Mathematics. — *"On the Geometry of the Group-manifold of simple and semi-simple groups".* By E. CARTAN and J. A. SCHOUTEN. (Communicated by Prof. JAN DE VRIES).

(Communicated at the meeting of March 27, 1926).

If the ∞^r transformations of a continuous group with r parameters are represented by points of an X_r, then the group-manifold (variété du groupe, Gruppenmannigfaltigkeit) arises. The parameters play the part of coordinates of the X_r. In the present paper we will prove that there exist in X_r three connexions (connexions, Uebertragungen) two of them being not symmetrical but having zero curvature, the third being symmetrical (à torsion nulle) and in the case of a simple or a semi-simple group Riemannian. The Riemannian connexions, formed in the above manner, are of a very particular type, having, as far as we know, not yet been examined except the case $r = 3$, where the curvature becomes constant. The main feature of these geometries is the possession of two absolute parallelisms, who in the case $r = 3$ pass into the well known parallelisms of CLIFFORD. In a subsequent article we will prove that this property is characteristic for the geometries of semi-simple groups, but for one exception, which is in close connexion with the non-associative numbersystem with 8 unities of GRAVES-CAYLEY.

§ 1. *The first displacement with zero curvature* (—).

The transformations of the group being given by the equations

$$'x^k = x^k\,(x^1, \ldots, x^n, \xi^{a_1}, \ldots, \xi^{a_r}); \; k = 1, \ldots, n, \quad . \quad . \quad . \quad (1)$$

the ξ^ν are coordinates in the group-manifold X_r.

T_ξ corresponding with ξ^ν and $T_{\xi+d\xi}$ with $\xi^\nu + d\xi^\nu$, the linear element $d\xi^\nu$ corresponds with the infinitesimal transformation

$$T_{\xi+d\xi}\, T_\xi^{-1} \; ^{1})$$

This infinitesimal transformation has not only significance in the point ξ^ν. In any other point η^ν it corresponds with the linear element, extending from T_ξ to $T_{\eta+d\xi}\, T_\xi^{-1}\, T_\eta$. By this correspondence, which may be characterized by the equation

$$T_{\xi+d\xi}\, T_\xi^{-1} = T_{\eta+d\eta}\, T_\eta^{-1} \; ^{2}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

[1]) This means that T_ξ^{-1} has to be executed first and then $T_{\xi+d\xi}$.

[2]) See note page 807.

a connexion in X_r is defined, and this connexion is integrable, since because of (2) with a definite linear element in any point corresponds one and only one linear element in any point of the neighbourhood. If this connexion, which we will represent by (—), were symmetrical, it would be possible to lay in X_r a Cartesian system of coordinates and X_r would be an E_r. We will, however, see that this is not the case in general.

T_u being the general transformation of a subgroup with one parameter, the equation

$$T_\xi = T_u \, T_{\xi_0} \quad \ldots \ldots \ldots \quad (3)$$

represents a geodesic through $\underset{0}{\xi^\nu}$. Writing (2) in the form

$$T_{\xi'} \, T_\xi^{-1} = T_{\eta'} \, T_\eta^{-1}, \quad \ldots \ldots \ldots \quad (4)$$

this equation defines the (—)-aequipollence of two finite segments of geodesics.

Besides the system of coordinates ξ^ν we introduce in every point of X_r a system of coordinates ε^k, geodesic in that point and we place these systems in such manner that two elements with the same coordinates $d\varepsilon^k$ in different points arise from one another by a (—)-parallel displacement and correspond consequently with the same infinitesimal transformation. Then in every point

$$\frac{\partial' x^k}{\partial \xi^\lambda} = \frac{\partial' x^k}{\partial \varepsilon^j} \frac{\partial \varepsilon^j}{\partial \xi^\lambda} \,{}^{1)} \quad \ldots \ldots \ldots \quad (5)$$

holds good. Now $\frac{\partial' x^k}{\partial \varepsilon^j}$ depends only on $'x^k$ and is independent of the ξ^ν because $d\varepsilon^k$ represents in every point the same infinitesimal transformation. $\frac{\partial \varepsilon^j}{\partial \xi^\lambda}$ depends only on the relative situation of the systems ξ^ν and ε^k, i.e. therefore on ξ^ν and is independent of the $'x^k$. Writing for these functions

$$\frac{\partial' x^k}{\partial \varepsilon^j} = \xi^k_j \, ('x) \quad ; \quad \frac{\partial \varepsilon^j}{\partial \xi^\lambda} = \psi^j_\lambda \, (\xi)$$

we obtain

$$\frac{\partial' x^k}{\partial \xi^\lambda} = \xi^k_j \, \psi^j_\lambda \quad \ldots \ldots \ldots \quad (6)$$

ψ^j_λ being nothing else but the affinor of unity, with regard to ε^k above and to ξ^λ beneath, the rank of ψ^j_λ is r and the equation (5) admits therefore an inversion of the form

$$\xi^k_i = a^\lambda_i \, \frac{\partial' x^k}{\partial \xi^\lambda} \quad \ldots \ldots \ldots \quad (7)$$

So we have obtained the first part of the first fundamental theorem

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[1]) We omit all signs Σ with regard to two differently placed indices.

of LIE [1]) and we see that this theorem is in close connexion with the displacement (—).

