Mathematics. - "On Riemannian Geometries admitting an absolute parallelism." By E. Cartan and J. A. Schouten. (Communicated by Prof. Jan de Vries).
(Communicated at the meeting of April 24, 1926).
We will say that a RiemanNian geometry admits an absolute parallelism, if it is possible to define the parallelism of two directions in two different points in a manner, which is absolute (viz independent of the choice of the coordinates) and satisfies the following conditions:

1. A geodesic is in all her points self-parallel;
2. The angle between two different directions in an arbitrary point $P$ is equal to the angle between the two parallel directions in another arbitrary point $Q$.

By the thus defined parallelism a connexion arises, which

1. leaves invariant the tensor $g_{i \mu}$
2. possesses the same geodesics as the given Riemannian connexion
3. has zero curvature.

This new connexion is not necessary symmetrical. In such a Riemannian geometry evidently through every non-singular point not situated on a given geodesic one and only one geodesic may be drawn, which is in each of her points parallel to the given one.

In a previous paper we proved, that with every simple or semi-simple group such a RIEMANNian geometry corresponds and that the geometries corresponding with simple groups admit two different absolute parallelisms. The most simple case is the geometry of the elliptical $S_{3}$, the two parallelisms being those of Clifford.

We will prove presently that, supposing the fundamental form definite, there exists besides these geometries corresponding with the mentioned groups only one other geometry with the designed property and that this geometry is in close connexion with the non-associative numbersystem of Graves-Cayley.
§ 1. Fundamental relations.
A connexion with the same geodesics as the given RiEMANNian geometry, has parameters of the form

$$
\begin{equation*}
\bar{\Gamma}_{\lambda, \mu}^{\nu}=\stackrel{0}{\Gamma}_{\lambda, \mu}^{\nu}+p_{\lambda,} A_{, \mu}^{\nu}+p_{\mu} A_{\lambda,}^{\prime}+S_{\lambda, \mu}^{\cdots \nu} \tag{1}
\end{equation*}
$$

where $p$; is an arbitrary vector and $S_{;}$an arbitrary in $\lambda_{\mu}$ alternating affinor and $\stackrel{I}{j} \%$, are the parameters of the Riemannian geometry. $_{0}$,

If we postulate

$$
\begin{equation*}
\bar{\nabla}_{\omega} \boldsymbol{g}_{i \mu \mu}=0 \tag{2}
\end{equation*}
$$

it follows immediately

$$
\begin{gather*}
p_{\lambda}=0  \tag{3}\\
S_{;, y, 2}=S_{|; \mu \nu\rangle} \tag{4}
\end{gather*}
$$

$S_{i, u}$ is therefore a trivector. For the quantity of curvature of the new connexion it follows from (1)

$$
\begin{equation*}
0=R_{\omega, \mu i}^{-\cdots \nu}=K_{\omega, \mu i}^{\cdots \nu}+2 \nabla_{[\omega} S_{\mu]} i^{\nu}-2 S_{\alpha[\omega} S_{\mu] i}^{\alpha} \tag{5}
\end{equation*}
$$

and from (5) after some calculations

From this latter equation it follows that $S_{\text {b,y, }}$ i, is a quadrivector. By differentiation we deduce from (6) and (7), using the identity

$$
\begin{equation*}
\left.\stackrel{0}{\nabla}, K_{\omega \mu j}^{\ldots \nu}=2 \nabla_{[\omega} K_{\mu] ;}{ }^{1}\right) \tag{8}
\end{equation*}
$$

the equation

$$
\begin{equation*}
\stackrel{0}{\nabla}_{\omega} K_{\mu \lambda}=0 \tag{9}
\end{equation*}
$$

§ 2. The case of constant real p-directions.
If a real $p$-direction is constant by the (0)-connexion, then the orthogonal ( $n-p$ )-direction is constant also and it is well-known that $V_{n}$ is composed by $\infty^{n-p} V_{p}$ which are totally geodesic and mutually (0)-parallel and also in $\infty^{p} V_{n-p}$ with the same properties orthogonal to the $V_{p}$. The trivector is the sum of two trivectors $S_{i \mu,}^{\prime}$, and $S_{i \mu \nu,}^{\prime \prime}$, which are entirely situated in the $V_{p}$ and $V_{n-p}$ resp. and the transformation (1) is composed by a geodesic transformation (viz. a transformation which leaves the geodesics invariant) of every $V_{P}$ in itself, given by $S_{j \mu,}^{\prime}$ and an analogous transformation of every $V_{n-p}$ in itself, given by $S_{i, \mu, j}^{\prime \prime}$. Indeed, $K_{\text {,, ui, }}$ is divided, as is well-known, into two parts, which are situated entirely in $V_{p}$ and $V_{n-p}$ resp. If therefore $v^{\nu}$ and $w^{\nu}$ are two vectors lying entirely in $V_{p}$ and $V_{n-p}$ resp., then from (6) it follows:

$$
\begin{equation*}
0=v^{\omega} \boldsymbol{w}^{\mu} v^{i} w^{\nu} K_{\omega \mu i \nu}=w^{\mu} v^{\prime} S_{\alpha, \mu i} w^{\nu} v^{\omega} S_{\nu j, j} \tag{10}
\end{equation*}
$$

The real vector $w^{\mu} v^{i} S_{\alpha, j ;}$ is therefore zero, whence it follows that $S_{i \mu \nu}$ is divided in the described manner.

