Mathematics. — *"A Representation of the Bisecants of a Rational Twisted Curve on a Field of Points."* By Prof. JAN DE VRIES.

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1. The points A of a rational twisted curve α^n may be brought into a projective correspondence with the points C of a conic γ^2. If C_1 and C_2 correspond to A_1 and A_2 and if B is the point of intersection of the lines c_1 and c_2 which touch γ^2 at C_1 and C_2, B may be considered as the *image* of the bisecant b which joins A_1 to A_2.

Now the point range (B) on c_1 is the image of the *cone* $(b)^{n-1}$ which projects α^n out of A_1 and C_1 is the image of the *tangent* at A_1.

The scroll of the bisecants which rest on a line l, has $(n-1)$ bisecants in common with the cone $(b)^{n-1}$; hence its image cuts the tangent c in $(n-1)$ points B and is a curve λ^{n-1}.

The curves λ^{n-1} and μ^{n-1} have $(n-1)^2$ points in common; accordingly on the lines l and m there rest $(n-1)^2$ bisecants and the scroll which is represented on λ^{n-1}, is of the degree $(n-1)^2$. As, evidently, a plane through l contains $\frac{1}{2}n(n-1)$ chords, l is a multiple directrix of which any point carries $\frac{1}{2}(n-1)(n-2)$ chords.

If l cuts the curve α^n in A, the said scroll consists of the cone $(b)^{n-1}$, that has A as vertex, and a scroll of the degree $(n-1)(n-2)$, etc.

2. We shall now consider the case $n=4$ more closely.

To l as directrix of bisecants b of an α^4 there corresponds a scroll Λ^9 represented on a λ^3. As λ^3 contains six points of γ^2, the *tangents* to α^4 form a scroll of the sixth order.

The curves λ^3 form a system ∞^4. Through four points B_k chosen at random there pass *two* curves, for the corresponding bisecants b_k have two transversals l.

To a secant s with point of intersection A there corresponds the system of a scroll Σ^6 and the cone $(b)^3$. Now the curve λ^3 consists of a conic σ^2 and a tangent c to γ^2. The curves σ^2 form a system ∞^3.

The hyperboloid through three bisecants has 2 more points in common with α^4; through each of these points there passes a secant s, which rests on the three lines b. Consequently three points B generally define *two* conics σ^2.

The chords b_1, b_2, which are represented in B_1, B_2, are cut by one chord d. For b_2 cuts the scroll $\triangle_1{}^3$ corresponding to b_1 in one point outside α^4 and through this point there passes one generatrix of $\triangle_1{}^3$, which is a chord resting on b_1 and b_2. Hence *any* straight line δ represents a *scroll* $\triangle^3$.

3. For a trisecant t the scroll Λ^9 consists of three cones which have the points of intersection A_k of t with α^4 as vertices.

The line t is represented in the angular points B_k of a triangle circumscribed about γ^2. As each point of α^4 carries only one trisecant, the corresponding points C_k on γ^2 form a cubic involution and the same is the case with the tangents c_k. Accordingly the trisecants are represented on a conic β^2, the "*involution curve*" of the second cubic involution.

The points of intersection of β^2 with a curve λ^3 form two triplets of the cubic involution of the image points B_k. Hence two trisecants rest on l and the lines t form a *quadratic scroll.*

The points of intersection of β^2 and γ^3 are the images of four trisecants which touch α^4.

Any conic σ^2 is an *involution curve*; for the planes through a secant s define a cubic involution on α^4, so that the images B of the chords in one of these planes form a group of an I^3 on σ^2. Evidently the conic β^2 belongs to the system of the σ^2; but it belongs to ∞^1 secants (these form the second scroll of the hyperboloid of the trisecants).

Also the curves λ^3 are involution curves, for any plane through l contains six chords and these are represented in the angular points of a quadrangle circumscribed about γ^2.

4. Through any point of the chord d there pass two chords b_1, b_2; the planes $b_1 b_2$ pass through the single directrix e of $\triangle^3$.

The pairs of points B_1, B_2 form an involution I^2 on the image line δ; its double points are the images of the torsal lines k of $\triangle^3$. These are bisecants which join the points of contacts of two intersecting tangents and which, therefore, lie in a double tangent plane to α^4. Hence the directrix e is the intersection of two double tangent planes.

To any line e there corresponds a definite bisecant d; it rests on the *tangent chords* k of the two double tangent planes.

For a bisecant k $\triangle^3$ is a scroll of CAYLEY; for in this case k is a directrix and at the same time a generatrix, hence a directrix e.

5. The image δ of a $\triangle^3$ contains the images K of two chords k; consequently the system of the tangent chords is represented on a conic k^2. As this has six points in common with a λ^3, the *tangent chords* form a scroll of the *sixth* degree.

To the image δ of a $\triangle^3$ there corresponds the image D of the chord d on which the bisecants b of $\triangle^3$ rest. Evidently D and δ correspond to each other in a polar correspondence. The image K of a chord k lies on the line δ^* corresponding to k; hence D and δ are polar relative to the conic k^2 and two points D_1 and D_2 harmonically separated by k^2, are the images of two intersecting bisecants.

A polar triangle corresponds to three bisecants which come together in a point outside α^4.

6. Two point triplets chosen at random on α^4, define a cubic involution I^3. Any group of the I^3 defines three straight lines b of a scroll

(b), the *involution scroll*. This is represented on an involution conic $(B)^2$ of γ^2. The conics $(B)^2$ form a system ∞^4; the bisecants which are represented in four arbitrary points B_k, define two I^3 on α^4; accordingly through four points there pass *two* conics $(B)^2$.

$(B)^2$ has six points in common with a λ^3: hence the involution scroll is a $(b)^6$.

Through the angular points of a triangle circumscribed about γ^2 and two arbitrary points B, there passes only one $(B)^2$; for an I^3 is quite defined by a triplet and two pairs.

Two triangles circumscribed about γ^2 also define only one $(B)^2$, hence one I^3 on α^4.

7. The involution scroll of an I^4 is represented on a curve $(B)^3$ which is circumscribed about ∞^1 quadrangles formed by tangents to γ^2. As $(B)^3$ has nine points in common with λ^3 the scroll (b) has the degree 9.

As two groups of the I^4 are equivalent to 6 pairs, the curves $(B)^3$ form a system ∞^6. Through the angular points of two quadrangles circumscribed about γ^2, there passes one $(B)^3$; also through the angular points of three arbitrary triangles circumscribed about γ^2.

8. Let φ^2 be an arbitrary conic in the image plane. The tangents c through its points to γ^2 define on this curve an involutory correspondence (2,2), hence also a (2,2) between the points of α^4. The bisecants through associated points form a scroll $(b)^6$, for φ^2 has six points in common with a curve λ^3.

An involutory (2,2) is defined by 5 pairs; consequently the conics φ^2 form a system ∞^5.

The bisecants resting on a conic ε^2 through four points E_k of α define a (2,2) on α^4. For the cone $(b)^3$ which has a point A of α^4 as vertex, cuts ε^2 outside E_k in two more points and therefore contains two chords which rest on ε^2.

The scroll $(b)^6$ of the bisecants resting on ε^2, has α^4 as double curve and ε^2 as triple curve.