Writing $e^{\nu}dt$ for $d\xi^{\nu}$ and consequently $e^{k}dt$ for $d\varepsilon^{k}$, e^{ν} is a (—)-constant vector in X_r, having consequently constant coordinates with regard to the systems ε^{k}. (5) gives, that $d\varepsilon^{k}$ corresponds with the infinitesimal transformation

$$d\,'x^{k} = \xi^{k}_{j}\, e^{j}\, dt$$

or

$$df = \xi^{k}_{j}\, e^{j} \frac{\partial f}{\partial' x^{k}} = e^{j}\, 'X_{j} f\, dt \quad ; \quad 'X_{j} f = \xi^{k}_{j}\, ('x) \frac{\partial f}{\partial' x^{\nu}}.$$

With e^{k} corresponds therefore the infinitesimal transformation $e^{j} X_{j} f$ or $e^{\mu} X_{\mu} f$ in x^{k}, where

$$X_{j} f = \xi^{k}_{j}\, (x) \frac{\partial f}{\partial x^{k}} \quad ; \quad X_{\mu} f = \psi^{j}_{\mu}\, \xi^{k}_{j}\, (x) \frac{\partial f}{\partial x^{k}}. \quad . \quad . \quad . \quad . \quad (8)$$

If $\underset{1}{v}^{\nu}$ and $\underset{2}{v}^{\nu}$ are two vectors which are (—)-constant and correspond therefore with two infinitesimal transformations

$$\delta^{-} \underset{1}{v}^{\nu} = 0 \quad ; \quad d\underset{1}{v}^{\nu} = - \overline{\Gamma}^{\nu}_{\lambda\mu}\, \underset{1}{v}^{\lambda}\, d\xi^{\mu}\, ; \; d\underset{1}{v}^{k} = 0,$$

$$\delta^{-} \underset{2}{v}^{\nu} = 0 \quad ; \quad d\underset{2}{v}^{\nu} = - \overline{\Gamma}^{\nu}_{\lambda\mu}\, \underset{2}{v}^{\lambda}\, d\xi^{\mu}\, ; \; d\underset{2}{v}^{k} = 0,$$

then we have

$$\underset{1}{df} = \underset{1}{v}^{j}\, X_{j} f \quad ; \quad \underset{2}{df} = \underset{2}{v}^{j}\, X_{j} f.$$

Now it is well known that

$$\underset{1}{v}^{i}\, \underset{2}{v}^{j}\, X_{i}\, X_{j} f - \underset{1}{v}^{i}\, \underset{2}{v}^{j}\, X_{j}\, X_{i} f = \underset{1}{v}^{i}\, \underset{2}{v}^{j}\, c_{ij}^{\cdot\cdot k}\, X_{k} f \quad . \quad . \quad . \quad (9)$$

where $c_{ij}^{\cdot\cdot k} = - c_{ji}^{\cdot\cdot k}$ are the $^1/_2\, r^2 (r-1)$ constants of LIE, which determine the structure of the group. The corresponding linear element is the difference of

$$\underset{2}{v}^{\nu} + \underset{1}{v}^{\nu} + d\, \underset{2\,1}{v}^{\nu} = \underset{2}{v}^{\nu} + \underset{1}{v}^{\nu} - \overline{\Gamma}^{\nu}_{\lambda\mu}\, \underset{1}{v}^{\lambda}\, \underset{2}{v}^{\mu}$$

and

$$\underset{1}{v}^{\nu} + \underset{2}{v}^{\nu} + d\, \underset{1\,2}{v}^{\nu} = \underset{1}{v}^{\nu} + \underset{2}{v}^{\nu} - \overline{\Gamma}^{\nu}_{\lambda\mu}\, \underset{2}{v}^{\lambda}\, \underset{1}{v}^{\mu}$$

so that

$$\underset{1}{v}^{i}\, \underset{2}{v}^{j}\, c_{ij}^{\cdot\cdot k} = - \underset{1}{v}^{\lambda}\, \underset{2}{v}^{\mu} \left(\overline{\Gamma}^{\nu}_{\lambda\mu} - \overline{\Gamma}^{\nu}_{\lambda\mu} \right) \psi^{k}_{\nu}$$

or

$$S_{\lambda\mu}^{\cdot\cdot\nu} = \overline{\Gamma}^{\nu}_{[\lambda\mu]} = - {}^1/_2\, c_{\lambda\mu}^{\cdot\cdot\nu} = - {}^1/_2\, A^{ij\nu}_{\lambda\mu k}\, c_{ij}^{\cdot\cdot k} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

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[1]) Comp. f.i. LIE-SCHEFFERS, p. 376. We have used ξ and α instead of A^{j}_{λ} and A^{λ}_{i} to make comparison more easy.