[^0]In consequence of this proposition the case, where $V_{p}$ admits constant real $p$-directions is reduced to the case, where these $p$-directions do not exist. We will therefore suppose in the following, that there exist no constant real p-directions.
§ 3. $K_{\mu i}=\mathrm{cg} g_{\mu i}$.
From this supposition it follows that $K_{\mu ;}$ must be equal to $g_{\mu ;}$ but for a constant factor. Indeed, if this were not true, the principal regions of $K_{y, j}$, would define constant $p$-directions in consequence of (9). The case $K_{\mu ;}=0$ is to be excluded immediately, for in consequence of (6) we would have $S_{\alpha, \bar{\beta},} S^{\alpha, \beta,}=0$ which is by a real trivector only possible for $S_{j, j, ~}=0$, giving the trivial case $K_{\omega, y ;,}=0$. We have therefore

$$
\begin{equation*}
K_{\mu \lambda}=c g_{\mu \lambda} ; c=\text { constant } \neq 0 \tag{11}
\end{equation*}
$$

or

$$
\begin{equation*}
S_{\alpha 3 \lambda} S_{{ }_{\mu}{ }_{\mu}^{3}}=-\mathrm{cg} g_{\mu} . \tag{12}
\end{equation*}
$$

§ 4. Bianchi's identity.
Applying Bianchi's identity on (6), we obtain

From $S_{j, u v}$ we derive the covariants
$\left.\begin{array}{l}\text { a) } c g_{\lambda \mu}=S_{\alpha \lambda}^{\ldots \beta} S_{\beta \mu}^{\ldots \alpha} \\ \text { b) } \quad g_{\lambda \mu \nu}=S_{\alpha \lambda}^{\ldots \beta} S_{\beta \mu}^{\cdots{ }_{\beta}} S_{\gamma,}^{\ldots \alpha}\end{array}\right\}$. etc.
and we remark that all these covariants admit cyclical permutation of the indices. From (13) it follows

$$
\begin{equation*}
c S_{i \alpha \mid \mu} S_{\cdot, \omega \xi \mid}^{\alpha}=-2 g_{i \alpha \mid!\mu} S_{\cdot, \omega \xi \mid}^{\alpha} \tag{15}
\end{equation*}
$$

By transvection of this equation with $\left.S^{\xi}\right)^{2,1}$ and $S^{(n) \xi}$ arises
and
hence

$$
\begin{equation*}
g_{r, \alpha, \beta} S^{\alpha, \beta y} S_{\gamma, y, \mu)}=-\mathrm{cg} g_{(, \mu, \mu} \tag{18}
\end{equation*}
$$

Now it follows from (14b) by differentiation

$$
\begin{equation*}
\nabla_{\omega,} g_{i \mu \nu}=-\frac{c}{2} \nabla_{\omega, \nu} S_{i \mu \nu} \tag{19}
\end{equation*}
$$

and therefore

$$
\begin{equation*}
\nabla_{\omega} g_{j, \mu,} S_{v}^{\alpha \beta}=\frac{c^{2}}{2} \nabla_{\omega,} g_{j, \mu}=0 \tag{20}
\end{equation*}
$$

Consequently $g_{j, j} S_{, \mu \beta}^{\alpha \beta}$ is a tensor, which can differ from $g_{i, \mu}$ only by a constant factor. Writing for this factor $-c \varrho, \varrho=$ constant, then it follows by substitution in (18)

$$
\begin{equation*}
g_{\lambda \mu \nu}=\varrho S_{\lambda \mu \nu} \tag{21}
\end{equation*}
$$

Substituting this value in (15), it appears, that two cases are possible, either

$$
\begin{equation*}
S_{i \times \mid, \prime} S_{., \omega \mid}^{\prime \prime}=0 \tag{22}
\end{equation*}
$$

or

$$
\begin{equation*}
c=-2 \varrho \tag{23}
\end{equation*}
$$

From (7) in relation with (22) we have

$$
\begin{equation*}
\bar{\nabla}_{\omega,} S_{\lambda \mu \nu}=\stackrel{0}{\nabla}_{\omega} S_{\lambda \mu \nu}=0 \tag{24}
\end{equation*}
$$

hence the first case gives again the geometries for which $c=+2 \varrho$ mentioned above and treated in our previous paper.

For both cases, $c= \pm 2 \varrho$, the following relations hold good:
a) $\nabla_{\xi} K_{\omega, \mu j, ~}=0$
b) $\nabla_{\xi} S_{\alpha(i|\nu|} S_{\mu,(),}^{\prime \alpha}=0$

which are all a consequence of BIANCHI's indentity.
We have therefore as yet to prove, that the case $c=-2 \varrho$ leads to the elliptic geometry in $S_{7}$. For this it is necessary to use some propositions of the theory of groups.
§ 5. The group of the Riemann-Christoffel affinor.
Just as in the previous communication we make use of a system of measure-vectors $e^{k}, i, k=1, \ldots, n$, that is constant by the connexion ( - ). To two surface-elements in two different points, each defined by two directions, which have in both points the same coordinates $x^{k}, y^{k}$ with reference to the system $e^{k}$, corresponds the same Riemannian curvature, defined analytically by the form

$$
R=K_{i j k l} x^{i} y^{j} x^{k} y^{l}
$$

$V_{n}$ admits a translation, whereby every point moves along a linear element, given by a (-)-constant vector ${ }^{1}$ ). By this translation the change of the vector $x^{k}$ is given by

$$
\begin{equation*}
d x^{k}=2 S_{i j}{ }^{k} x^{i} d \xi^{j} \tag{26}
\end{equation*}
$$

