frac{1}{R_1}du+\frac{\partial}{\partial v}\frac{1}{R_1}dv=0;$$

the movement which lets d be at rest in Π_f, is expressed by (cf. 14):

$$\sqrt{E}\frac{\partial}{\partial u}\frac{1}{R_1}du+\sqrt{G}\frac{\partial}{\partial u}\frac{1}{R_2}dv=0.$$

These two movements are the same as:

$$\sqrt{E}\frac{\partial}{\partial v}\frac{1}{R_1}-\sqrt{G}\frac{\partial}{\partial u}\frac{1}{R_2}=0,$$

or as (cf. 3^a):

$$\sqrt{EG}\left(\frac{1}{R_1^2}+\frac{1}{R_2^2}\right)=0.$$

If we restrict ourselves to real movements, this is only the case if $\frac{1}{R_1}=\frac{1}{R_2}=0$.

Only if all the movements are translations, the displacements for which d is at rest in Π_m coincide with those for which d is at rest in Π_f ; except in this trivial case they are independent from each other.

In this way also the second part of the theorem in question has been proved.

The only point which is left is to draw the attention to the elegant construction of KOENIGS for these systems M_2 in which the motion of d depends on one parameter. Choose a curve $Ç_f$ in Π_f and a curve $Ç_m$ in Π_m both of which touch d; let alternately d roll over $Ç_f$ and $Ç_m$ over d; the system M_2 is linearly composed of these displacements.

§ 10. We shall return a moment to the quantities of the second order which appear in the movements of a system M_2, which we now again assume arbitrary.

In an initial position chosen at random O has again a definite situation in Π_f , OX touches the u-curve through that point; the tangent to the path described by an arbitrary point of Π_m, only depends on the ratio of du and dv, the increments of the coordinates of O. We can begin by making O describe different paths which all have the same tangent in the initial position; now the corresponding paths of the other points of Π_m also have the same tangents in the initial position but different curvatures.

In the future we shall only consider infinitesimal displacements for which

$$\frac{dv}{du}=\lambda$$

has a given value, whereas

$$\frac{d^2v}{du^2}=\lambda'$$

may assume any value. The pole of rotation I is always the same point.

§ 11. In the movement of a plane fixed system Π_m of which the position is defined by one parameter, which for the sake of more convenient expression we shall for the moment identify with time and indicate by t, the quantities of the first order are defined by the pole of rotation I, those of the second order by a point K^1, the other extreme of the diameter of the inflectional circle through I.

For the velocity- and acceleration-components of a point fixed to Π_m, we have resp.

$$\left.\begin{array}{l} V_x \quad \xi - \omega y \\ V_y = \eta + \omega x \end{array}\right\}$$

and

$$\left.\begin{aligned} J_x &= \frac{d\xi}{dt} - \omega\eta - \frac{d\omega}{dt}\,y - \omega^2 x \\ J_y &= \frac{d\eta}{dt} + \omega\xi + \frac{d\omega}{dt}\,x - \omega^2 y \end{aligned}\right\}^{1)}$$

The equation of the inflectional circle is in this case

$$\begin{vmatrix} V_x & V_y \\ J_x & J_y \end{vmatrix} = 0$$

or

$$\omega^3 (x^2 + y^2) + \left(\frac{d\omega}{dt}\,\xi - \frac{d\xi}{dt}\,\omega + 2\omega^2\eta\right)x + \left(\frac{d\omega}{dt} - \omega\,\frac{d\eta}{dt} - 2\,\omega^2\,\xi\right)y + \ldots = 0.$$

This gives us at once the center of the inflectional circle and at the same time the point K^1, as $I\left(-\frac{\eta}{\omega}, \frac{\xi}{\omega}\right)$ is also known.

We find

$$x_{K^1} = \frac{-\frac{d\omega}{dt}\,\xi + \frac{d\xi}{dt}\,\omega - \omega^2\eta}{\omega^3} \quad ; \quad y_{K^1} = \frac{-\frac{d\omega}{dt}\,\eta + \frac{d\eta}{dt}\,\omega - \omega^2\xi}{\omega^3}.$$

If we return to the case under consideration we must make the following substitutions:

$$\xi = \sqrt{E}\,;\ \eta = \lambda\sqrt{G}\,;\ \omega = \frac{\sqrt{E}}{R_1} + \lambda\,\frac{\sqrt{G}}{R_2}$$

$$\frac{d}{dt} = \frac{\partial}{\partial u} + \lambda\,\frac{\partial}{\partial v}.$$

Then we must consider λ as a constant, λ' as a parameter. In this way we find the locus of K^1 represented by

$$\left(\frac{\sqrt{E}}{R_1} + \lambda\,\frac{\sqrt{G}}{R_2}\right)^3 x = -E\,\frac{d}{du}\frac{1}{R_1} - \lambda\sqrt{EG}\,\frac{d}{du}\frac{1}{R_2} - \lambda\,\frac{\sqrt{E}}{R_2}\frac{d\sqrt{G}}{du} + \\ + \lambda\,\frac{\sqrt{G}}{R_2}\frac{d\sqrt{E}}{du} + \lambda'\,\frac{\sqrt{EG}}{R_2}$$

$$\left(\frac{\sqrt{E}}{R_1} + \lambda\,\frac{\sqrt{G}}{R_2}\right)^3 y = -\lambda\sqrt{EG}\,\frac{d}{du}\frac{1}{R_1} - \lambda^2\,G\,\frac{d}{du}\frac{1}{R_2} + \lambda\,\frac{\sqrt{E}}{R_1}\frac{d\sqrt{E}}{du} - \\ - \lambda\,\frac{\sqrt{G}}{R_1}\frac{d\sqrt{E}}{du} + \lambda'\,\frac{\sqrt{EG}}{R_1}$$

$$\left(\text{where } \frac{d}{du} = \frac{\partial}{\partial u} + \lambda\,\frac{\partial}{\partial v}\right).$$

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[1]) G. KOENIGS: Leçons de Cinématique; p. 142; A. HERMANN, Paris; 1897.

If we eliminate λ' the equation of this locus is

$$\left(\frac{\sqrt{E}}{R_1}+\frac{\sqrt{G}}{R_2}\right)^2\left(\frac{x}{R_1}+\frac{y}{R_2}\right)+\sqrt{E}\frac{d}{du}\frac{1}{R_1}+\sqrt{G}\frac{d}{du}\frac{1}{R_2}=0.$$

This represents a line h perpendicular to d; its intersection with d we call H.

§ 12. If now we consider λ as a variable, I and H move along d; the correspondence between I and H has been elaborately treated by KOENIGS. We shall not enter into it but we shall only point out the following properties.

Again we suppose λ to be a constant; I and H are fixed points of d and h is the locus of K^1. As IK^1 is always a diameter of the inflectional circle we have:

for all the displacements with the same pole of rotation the inflectional circles form a pencil with I and H as base points; in each of these displacements H describes a point of inflection in which the tangent coincides with h.

For each of these displacements on d the pole of rotation as well as the point of intersection with the inflectional circle are fixed; hence:

In all these displacements all points of d have the same radius of curvature.