$S_{\lambda\mu}^{\cdot\cdot\nu}$ is the quantity which makes the connexion asymmetrical. The connexion (—) is defined by the equation

$$\delta v^\nu = dv^\nu + \overset{0}{\Gamma}{}_{\lambda\mu}^{\nu} v^\lambda d\xi^\mu + S_{\lambda\mu}^{\cdot\cdot\nu} v^\lambda d\xi^\mu \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The $c_{ij}^{\cdot\cdot k}$ being constants, $S_{\lambda\mu}^{\cdot\cdot\nu}$ is a (—)-constant quantity

$$\overline{\nabla}_\omega S_{\lambda\mu}^{\cdot\cdot\nu} = 0 \quad ; \quad \delta^- S_{ij}^{\cdot\cdot k} = dS_{ij}^{\cdot\cdot k} = 0 \quad . \quad . \quad . \quad . \quad (12)$$

$S_{\lambda\mu}^{\cdot\cdot\nu}$ is only then equal to zero, if all the expressions $(X_i X_j)$ vanish.

§ 2. *The second connexion with zero curvature* (+).

A second connexion may be defined by means of the equation

$$T_\xi^{-1}\, T_{\xi+d\xi} = T_\eta^{-1}\, T_{\eta+d\eta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

This connexion being equally integrable, the curvature is here zero as well. A geodesic passing through $\overset{}{\underset{0}{\xi^\nu}}$ is given by the equation

$$T_\xi = T_{\xi_0}\, T_u \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and is at the same time a geodesic of the (—)-connexion, since

$$T_{\xi_0}\, T_u = (T_{\xi_0}\, T_u\, T_{\xi_0}^{-1})\, T_{\xi_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and $T_{\xi_0} T_u T_{\xi_0}^{-1}$ represents also a group with one parameter. Writing (13) in a finite form

$$T_\xi^{-1}\, T_{\xi'} = T_\eta^{-1}\, T_{\eta'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

it follows that the segments $\xi\xi'$ and $\eta\eta'$ are (+)-aequipollent. We will deduce this second connexion in another way and will find a formula corresponding with (11).

Introducing in the equation

$$'x = Tx$$

new variables on both sides,

$$'x' = S\,'x \quad ; \quad x' = S\,x$$

we have

$$'x' = S\,T\,S^{-1}\,x'.$$

S and T being transformations of the group, applying the transformation T on x is therefore the same as applying STS^{-1} on Sx. If we take S and T both infinitesimal

$$Sx^k = x^k + s^j X_j\, x^k\, dt + \ldots$$

$$Tx^k = x^k + t^j X_j\, x^k\, dt + \ldots$$

in which s^j and t^j are (—)-constant vectors, the transition from T into STS^{-1} corresponds with the linear element $s^j dt$ or, because

$$S^{-1} = x^k - s^j X_j x^k dt + \dots$$
$$TS^{-1} x^k = x^k - s^j X_j x^k dt + t^j X_j x^k dt - t^i X_i s^j X_j x^k dt^2 + \dots$$
$$STS^{-1} x^k = x^k + t^j X_j x^k dt - t^i s^j (X_i X_j - X_j X_i) x^k dt^2 + \dots$$

with the transition from t^k into $t^k + s^i t^j c_{ij}^{\cdot\cdot k} dt$. So we obtain a new connexion

$$\delta^+ t^k = - s^i t^j c_{ij}^{\cdot\cdot k} dt = t^i s^j c_{ij}^{\cdot\cdot k} dt = \delta^- t^k + t^i s^j c_{ij}^{\cdot\cdot k} dt \quad . \quad (17)$$

or in a more general form

$$\left.\begin{aligned} \delta^+ v^\nu &= \delta^- v^\nu + c_{\lambda\mu}^{\cdot\cdot\nu} v^\lambda d\xi^\mu \\ &= dv^\nu + \overset{0}{\Gamma}{}_{\lambda\mu}^{\nu} v^\lambda d\xi^\mu + {}^1/_2\, c_{\lambda\mu}^{\cdot\cdot\nu} v^\lambda d\xi^\mu \\ &= dv^\nu + \overset{0}{\Gamma}{}_{\lambda\mu}^{\nu} v^\lambda d\xi^\mu - S_{\lambda\mu}^{\cdot\cdot\nu} v^\lambda d\xi^\mu \end{aligned}\right\} \quad . \quad . \quad . \quad (18)$$

By the (+)-connexion the infinitesimal transformations of the group undergo a transformation and these transformations form the *adjoint group*. Both connexions correspond therefore exactly with the two fundamental relations between the infinitesimal transformations which occur in the classical theory [1]).