[^1]Since the Riemannian curvature of a surface-element dozs not change by this translation, it follows easily, that the form is invariant by all infinitesimal transformations

$$
\begin{equation*}
X_{i} f=S_{i j}^{. k} x^{j} \frac{\partial f}{\partial x^{k}} ; \quad Y_{i} f=S_{i j}^{i^{k}} y^{j} \frac{\partial f}{\partial y^{k}} \tag{27}
\end{equation*}
$$

applied simultaneously on the vectors $x^{k}$ and $y^{k}$.
The transformations (27), whose coefficients because of $c=-2 \varrho$ change by the transition from one point of $V_{n}$ to another, form no group. But they are contained in the group $I$ of all rotations which leave the form $R$ invariant. Especially the alternated combinations ( $X_{i} X_{j}$ ) belong to this group ${ }^{1}$ ):

$$
\begin{equation*}
\left(X_{i} X_{j}\right)=\left(S_{i k}^{\ddot{i}^{\lambda}} S_{j \lambda}^{\check{l}}-S_{j j^{\lambda}}^{\ddot{n}_{i j}^{\prime}} S^{\prime}\right) x^{k} \frac{\partial f}{\partial x^{l}} \tag{28}
\end{equation*}
$$

The differential

$$
d x^{k}=S_{i j}^{* k} x^{i} d \xi^{J}
$$

obtained by displacing the vector $x^{k}(0)$-parallel, is half the differential (26), whence it follows, that the Riemannian curvature is invariant also by this displacement. Hence the $V_{n}$ belongs to an important class of $V_{n}$, viz those in which the curvature is invariant by pseudo-parallel displacements.

Particularly the rotation, corresponding with a surface-element, will belong to the group $I$. This is confirmed by calculation. With a surfaceelement with the coordinates $p^{i j}$ corresponds the rotation

$$
\begin{align*}
& p^{i j} K_{i j}^{\cdots k^{l}} x^{k} \frac{\partial f}{\partial x^{l}}=p^{i j}\left(1 / 3 S_{k j}^{\cdots \lambda} S_{i, \lambda}^{l}-2 / 3 S_{j i^{\lambda}}^{\prime \lambda} S_{k, \lambda}^{l}+\right.  \tag{29}\\
& \left.+1 / 3 S_{i k^{\lambda}} S_{j, \lambda}^{l}\right) x^{k} \frac{\partial f}{\partial x^{l}}=2 / 3 p^{i j} S_{i j}^{k} X_{k} f-1 / 3 p^{i j}\left(X_{i} X_{j}\right)
\end{align*}
$$

Hence it follows that the group of holonomy of $V_{n}$ is the group $\Gamma$ or one of his subgroups ${ }^{2}$ ).
§ 6. The group $\Gamma$ leaves invariant no p-direction.
It has already been proved in § 2 that there cannot exist a real by $\Gamma$ invariant $p$-direction. If there were an invariant imaginary $p$-direction, the conjugate imaginary $p$-direction and the orthogonal $(n-p)$-direction would also be invariant. Hence it would follow, that there were an invariant real $q$-direction, except in the case $n=2 p$ and that the $p$ direction were totally isotropical. In this latter supposition we choose the system in such a way, that

$$
2 x^{1} x^{2}+\ldots \ldots+2 x^{n-1} x^{n}
$$

[^2]is the fundamental form, the given $p$-direction is defined by
$$
x^{1}=x^{3}=\ldots=x^{n-1}=0
$$
and the conjugate $p$-direction by
$$
x^{2}=x^{4}=\ldots .=x^{n}=0 .
$$

Every infinitesimal transformation of $\Gamma$ contains then only coefficients $a_{i}{ }^{j}{ }^{1}$ ), whereby $i+j$ is even and therefore only coefficients $a_{i j}$ whereby $i+j$ is odd. One of the $S_{i j k}$ can therefore only then be equal to zero, if not only $i+j$, but also $j+k$ and $k+i$ are odd.

We find therefore $S_{i j k}=0$.
§ 7. The group I' is simple.
Every linear group which leaves no p-direction invariant is either simple or semi-simple ${ }^{2}$ ).

In the latter case $I$ may be obtained from two simple or semi-simple groups $\gamma_{1}$ and $\gamma_{2}$ in $q$ variables $x^{1}, \ldots, x^{q}$, resp. $s$ variables $y^{1}, \ldots, y^{s}$, which leave invariant no p-direction. The group is then the group of transformations of the $n=q s$ products $z^{(i z)}=x^{i} y^{z}, i=1, \ldots, q, \alpha=1, \ldots, s$ by the transformations of $\gamma_{1}$ and $\gamma_{2}$. Every transformation of $I$ is of the form

$$
\alpha_{i}^{j} \boldsymbol{z}^{(i \lambda)} \frac{\partial f}{\partial z^{(j \lambda)}}+b_{\alpha}^{\cdot \hat{\beta}} z^{(k \alpha)} \frac{\partial f}{\partial z^{(k, j)}} .
$$