The quantity $S_{\lambda\mu}^{\cdot\cdot\nu}$ is constant by the (+)-connexion also. Indeed we have

$$\left.\begin{aligned} \overset{+}{\nabla}_\omega S_{\lambda\mu}^{\cdot\cdot\nu} &= \overline{\nabla}_\omega S_{\lambda\mu}^{\cdot\cdot\nu} - c_{\lambda\omega}^{\cdot\cdot\alpha} S_{\alpha\mu}^{\cdot\cdot\nu} - c_{\mu\omega}^{\cdot\cdot\alpha} S_{\lambda\alpha}^{\cdot\cdot\nu} + c_{\alpha\omega}^{\cdot\cdot\nu} S_{\lambda\mu}^{\cdot\cdot\alpha} \\ &= 0 + \tfrac{2}{3}\, c_{[\omega\lambda}^{\cdot\cdot\alpha} c_{\mu]\alpha}^{\cdot\cdot\nu} = 0 \end{aligned}\right\} \quad . \quad . \quad (19)$$

since the JACOBIAN identity gives:

$$c_{[ij}^{\cdot\cdot m} c_{l]m}^{\cdot\cdot k} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

The integrability of the (+)-connexion follows also by applying the general formula for the transformation of the quantity of curvature:

$$'R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} - 2 \nabla_{[\omega} A_{|\lambda|\mu]}^{\cdot\cdot\nu} + 2\Gamma_{[\omega\mu]}^{\alpha} A_{\lambda\alpha}^{\cdot\cdot\nu} - 2 A_{\alpha[\omega}^{\cdot\cdot\nu} A_{|\lambda|\mu]}^{\cdot\cdot\alpha}, \quad . \quad (21)$$

valid by the transformation of parameters

$$'\Gamma_{\lambda\mu}^{\nu} = \Gamma_{\lambda\mu}^{\nu} + A_{\lambda\mu}^{\cdot\cdot\nu}.$$

§ 3. *The symmetrical displacement* (0).

$S_{\lambda\mu}^{\cdot\cdot\nu}$ being an affinor in X_r, the equation

$$\delta^0 v^\nu = dv^\nu + \overset{0}{\Gamma}{}_{\lambda\mu}^{\nu} v^\lambda d\xi^\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

defines a third displacement (0), which is symmetrical.

$S_{\lambda\mu}^{\cdot\cdot\nu}$ is also constant by (0):

$$\overset{0}{\nabla}_\omega S_{\lambda\mu}^{\cdot\cdot\nu} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

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[1]) The two expressions (2) and (13) occur for the first time side by side in CARTANs paper Bull. d. Sc. Math. 34. 1910, page 250—283.

in consequence of the JACOBIAN identity. By applying (21), we have for the quantity of curvature

$$\left.\begin{aligned} \overset{0}{R}{}_{\omega\mu\lambda}^{\cdot\cdot\cdot\,\nu} &= -2\,S_{\omega\mu}^{\cdot\cdot\,\alpha}\,S_{\lambda\alpha}^{\cdot\cdot\,\nu} - 2\,S_{\alpha[\omega}^{\cdot\cdot\,\nu}\,S_{|\lambda|\mu]}^{\cdot\cdot\,\alpha} \\ &= -2\,S_{\omega\mu}^{\cdot\cdot\,\alpha}\,S_{\lambda\alpha}^{\cdot\cdot\,\nu} - S_{\lambda\mu}^{\cdot\cdot\,\alpha}\,S_{\alpha\omega}^{\cdot\cdot\,\nu} + S_{\lambda\omega}^{\cdot\cdot\,\alpha}\,S_{\alpha\mu}^{\cdot\cdot\,\nu} \\ &= \quad S_{\omega\mu}^{\cdot\cdot\,\alpha}\,S_{\alpha\lambda}^{\cdot\cdot\,\nu} \end{aligned}\right\} \quad . \quad . \quad (25)$$

from which it follows that $\overset{0}{R}{}_{\omega\mu\lambda}^{\cdot\cdot\cdot\,\nu}$ is constant for all the three connexions.

The part of the parameters that is symmetrical in $\lambda\mu$ being equal for the three connexions, it follows that they have the same geodesics. The geometrical link between the three connexions is as follows. If $\underset{1}{d}\xi^\nu$ and $\underset{2}{d}\xi^\nu$ are two linear elements in ξ^ν, and if $\underset{1}{d^0}\xi^\nu$, $\underset{1}{d^+}\xi^\nu$ and $\underset{1}{d^-}\xi^\nu$ arise by (0)-, (+)- and (—)-displacement of $\underset{1}{d}\xi^\nu$ along $\underset{2}{d}\xi^\nu$ and $\underset{2}{d^0}\xi^\nu$, $\underset{2}{d^+}\xi^\nu$ and $\underset{2}{d^-}\xi^\nu$ by the analogous displacement of $\underset{2}{d}\xi^\nu$ along $\underset{1}{d}\xi^\nu$, then $\underset{1}{d^0}\xi^\nu$ and $\underset{2}{d^0}\xi^\nu$ form together with $\underset{1}{d}\xi^\nu$ and $\underset{2}{d}\xi^\nu$ a closed figure, as well as $\underset{1}{d^+}\xi^\nu$ and $\underset{2}{d^-}\xi^\nu$ and $\underset{1}{d^-}\xi^\nu$ and $\underset{2}{d^+}\xi^\nu$, while among the three points opposite to ξ^ν, the point belonging to $\underset{1}{d^0}\xi^\nu$ and $\underset{2}{d^0}\xi^\nu$ lies in the midst between the other two. If $f^{\lambda\mu}\,d\sigma$ is an infinitesimal bivector, and if we call generally $f^{\lambda\mu}\,d\sigma\,\Gamma_{[\lambda\mu]}^{\nu}$ the corresponding torsion, it follows that the (+)-torsion and the (—)-torsion are equal and of opposite sign, the (0)-torsion being zero. If the bivector is formed by the two infinitesimal transformations X_i and X_k, then the torsion has the direction of $(X_i\,X_k)$.

§ 4. *The adjoint group.*

If we write always e^k for a (—)-constant vector, which corresponds therefore with an infinitesimal transformation $e^k X_k$, the (+)-differential is according to (18) given by the equation

$$\delta^+ e^k = e^i\, c_{ij}^{\cdot\cdot\,k}\, d\xi^j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Any function f of e^k undergoes by an infinitesimal transformation of the adjoint group a change

$$df = \frac{\partial f}{\partial e^k}\, e^i\, c_{ij}^{\cdot\cdot\,k}\, d\xi^j\,,$$

so that we get for the symbol of LIE of the infinitesimal transformations the well known equation

$$E_j\, f = e^i\, c_{ij}^{\cdot\cdot\,k}\, \frac{\partial f}{\partial e^k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

If we call *constant quantities* such quantities, which are constant by (+), (—) and (0), then the following propositions hold good:

A constant simple co- or contravariant p-vector determines a $(n-p)$-fold, resp. p-fold invariant subgroup of the adjoint group.

The system of equations, obtained by combining in an invariant way a constant quantity with one or more vectors e^k and putting this combination equal to zero, is invariant by the transformations of the adjoint group. If the system is linear in e^k, it defines an invariant subgroup of the adjoint group, which may be however zero.

Examples:

$$v_{i_1 \ldots i_p}^{\ldots\ldots k_1 \ldots k_q} \, e^{i_1} \ldots e^{i_s} = 0 \quad ; \quad s \leqq p$$

$$v_{i_1 \ldots i_p}^{\ldots\ldots [k_1 [k_2 \mid k_3 \ldots k_q \mid} e^{l_1]} e^{l_2]} = 0.$$

Constant quantities are among others $c_{ij}^{\cdot\cdot k}$ and $\overset{0}{R}{}_{lij}^{\cdot\cdot\cdot k}$ and all their co-variants. The most important ones are

$$\overset{0}{R}_{ij} = \overset{0}{R}{}_{lij}^{\cdot\cdot\cdot l} \quad ; \quad R_{[ij]} \quad ; \quad R_{(ij)} \tag{28}$$

$$g_i = {}^1/_2 \, c_{ia}^{\cdot\cdot a}$$

$$c \, g_{ij} = {}^1/_4 \, c_{ai}^{\cdot\cdot b} \, c_{bj}^{\cdot\cdot a} \, ; \; c = \text{constant} \tag{29}$$

$$g_{ijk} = - \, {}^1/_8 \, c_{ai}^{\cdot\cdot b} \, c_{bj}^{\cdot\cdot c} \, c_{ck}^{\cdot\cdot a} \; {}^{1)}$$

All quantities (29) admit cyclical permutation of the suffixes:

$$g_{ijk} = g_{jki} = g_{kij} \, .$$

Important invariant subgroups are:

a. the group of g_i

$$e^j g_j = 0.$$

b. the central group, containing all infinitesimal transformations whose alternated combinations (Klammerausdruck, crochet) with every other vanish

$$e^i c_{ij}^{\cdot\cdot k} = 0.$$

c. the derived group, containing all infinitesimal transformations, which may be written as alternated combination of two others

$$c_{ij}^{\cdot\cdot [k} e^{l]} = 0.$$

d. the group of g_{ij}

$$e^i g_{ij} = 0.$$

The tensor g_{ij} has remarkable properties. In the first place it follows from (25)

$$R_{ij} = c \, g_{ij} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \tag{30}$$

R_{ij} is therefore symmetrical, i. c. the connexion (0) is equiaffine (inhaltstreu). The connexions (+) and (—) are integrable and have therefore the same property. HURWITZ [2]) has used the equiaffinity of (—) to

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1) The constants c, ${}^1/_4$, etc. are introduced with regard to calculations to be executed further on.

2) Göttinger Nachrichten 1897 p. 71—90.

form volume-integrals in the group-manifold of an arbitrary finite continuous group. If we start from an unit of volume in ξ^ν and denote by $d\tau^0$, $d\tau^+$ and $d\tau^-$ the volumes of a space-element, measured with these unit, then generally $d\tau^0$, $d\tau^+$ and $d\tau^-$ are unequal. Moreover, $d\tau^+ = d\tau^-$ in the E_{r-1} of the invariant subgroup of g_i passing by ξ^ν, whilst always $d\tau^0 = \sqrt{d\tau^+ d\tau^-}$. Only in the case $g_i = 0$ the measured volumina get equal in every point.