Consequently in every vanishing coefficient $A_{(i, i)}^{(j)}$ of an infinitesimal transformation of $I^{\prime}$ either $i=j$ or $\alpha=\beta$, moreover, if $i \neq j$ then we have

$$
A_{(i \alpha)}{ }^{(j \mu)}=A_{(i \lambda)}{ }^{(j \lambda)}
$$

for every value of $\lambda$.
The group $\Gamma$ being a group of rotations, it leaves invariant a nondegenerate quadratic form $\Phi\left(z^{(i x)}\right)$. Two cases are possible, either $\Phi$ vanishes identically or $\Phi$ dont vanish. In the first case by giving to the $y^{\alpha}$ fixed values, we obtain a quadratic form $f\left(x^{i}\right)$, invariant by $\gamma_{1}$, which is determined uniquely but for a constant factor and is not degenerate, since otherwise $\gamma_{1}$ would leave invariant a $p$-direction. In the same manner a quadratic form $q\left(y^{2}\right)$ may be obtained, so that $\Phi\left(x^{i} y^{\alpha}\right)=f\left(x^{i}\right) \varphi\left(y^{x}\right)$. We may therefore suppose, that $\Phi$ has the form

$$
\begin{equation*}
\Phi=\sum_{i, \alpha}\left(z^{(i \alpha)}\right)^{2} \tag{30}
\end{equation*}
$$

[^3]In the second case, setting

$$
\Phi\left(z^{(i \alpha)}\right)=g_{(i x)(j 3)} z^{(i x)} z^{(j i)}
$$

it follows

$$
g_{(i x)(j:)}+g_{(i z)(j x)}=0
$$

The bivector $a_{i j}=g_{(i \alpha)(j i)}$, in which expression to $\alpha$ and $\beta$ fixed values are to be given, is therefore invariant by $\gamma_{1}$ and so is by $\gamma_{2}$ the bivector $b_{u 3}=g_{(i x)(j ;)}$, in which $i$ and $j$ have fixed values. These two bivectors are uniquely determined but for constant factors and non-degenerate, for otherwise $\gamma_{1}$ resp. $\gamma_{2}$ would leave invariant a $p$-direction. The constant factors may be chosen so that for $\Phi$ holds good

$$
g_{(i x)(j \xi)}=a_{i j} b_{x, j} .
$$

Hence it may be supposed that $\Phi$ has the form:

$$
\left.\begin{array}{c}
\left.\Phi=\sum_{i, \alpha} z^{(2 i-1.2 x-1)} z^{(2 i, 2 x)}-z^{(2 i-1.2 x)} z^{(2 i .2 x-1)},\right) \\
i=1, \ldots, \frac{q}{2} ; \quad a=1, \ldots, \frac{s}{2} . \tag{31}
\end{array}\right\}
$$

Now we will prove that the cases (30) and (31) cannot occur. In the case (30) we have

$$
S_{(i \lambda)(j z)(k \gamma)}=S_{(i i)(j ;)^{(k \gamma)}} .
$$

If the coefficient $S_{(i x)(j i)(k \gamma)}$ is unequal to zero, then either $j=k$ or $\beta=\gamma$, likewise either $i=j$ or $\alpha=\beta$ and either $i=k$ or $\alpha=\gamma$, therefore either $i=j=k$ or $\alpha=\beta=\gamma$. Now supposing f.i. $i=j=k$ and $\alpha \neq \beta$, then we have for every value of $k$

$$
S_{(i x)\left(i^{i}\right)(i \gamma)}=S_{(k x)(k, 3)(i \gamma)}
$$

For $i \neq k$ the condition mentioned above is not satisfied on the right side of this equation and the trivector $S$ must therefore be equal to zero.

In the case (31) we have

$$
S_{(i \alpha)(j \dot{ })(k \gamma)}=(-1)^{k+\gamma} S_{(i \dot{z})(j ; j)^{\left(k^{\prime} \gamma^{\prime}\right)}}
$$

where

$$
k^{\prime}=\left\{\begin{array}{l}
k+1, k \text { odd } \\
k-1, k \text { even }
\end{array} \quad ; \quad \gamma^{\prime}=\left\{\begin{array}{l}
\gamma+1, \gamma \text { odd } \\
\gamma-1, \gamma \text { even. }
\end{array}\right.\right.
$$

Hence if the coefficient $S_{(i x)(j i)(k \gamma)} \neq 0$ we have

$$
j=k^{\prime} \text { òr } \beta=\gamma^{\prime} ; \quad i=k^{\prime} \text { oे } \alpha=\gamma^{\prime} ; \quad i=j^{\prime} \text { òr } \alpha=\beta^{\prime} .
$$

Now supposing f.i. $i=j=k^{\prime}$ with $\alpha=\beta^{\prime}$, then we have for every value of $\lambda$

$$
S_{(i \alpha)\left(i^{\prime}\right)\left(i^{\prime} \gamma\right)}=(-1)^{\alpha+\lambda} S_{\left.(i \lambda)\left(i^{\prime} \lambda^{\prime}\right) i^{\prime} \gamma\right)} .
$$

For $\lambda \neq \gamma$ we have for every value of $k$

$$
S_{\left(i^{\prime}\right)\left(i^{\prime}\right)^{\prime}\left(i^{\prime} \gamma\right)}=(-1)^{i+k} S_{(i \lambda)\left(k^{\prime}\right)\left(k^{\prime} \gamma\right)}
$$

Taking $k \neq i$, we must have $\lambda=\gamma^{\prime}$, i.e. every index $\lambda$ different from
$\gamma$, must be equal to $\gamma^{\prime}$. This however is only possible if $s=2$. If $\lambda=\gamma^{\prime}$ then we have, supposing $k \neq i$,

$$
S_{\left(i \gamma^{\prime}\right)(k \gamma)\left(k^{\prime} \gamma\right)}=-S_{(i \gamma)(k \gamma)\left(k^{\prime} \gamma^{\prime}\right)}
$$

from which it follows $k=i^{\prime}$. From this we derive that $q=2$ also.
The only possible semi-simple group is therefore the orthogonal group in four variables. The following developments suppose $\Gamma$ to be simple. They would however be solid also for the orthogonal group in 4 variables. We may however easily verify, that the case $n=4$ cannot occur. Indeed for $n=4$ it follows from (7) that $S_{, y, y}=0$, by which we return to the case $c=+2 \varrho$.
§ 8. The form $R$ derived from the infinitesimal transformations of the group.

Be

$$
Z_{I f}=a_{i_{j}}{ }^{k} x^{j} \frac{\partial f}{\partial x^{k}} ; \quad I, J, K, L=1, \ldots, t .
$$

a system of $r$ infinitesimal transformations with real coefficients of $\Gamma$. To each of these transformations we adjoin the bilinear form

$$
\zeta_{I}=a_{I j k} x^{j} y^{k}=1 / 2 a_{I j k} p^{j k}
$$

From the equation

$$
\begin{equation*}
\left(Z_{I} Z_{J}\right)={\underset{I}{I J}}^{\kappa} \quad Z_{k} t \tag{32}
\end{equation*}
$$

it follows that

$$
Z_{1} \zeta_{J}=c_{I J}^{\kappa} \zeta_{K}
$$

Now we introduce the fundamental tensor

$$
\begin{equation*}
\left.G_{I I}=1 / 2 a_{I j k} a_{j}^{j k_{1}}\right) \tag{33}
\end{equation*}
$$

and by using this tensor we obtain from $c_{i j}{ }^{K}$ the covariant components $c_{I J K}$. From (32) it follows that

$$
a_{i \ddot{i}}{ }^{k} a_{J k j}-a_{\ddot{j} i}^{k} a_{I k j}=c_{I J}^{\cdot{ }^{K}} a_{K i j} .
$$

from which we obtain by transvection with $a_{L}{ }^{i j}$

$$
\begin{equation*}
-a_{i \ddot{i}}^{j} a_{j_{j}}^{k} a_{K k}{ }^{i}=c_{I J K} \tag{34}
\end{equation*}
$$

The $c_{I J K}$ are therefore the components of a trivector.
The form

$$
R=K_{i j k l} x^{i} y^{j} x^{k} y^{l}=1 / 4 K_{i j k l} p^{i j} p^{k l}
$$

is a quadratic form in $\zeta_{1}, \ldots, \zeta_{r}$. Indeed it follows from (29)

$$
\begin{equation*}
\frac{1}{2} \frac{\partial R}{\partial p^{i j}}=\frac{2}{3} S_{i j}^{.^{k}} \xi_{k}-1 / 3 \xi_{i j} \tag{35}
\end{equation*}
$$

where $\xi_{k}$ and $\xi_{i j}$ are forms which correspond to the infinitesimal trans-
${ }^{1}$ ) For this introduction it is evidently necessary that $g_{i, j}$ be known.
formations $X_{i} f(27)$ and ( $X_{i} X_{j}$ ) (28). Now we deduce the general form of a quadratic form $R\left(\zeta_{I}\right)$, invariant by the group $I$. If

$$
R\left(\zeta_{I}\right)=A^{I I} \zeta_{I} \zeta_{I}
$$

then we have

$$
Z_{I}(R)=A^{\prime K} \zeta_{K} c_{I J}{ }^{L} \zeta_{L}
$$

and therefore

$$
C_{I J}^{. .} A^{J K}+C_{I J}^{K} \quad A^{I L}=0
$$

These relations express that the adjoint group of $\Gamma$, which is generated by (est engendré par, erzeugt wird durch) the transformations

$$
E_{1} f=c_{i j}^{K} \quad e^{J} \frac{\partial f}{\partial e^{K}}
$$

leaves invariant the form $A_{I J} e^{I} e^{I}$. Consequently, $I$ being simple and the adjoint group leaving invariant the form $G_{I I} e^{I} e^{I}$, we have

$$
\mathrm{A}_{I I}=h \mathrm{G}_{I I}
$$

therefore

$$
\begin{equation*}
R\left(\zeta_{I}\right)=h G^{I J} \zeta_{I} \zeta_{J}=h \zeta^{I} \zeta_{I} \tag{36}
\end{equation*}
$$

From this equation it follows that

$$
\begin{equation*}
K_{i j}^{i j}=2 h r=-n c ; \quad h=-\frac{n c}{2 r} \tag{37}
\end{equation*}
$$