Applying the JACOBIAN identity, we have

$$\left.\begin{aligned} S_{ij}^{\cdot\cdot l}\, g_{lk} &= {}^1/_c\, S_{ij}^{\cdot\cdot l}\, S_{lm}^{\cdot\cdot n}\, S_{kn}^{\cdot\cdot m} \\ &= -\frac{1}{c}\, S_{jm}^{\cdot\cdot l}\, S_{li}^{\cdot\cdot n}\, S_{kn}^{\cdot\cdot m} \\ &\quad -\frac{1}{c}\, S_{mi}^{\cdot\cdot l}\, S_{lj}^{\cdot\cdot n}\, S_{kn}^{\cdot\cdot m} = \\ &= -\frac{1}{c}\, g_{jik} + \frac{1}{c}\, g_{ijk} = -\frac{1}{c}\, g_{ikj} + \frac{1}{c}\, g_{ijk} \end{aligned}\right\} \quad . \quad . \quad (31)$$

whence it follows that $S_{ij}^{\cdot\cdot l}\, g_{lk}$ is alternating in jk and therefore is a trivector.

§ 5. *Semi-simple and simple groups.*

A simple group is a group, which does not possess invariant subgroups. A group is called semi-simple, if it possesses no integrable invariant subgroups. CARTAN has proved [1]) that a group is then and only then semi-simple, if the tensor $g_{\lambda\mu}$ has the rank r. There exists therefore in that case a tensor of rank r, which is (—)-constant and it follows that for a semi-simple group the connexion (0) is a Riemannian one with $g_{\lambda\mu}$ as fundamental tensor [2]). For the constant c it follows from (30):

$$c = \frac{1}{n} K = -(n-1) K_0 \quad ; \quad K = K_{\alpha\beta}^{\cdot\cdot\beta\alpha}; \quad K_{\omega\omega\lambda}^{\cdot\cdot\cdot\nu} = R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} \quad . \quad . \quad (32)$$

Moreover it follows that g_λ is zero.

If the group is semi-simple, there exist invariant sub-groups. Such a subgroup determines a real p-direction which is constant by the three displacements [3]). These p-directions determine a system of $\infty^{n-p}\ V_p$ which are totally geodesic and mutually parallel. The $(n-p)$-direction orthogonal to these V_p is also constant and determines in the same way ∞^p totally geodesic mutually parallel V_{n-p}. The coordinates may be

[1]) Thèse, Paris 1894.

[2]) L. P. EISENHART, Proc. Nat. Acad. 11, 1925, p. 246—250, has proved that every simple transitive group is connected with a symmetrical connexion and with one or more symmetrical quantities of degree 2.

[3]) Compare the division of semi-simple groups by CARTAN, Thèse, p. 52, especially for the demonstration that the case where the direction would have in common a p-direction with the orthogonal $(n-p)$-direction, can not occur.

chosen in such a way that their congruences are entirely in the V_p and V_{n-p}, so that the linear element decomposes into

$$ds^2 = \underset{1}{ds^2} + \underset{2}{ds^2}$$

where $\underset{1}{ds}$ represents the linear element of V_p and $\underset{2}{ds}$ the linear element of V_{n-p}. $S_{\lambda\mu\nu}$ and $R_{\omega\mu\lambda\nu}$ decompose into two separately constant parts which lie totally in V_p resp. V_{n-p}. If there are more invariant subgroups, then it is possible to repeat this decomposition, so that V_r finally decomposes into a certain number of V_{p_1}, V_{p_2} etc., which are all geodesic, parallel and mutually orthogonal. Every system belongs to a simple group, which is an invariant subgroup of the given group, and every V_p of this system is groupmanifold of this subgroup. The geometry of the V_r decomposes into the geometries of these simple invariant subgroups. The geometry of a semi-simple group is therefore reduced to the geometries of simple groups. [1])

§ 6. *Some geometrical properties.*

Starting from an arbitrary geodesic we obtain by means of the connexion (+) a congruence of geodesics which are mutually (+)-parallel, and in the same manner we obtain a congruence of mutually (—)-parallel geodesics. For $r = 3$ it follows from (30) that V_3 is an S_3. The two parallelisms coincide then with the parallelisms of CLIFFORD. [2])

If v^ν is a tangential vector in a point of the first geodesic, then the first congruence is given by the equation

$$\overset{+}{\nabla}_\mu v^\nu = \overset{0}{\nabla}_\mu v^\nu + S_{\lambda\mu}^{\cdot\cdot\,\nu} v^\lambda = 0,$$

from which it follows that v_λ is a solution of the equation of KILLING

$$\nabla_{(\mu} v_{\lambda)} = 0. \text{ [3])}$$

The transformation

$$'\xi^\nu = \xi^\nu + v^\nu\, dt$$

is therefore a *motion*, and even a *translation* since the length of the vector v^ν is the same in all points of V_r. The same consideration being

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[1]) H. LEVY, Rend. Acc. Linc. 3, 1926, p. 65—69 (see also p. 124—129) has found that every Riemannian geometry with $\nabla_\xi K_{\omega\mu\lambda\nu} = 0$ decomposes into Riemannian geometries with zero curvature. This does not agree with our results.