§ 9. The order $r$ of $\Gamma$ is equal to $3 n$.
Taking for the infinitesimal transformations of $\Gamma$ the $n$ transformations $X_{i} f$ and $r-n$ other independent ones, then we have for $i, j, k \leqq n$ in consequence of (12)

$$
\begin{equation*}
G_{i j}=1 / 2 S_{i \alpha 3} S_{j}^{\alpha j}=-1 / 2 c g_{i j} \tag{38}
\end{equation*}
$$

On the other hand it follows from (35) and (36) that

$$
\frac{1}{2} \frac{\partial R}{\partial p^{i j}}=h \zeta_{I} \frac{\partial \zeta^{I}}{\partial p^{i j}}=\frac{2}{3} S_{i j}^{\check{i j}^{k}} \zeta_{k}-{ }^{1 / 3}{c_{i j}^{\prime!}}^{I} \zeta_{I}
$$

Hence

$$
\begin{array}{rlr}
h \frac{\partial \zeta^{I}}{\partial p^{i j}} & =\frac{2}{3} \varepsilon_{I} S_{i j}^{\cdot}{ }^{I}-1 / 3 c_{i j}^{\prime \prime} & \varepsilon_{I}=\left\{\begin{array}{l}
1, I=n \\
0, I>n
\end{array}\right. \\
h \frac{\partial \zeta_{I}}{\partial p^{i j}} & =\frac{2}{3} G_{I k} S_{i j}^{\prime \cdot}-1 / 3 G_{I J} c_{i j}^{\prime \cdot} . &
\end{array}
$$

By (38) we have

$$
h \frac{\partial \zeta_{k}}{\partial p_{i j}}=h S_{k i j}=-1 / 3 c S_{i j k}-1 / 3 c_{i j k}
$$

hence

$$
\begin{equation*}
c_{i j k}=-(c+3 h) S_{i j k} . \tag{39}
\end{equation*}
$$

Now it follows from (34) that $c_{i j k}, i, j, k=n$, is identical with the quantity $g_{i j k}$ defined by (14) so that we have derived in another way once more the identity (21)

$$
\begin{equation*}
g_{i j k}=\varrho S_{i j k} \tag{21}
\end{equation*}
$$

obtaining at the same time as a new result

$$
\varrho=-(c+3 h) .
$$

Now it was proved in § 4 that $c= \pm 2 \varrho$. Hence from (37) follows that the only possibilities are

$$
c=+2 \varrho, \quad h=-\frac{c}{2}, r=n
$$

giving the $V_{n}$ of the simple groups, and

$$
c=-2 \varrho, \quad h=-\frac{c}{6}, \quad r=3 n
$$

§ 10. The group I' is for $c=-2 \varrho$ the orthogonal group in 7 variables.
Now rests only to examine for which types of simple groups, leaving invariant a non-degenerate quadratic form but no linear manifold, $r$ may be equal to $3 n$. Previously we remark that the roots of the characteristic equation of an orthogonal group are in pairs equal and opposite. These roots are called by CARTAN the weights of the group ${ }^{1}$ ). It may be remembered that every simple group, which leaves invariant no linear manifold, is entirely determined by her principal weight (poids dominant).

Type A. If the rank of the group is $l$, then the order is $r=l(l+2)$ and every weight is of the form

$$
m_{1} \omega_{1}+\ldots \ldots+m_{l+1} \omega_{l+1}
$$

in which the sum of the rational coefficients $m_{i}$ is zero. The difference between two coefficients, corresponding with the same weight or with two different weights is a whole number. Since with every weight corresponds an equal and opposite weight, the coefficients $m_{i}$ are either all whole numbers or all half odd numbers. If the $m_{i}$ are all whole numbers, the group contains all weights $\omega_{i}-\omega_{j}$ and 0 and the number of the variables is therefore at least equal to $t$, which is not possible if $r=3 n$. If the $m_{i}$ are fractions, then $l$ is odd and all weights of the form

$$
1 / 2\left(\omega_{1}+\ldots \ldots+\omega_{\frac{l+1}{2}}-\omega_{\frac{l+1}{2}+1}-\ldots \ldots-\omega_{l+1}\right)
$$

exist. Their number is therefore

$$
\frac{(l+1)!}{\left(\frac{l+1}{2}\right)!\left(\frac{l+1}{2}\right)!}>\frac{l(l+2)}{3}
$$

from which it would follow $n>\frac{r}{3}$.

[^4]Type B. Here is $l=2, r=l(2 l+1)$ and the weights are of the form

$$
m_{1} \omega_{1}+\ldots \ldots+m_{l} \omega_{l}
$$

in which the $m_{i}$ are whole numbers or half whole numbers. In the latter case is $n \geqq 2 l>\frac{l(2 l+1)}{3}=\frac{r}{3}$. In the first case the existence of a weight, which is not of the form $\pm \omega_{i}$ has as a consequence that all weights of the form $\pm \omega_{i} \pm \omega_{j}$ exist, whence it follows that $n \equiv 2 l(l-1)>\frac{l(2 l+1)}{2}=\frac{r}{3}$. There remains therefore only the case where the weights have the form $\pm \omega_{i}$. This case corresponds with the orthogonal group in $2 l+1$ variables. From the identity $2 l+1=\frac{l(2 l+1)}{3}$ it follows that $l=3$, so that the orthogonal group in 7 variables appears to be possible.