[2]) A treatment of the geometry in S_3 and of CLIFFORD's parallelisms from the point of view of the different connexions is to be found by CARTAN, Récentes généralisations de la notion d'espace, Bull. Sc.. Math. 48, 1924, p. 294—320 and from another point of view by EN. BORTOLOTTI, Parallelismo assoluto e vincolato negli S_3 etc. Atti Veneto 84, 1925, p. 821—858.

[3]) Der Ricci Kalkül, SPRINGER 1924, p. 212.

valid for the displacement (—), two kinds of translations are possible in V_r.

The first kind may be given also by the equation

$$T_{'\xi} = T_a \, T_\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

where T_a represents a definite transformation, the second by

$$T_{'\xi} = T_\xi \, T_a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

whence it appears that the translations belong to the first and second parametergroup. The described paths are (—)-aequipollent, while a segment passes into a (+)-aequipollent segment. The same holds m.m. for the second transformation, and for all continuous groups, not only for the semi-simple ones. Transformations of the form

$$T_{'\xi} = T_a \, T_\xi \, T_b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

leave invariant all three displacements, while (+)- as good as (—)-parallel segments pass into segments of the same kind. We may ask if there exist yet other pointtransformations of X_r, which leave invariant the connexions (+) and (—). Determining first the transformations wich leave invariant the group-structure, i.e. which transform $T_\xi \, T_\eta = T_\zeta$ into $T_{'\xi} \, T_{'\eta} = T_{'\zeta}$ [1]), to these transformations only the transformations (33) or (34) have to be added to obtain the desired transformations. The transformations (35) form an invariant subgroup of the so obtained group.

The connexions (+) and (—) determine each the structure of the group entirely. On the contrary, the group being not simple or semi-simple, it may occur that the structure is not entirely determined by (0). In the first place the group of transformations of (0) in itself contains among others the transformations $T_{'\xi} = T_\xi^{-1}$ that changes (+) in (—) v.v. The group being not simple or semi-simple there may exist moreover transformation of (0) in itself, conserving not the total of (+) and (—) parallelism. If this case occurs, there exist different absolute parallelisms. A simple example is given by the group

$$(X_1 \, X_2) = X_3 \quad ; \quad (X_1 \, X_3) = (X_2 \, X_3) = 0.$$

The (0)-connexion is here an ordinary affine one. In the corresponding E_3 besides the ordinary parallelisms there can be defined an infinity of absolute parallelisms with the straight lines as geodesics. With the given group is connected the parallelism by which the direction (a, β, γ) in 0 corresponds with the direction $(\alpha, \beta, \gamma + \alpha y - \beta x)$ in (x, y, z).

We can indicate also in which case a given integrable connexion is

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[1]) For the semi-simple groups this investigation has been executed by CARTAN, Bull. des Sc. Math. 49, 1925, p. 361—374.

(—)- or (+)-connexion of a group. If $A\,B\,C$ is a triangle of geodesics in X_r and if the connexion belongs to a group, then $A'\,C'$ is aequipollent with $A\,C$, if this property holds good for $A'\,B'$ and $A\,B$ as well as for $B'\,C'$ and $B\,C$, for every choice of A'.

The reciprocal is also true. If we make correspond with each segment $\vec{a}$ in X_r and its aequipollent segments the transformation, which transforms an arbitrary point X into $'X$, so that $\overrightarrow{X\,'X} = \vec{a}$, then these transformations form a continuous group with r parameters and the given connexion is the (—)-connexion of this group. Infinitesimally the condition is as follows. The connexion belongs then and only then to a group, if the expression $f^{\lambda\mu} d\sigma\, S_{\lambda\mu}^{\cdot\cdot\,\nu}$ does not change if the surface element is replaced by an aequipollent element, or, which is the same, if $S_{\lambda\mu}^{\cdot\cdot\,\nu}$ is a quantity constant by the given connexion,

Starting from two vectors v^ν and w^ν in ξ^ν, we may construct a congruence v^ν by means of the connexion (+) and a congruence w^ν by means of the connexion (—). These two congruences are V_2-building [1]), since

$$v^\mu \overset{0}{\nabla}_\mu w^\nu - w^\mu \overset{0}{\nabla}_\mu v^\nu = v^\mu S_{\lambda\mu}^{\cdot\cdot\,\nu} w^\lambda + w^\mu S_{\lambda\mu}^{\cdot\cdot\,\nu} v^\lambda = 0.$$

In any of these V_2 the angle between two vectors v^ν and w^ν in every point is equal, since

$$v^\mu \overset{0}{\nabla}_\mu v_\lambda w^\lambda = v^\mu S_{\lambda\mu}^{\cdot\cdot\,\nu} v_\nu w^\lambda + v^\mu S_{\mu\lambda}^{\cdot\cdot\,\nu} w^\lambda v_\nu = 0$$

$$w^\mu \overset{0}{\nabla}_\mu v_\lambda w^\lambda = 0.$$

In the V_2 obtained in this manner two translations are possible whence it follows that their geometry must be euclidian. For $r=3$ the V_2 are the well known surfaces of CLIFFORD.

We may generalise these considerations. If T_u is the general transformation of a p-fold continuous group, $T_u\,T_\xi$ is a total geodesic X_p passing through ξ^ν. Comparing this X_p with the X_p passing through η^ν given by $T_u\,T_\eta$, with every geodesic in one X_p corresponds a (—)-parallel in the other. Indeed, T_u and T_v being two general transformations of the subgroup, the segment $T_v\,T_\xi\,(T_u\,T_\xi)^{-1}$ is (—)-aequipollent with the segment $T_v\,T_\eta\,(T_u\,T_\eta)^{-1}$ since

$$T_v\,T_\xi\,T_\xi^{-1}\,T_u^{-1} = T_v\,T_u^{-1} = T_v\,T_\eta\,T_\eta^{-1}\,T_u^{-1}.$$

We may therefore say that X_p displaces itself (—)-parallel if ξ^ν changes. Choosing η^ν in the first X_p it follows that, being given a geodesic in that X_p passing through ξ^ν, through every other point of that X_p passes a (—)-parallel geodesic, situated entirely in X_p. In the same way $T_\xi\,T_v$

[1]) Ricci Kalkül, p. 107.

gives a totally geodesic X_p through ξ^ν, for which the same holds good with respect to the displacement (+). These two X_p coincide if T_ξ is a not singular transformation of the subgroup, since

$$T_\xi \, T_u = \left(T_\xi \, T_u \, T_\xi^{-1}\right) T_\xi$$

and $T_\xi \, T_u \, T_\xi^{-1}$ represents a general transformation of the subgroup also. The property of parallel motion of geodesics exists also for the (+)-parallelisms.

The reciprocal holds good: if it is possible to make correspond to every geodesic X_p through every point a (—)- (or (+)-) parallel, which lies entirely in X_p, then X_p belongs to a subgroup.

The X_p arising from the X_p through ξ^ν by parallel motion are generally different, according as the (+)- or (—)-displacement is used.

The invariant subgroups distinguish themselves by the coincidence of the two systems of X_p, which are obtained from the X_p in ξ^ν by means of (+) and (—).

In another kind of totally geodesic X_p the (+)- or (—)-parallel of a geodesic through a point of the X_p not situated on that line lies not in general in X_p. (A (o)-parallel direction is always situated in X_p). If we construct through a point not situated in X_p all geodesics, which are (+)- or (—)-parallel with all geodesics through a definite point of X_p, then there arises a totally geodesic X_p, but this X_p is different according as the point in the first X_p is chosen differently. In an S_3 exist only real geodesical S_2 of the second kind, since the corresponding group of the ordinary rotations in R_3 contains no subgroups with two parameters. The isotropical planes belong however to the first kind.

§ 7. *Classification of the geometries belonging to the simple groups.*

For a classification of the geometries of simple groups we have to start with the classification of these groups given by CARTAN[1]). By this classification a great part is played by a number l that has the following geometrical signification. If v^ν is a definite vector, which has not a singular direction, the vectors w^ν, satisfying the equation

$$v^\omega \, w^\mu \, R_{\omega\mu\lambda\nu} = 0$$

fill an R_l. There exist four normal types, A, B, C and D:

A.	$l = 1, 2, ..$	$r = l(\,l + 2)$	$=$ 3, 8, 15, 24, 35
B.	$l = 3, 4, ..$	$r = l(2l + 1)$	$=$ 21, 36, 55, 78, 105
C.	$l = 2, 3, ..$	$r = l(2l + 1)$	$=$ 10, 21, 36
D.	$l = 4, 5, ..$	$r = l(2l - 1)$	$=$ 28, 45, 66, 91, 120

and five abnormal ones:

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1) Thèse 1894.

E.	$l=6$	$r=78$
E.	$l=7$	$r=133$
E.	$l=8$	$r=248$
F.	$l=4$	$r=52$
G.	$l=2$	$r=14$.

For each type the form of $S_{\lambda\mu\nu}$ may be deduced and the corresponding geometry constructed [1]).

We will draw the attention on some important questions. Realising V_r in R_{r+m} it would be important to know the minimum value of m and to have a method to obtain such a realisation in the most simple manner. Also there may be asked for the necessary and sufficient conditions for a given symmetrical connexion being (o)-connexion of a group.

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[1]) Compare WEYL. Math. Zeitschr. 24, 1925 p. 354 f.