Type C. Here is $l \equiv 3, r=l(2 l+1)$ and the weights have the form

$$
m_{1} \omega_{1}+\ldots+m_{l} \omega_{l}
$$

in which the $m_{i}$ are whole numbers. The group, for which the weights are $\pm \omega_{i}$, has $2 l$ variables and $2 l \neq \frac{l(2 l+1)}{3}=\frac{r}{3}$; the other groups have at least $r$ variables.

Type $D$. Here is $l \equiv 4, r=l(2 l-1)$ and the weights have the form $m_{1} \omega_{1}+\ldots+m_{l} \omega_{l}$. Is $l \equiv 5$, then the $m_{i}$ may be half odd numbers, and we have $n \geqslant 2^{l-1}>\frac{l(2 l-1)}{3}=\frac{r}{3}$. If the $m_{i}$ are whole numbers, then we have in the first place the system of weights that corresponds with $n=2 l \neq \frac{l(2 l-1)}{3}=\frac{r}{3}$ and further other groups, for which $n \equiv \boldsymbol{r}$.

The types $E, F, G$ give groups for which $n>\frac{r}{3}$.
There is consequently but one possibility, $I$ is the group of all orthogonal transformations in seven variables. From the expression for $R$ it follows that $V_{7}$ is an elliptical $S_{7}$.
§ 11. The absolute parallelism in $S_{7}$.
It is indeed very easy to indicate in $S_{7}$ an infinity of parallelisms which satisfy the prescribed conditions. In the projective space of 7 dimensions the absolute be defined by

$$
x_{0}^{2}+x_{1}^{2}+\ldots .+x_{7}^{2}=0 .
$$

A point be given by 8 coordinates whose squares have the sum 1 . The distributive, but not associative numbersystem of Graves-Cayley with the unities $1, e_{1}, \ldots, e_{7}$ is given by the rules of multiplication

$$
\begin{align*}
& e_{i}^{2}=-1 ; e_{i}=e_{i+1} e_{i+3}=-e_{i+3} e_{i+1}=e_{i+2} \quad e_{i+6}=  \tag{40}\\
& \left.=-e_{i+6} e_{i+2}==e_{i+4} e_{i+5}=-e_{i+5} e_{i+4} \quad ; \quad e_{i}=e_{i+7}\right\rangle
\end{align*}
$$

With the point $x_{0}, \ldots, x_{7}$ corresponds the number

$$
X=x_{0}+\sum_{i} x_{i} e_{i}
$$

with the module $\sqrt{x_{0}{ }^{2}+x_{1}{ }^{2}+\ldots+x_{7}{ }^{2}}=1$.
$\overrightarrow{X Y}$ being an arbitrary segment of a geodesic and $X^{\prime}$ an arbitrary point, we will call the segment $\overrightarrow{X Y}$ and $\vec{X}^{\prime} Y^{\prime}$ aequipollent, if

$$
\begin{equation*}
Y^{\prime} X^{\prime-1}=Y X^{-1} ;\left(X^{-1}=x_{0}-\Sigma x_{i} e_{i}\right) \tag{41}
\end{equation*}
$$

Equalising the scalar parts of both members of (41) we obtain

$$
x_{0}^{\prime} y_{0}^{\prime}+\ldots+x_{7}^{\prime} y_{7}^{\prime}=x_{0} y_{0}+\ldots+x_{7} y_{7}
$$

from which tollows the equality of length of the segments $\vec{X} Y$ and $\overrightarrow{X^{\prime}} Y^{\prime}$. If $X$ and $X^{\prime}$ are given, the relation between the $y_{i}$ and the $y_{i}^{\prime}$ is linear and such, that with a vector in $X$ corresponds a vector in $X^{\prime}$ with the same length.

The aequipollence is therefore conformal. The geodesics of $S_{7}$ are selfparallel. Indeed, if we put

$$
\left.Y X^{-1}=Z, \text { whence } Y=Z X^{1}\right)
$$

and if $X^{\prime}$ is situated on the geodesic $\vec{X} Y$ :

$$
X^{\prime}=\lambda X+\mu Y=\lambda X+\mu Z X
$$

then we have

$$
Y^{\prime}=Z X^{\prime}=\lambda Z X+\mu Z(Z X)
$$

so that

$$
\begin{aligned}
& =\lambda Y+\mu\left(2 z_{0}-Z^{-1}\right) Z X \\
& =\left(\lambda+2 \mu z_{0}\right) Y-\mu X
\end{aligned}
$$

We get another absolute parallelism by putting

$$
\begin{equation*}
X^{\prime-1} Y^{\prime}=X^{-1} Y \tag{42}
\end{equation*}
$$

If generally $A$ is an arbitrary fixed number of the numbersystem, then we have two families of absolute parallelisms, each depending from 7 parameters, by the equations

$$
\begin{align*}
& Y^{\prime}\left(X^{\prime-1} A\right)=Y\left(X^{-1} A\right)  \tag{43}\\
& \left(A X^{\prime-1}\right) Y^{\prime}=\left(A X^{-1}\right) Y \tag{44}
\end{align*}
$$

It may be foreseen that there exists an infinity of absolute parallelisms.

[^5]Indeed, by the infinitesimal translation (26) two (-)-parallel vectors in two different points are transferred into two vectors which are no more (-)-parallel. The metrical properties of $S_{7}$ being invariant by translation, the ( - )-parallelism is transformed into a ( - -parallelism different from the first, so that there exists a continuous family of (-)-parallelisms.

In $S_{7}$ there do not exist other absolute parallelisms than those defined by (43) and (44). Let us consider a determined absolute parallelism and a congruence of geodesics invariant by this absolute parallelism. We get a translation by moving every point on the corresponding geodesic over a distance equal for all points. In an $S_{n}$, wherein the equation of the absolute quadric is $x_{1}{ }^{2}+\ldots+x_{n}{ }^{2}=0$, every infinitesimal translation may be reduced to the form

$$
x_{1} \frac{\partial f}{\partial x_{2}}-x_{2} \frac{\partial f}{\partial x_{1}}+x_{3} \frac{\partial f}{\partial x_{4}}-x_{4} \frac{\partial f}{\partial x_{3}}+\ldots
$$

and the trajectories of the translation all intersect the two plane manifolds in the absolute

$$
\begin{aligned}
& x_{1}+i x_{2}=x_{3}+i x_{4}=\ldots \ldots=0 \\
& x_{1}-i x_{2}=x_{3}-i x_{4}=\ldots \ldots=0 .
\end{aligned}
$$

For $n=7$ these two manifolds $P_{3}$ are situated on the absolute quadric, there exist two different families and two conjugate imaginary manifolds belong to the same family. The $P_{3}$ of the first family $\left(P_{3}^{+}\right)$are characterised by the property that their equations being written in the form

$$
y_{0}+i y_{1}=y_{2}+i y_{3}=\ldots .=y_{6}+i y_{7}=0
$$

the determinant of the coefficients of $x_{0}, \ldots, x_{7}$ in $y_{0}, \ldots, y_{7}$ is positive. By the other family $\left(P_{3}\right)$ this determinant is negative.

Reciprocally to every $P_{3}$ and his conjugate corresponds a determined congruence of geodesic lines, to which belongs a group of translations with one parameter.

An absolute parallelism may be obtained by choosing $\infty^{7}$ manifolds $P_{3}$ such that the corresponding congruences are isogonal. It may be proved, that in this manner no other absolute parallelisms are obtained than those which are defined by (43) and (44). The $(+)$ - and ( - ) parallelisms are obtained by means of manifolds from $\left(P_{3}^{+}\right)$resp. $\left(P_{3}\right)$.

The points of $S_{7}$, the $(+)$ - and (-)-parallelisms may be considered as elements of $S_{7}$. In the same way as we extend in the ordinary projective space the group of projective point-transformations by adjunction of correlations, the group of motions and reflexions in $S_{7}$ may be extended by adjunction of the four continuous families of transformations, which transform points in $(+)$ - and ( - -parallelisms. In this manner we have in $S_{7}$ a triality ${ }^{1}$ ) by which it is possible to define the distance of two ( + )- or two ( - )-parallelisms, etc.

[^6]
## § 11. General conclusion.

In a Riemannian geometry, in which the linear element is the sum of $h$ linear elements, corresponding with geometries of finite groups and $k$ linear elements, corresponding with the geometry of $S_{7}$, there exist, for $k=0,2^{h}$ absolute parallelisms and, for $k>0,2^{h+k}$ continuous families of $\infty^{7 k}$ absolute parallelisms.

This result remains valid, if a euclidian linear element of an arbitrary number of dimensions is added.
P.S. We remark, that an error has been made in the deduction of (10) in the first note, the linear element having really the opposite sign, hence

$$
S_{i, j}^{\prime \prime}=+1 / 2 c_{i, \mu}
$$

instead of $-1 / 2 c_{;, j}$. Also on page 807 the linear element $s^{j} d t$ corresponds with the transition from $t^{k}$ into $t^{k}-s^{i} t^{j} c_{i j}{ }^{*} d t$, hence in (18) $c_{i,}^{\prime,}{ }^{\prime \prime}$ ought to be substituted by $-c_{;, \%}$. The error has had no serious consequences.


[^0]:    ${ }^{1}$ ) R. K. p. 168.

[^1]:    ${ }^{1}$ ) It may easily be proved that the problem treated above is equivalent to the determination of the $V_{n}$, which admit $n$ infinitesimal isogonal translations, not situated in one ( $n-1$ )-direction.

[^2]:    ${ }^{1}$ ) It may be verified easily that $X_{i}$ and ( $X_{i} X_{j}$ ) does form a group. but we will not use this property.
    ${ }^{2}$ ) E. Cartan, Ann. Ec. Norm. 3,42 (25) p. 21 . It may be proved, that $\Gamma$ is the group of holonomy itself.

[^3]:    ${ }^{1}$ ) By coefficients of the infinitesimal transformation $\mathrm{e}^{k} X_{k} f$ we mean the $n^{2}$ expressions $\mathrm{e}^{k}, S_{k}{ }_{i}{ }^{j}, i, j=1, \ldots, n$.
    ${ }^{2}$ ) To be compared by this and the following $\S \S$. E. CARTAN, Les groupes projectifs, qui ne laissent invariante aucune multiplicité plane. Bull. Soc. Math. 41 (13) p. 53-96.

[^4]:    ${ }^{1}$ Loc. cit. § 1, No. 3. We make use of the notations used there.

[^5]:    ${ }^{1)}$ This result remains valid. although the multiplication is no longer associative.

[^6]:    ${ }^{1}$ ) E. Cartan, Bull. Sc. Math. 2, 49 (25) p. 361-371.

