Mathematics. — *"On a Function which in any Interval Assumes any Value a Non-Enumerable Number of Times, and on a Function Representing a Rectifiable Curve which in any Interval is Non-Differentiable a Non-Enumerable Number of Times".* By Prof. J. A. Barrau. (Communicated by Prof. Jan de Vries).

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The communication of J. Wolff: "On a Function which in any Interval Assumes any Value on a Non-Enumerable Set of Points",[1] led me to observe that such a function may also be defined in a more elementary way; also M functions y of N variables x so that in the representation of the x space on the y space defined in this way, the image of any N-dimensional region of the former space covers the whole y space a non-enumerable number of times.

The former is done by the aid of the theorem: [2])

"Any number $x\,(0 \leqq x \leqq 1)$ is developed into a binary fraction p (if two developments are possible we choose the one which ends in a repeating zero, not the one with repeating one). If u_n is the arithmetic mean of the first n figures behind the comma in p, for any x the sequence $u_n\,(n = 1, 2, 3, \ldots$ etc.) has an upper limit y; y is considered as a function of x.

On any sub-interval of the interval $0 \leqq x \leqq 1$ this function assumes all values from 0 to 1 on a set of values of x which has the same power as the continuum" [3]).

For u_n we may choose as well for instance the mean of the first n figures with *odd* order numbers; in this case an arbitrary choice of the figures on the *even* places after a definite order number (hence in a definite interval), has no influence on the value of y (which by a proper choice of the odd places may be made equal to any number from 0 to 1); in this way it is easily seen that the x-values corresponding to the same y, are non-enumerable in the interval in question.

In order to define in a similar way M functions y_i of N variables x_j, we understand by $(u_i)_n$ the mean of the first Nn of those figures in the N developments p_j (for each j the first n) of a point of the x-space of which the order number

$$\varrho \equiv i\ (mod.\ M + 1)\ ^{4})$$

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[1]) These Proceedings **29**, p. 127.

[2]) Published as N⁰. 44, deel XIV, Wiskundige opgaven, Amsterdam, where a proof will be given afterwards.

[3]) This function has the property that it has any period $\omega = 2^{-N}$ (N a positive integer).

[4]) We may also assume $\rho \equiv i$ (*mod.* M).

and we put

$$y_i = \overline{\lim_{n=\infty}} (u_i)_n .$$

After a certain order number (hence in a certain sub-hyper cube of the hyper cube $0 \leqq x_j \leqq 1$), we may choose the figures with order numbers $\varrho \equiv i \neq 0$ in such a way that any y_i assumes any desired value from 0 to 1; the choice of the figures with order numbers $\varrho \equiv i = 0$ has no influence on the y_i; hence the power of the set of points x in the sub-hyper cube (and a fortiori in a region containing this), to which there corresponds a given point y, is the same as that of the continuum.

The restriction of x_j and y_i to the values from 0 to 1 is immaterial and is e.g. annulled by repeating the unit-hyper cube of x-space in the directions of the edges and by taking the function

$$z_i = \frac{1-y_i}{y_i} - \frac{y_i}{1-y_i}$$

in stead of any y_i.

A function $y = f(x)$ which represents a rectifiable curve of given length $\lambda > 1$ between the points A and B, (the points 0 and 1 of the X-axis), and which above any sub-interval of AB contains a non-enumerable number of points without tangent, may be defined in the following way:

Between A and B we construct a series of broken lines P_n ($n = 0$, 1, 2, ... etc.), with lengths

$$l_n = \lambda^{1-2^{-n}}, \quad \text{so that} \quad k_n = \frac{l_n}{l_{n-1}} = \lambda^{2^{-n}}$$

and:

$$\lim_{n=\infty} l_n = \lambda \quad ; \quad \lim_{n=\infty} k_n = 1.$$

P_0 is the line AB.

The odd angular points of P_1 lie on the X-axis in the points 0, $\frac{1}{2}$, $\frac{3}{4}$, ..., $1-2^{-m}$, ...

The even angular points lie above the middles of these successive segments, so that P_1 consists of the legs of a series of isosceles triangles. The vertices of these triangles are chosen so that the sum of the legs is successively a_1, $a_1 x_1$, $a_1 x_1^2$, ..., $a_1 x_1^m$, ... times their base. Here a_1 is an arbitrary number so that $1 < a_1 < k_1 = l_1$ and x_1 is defined by the condition that P_1 has the length l_1; hence:

$$\frac{a_1}{2} + \frac{a_1 x_1}{4} + \frac{a_1 x_1^2}{8} + \ldots = k_1 = l_1$$

$$2 > x_1 = \frac{2k_1 - a_1}{k_1} = \frac{2\,l_1 - a_1}{l_1} > 1.$$

If φ_m and ψ_m are the angles which (from left to right) resp. the ascending and the descending leg of the m^{th} triangle make with the X-axis,

$$\lim_{m=\infty} \varphi_m = \frac{\pi}{2} \quad ; \quad \lim_{m=\infty} \psi_m = \frac{\pi}{2}.$$

There is, therefore, a value M_1 of m for which and above which $\varphi > a$, $\psi > a$, if a represents an arbitrary angle in the first quadrant, e.g. $\frac{\pi}{4}$.

On each side of P_1, now from right to left, we choose again the dividing points on $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots$ of this side as odd angular points of a branch of P_2 (with the right extremity of this side as first angular point). The even angular points of this branch of P_2 lie again perpendicularly above the middles of the segments between the odd ones, so that P_2 is formed by the legs of a series of triangles which, in this case, are not isosceles. The vertices of these triangles are chosen so that the proportion of the sum of the legs to the base is successively (from right to left)

$$a_2, \quad a_2 x_2, \quad a_2 x_2^2, \ldots, \quad a_2 x_2^m, \ldots$$

Again

$$1 < a_2 < k_2$$

and

$$2 > x_2 = \frac{2k_2 - a_2}{k_2} > 1,$$

so that the length of P_2 is indeed l_2.

For the angles ψ_m and φ_m which, counted from a sufficiently large m, are surely positive, we have again as above:

$$\lim \psi_m = \frac{\pi}{2} \quad ; \quad \lim \varphi_m = \frac{\pi}{2},$$

hence for $m \geqq M_2$,

$$\psi > a \quad ; \quad \varphi > a.$$

Thus we go on, starting alternately from the left and from the right and always choosing $1 < a_n < k_n$ and $x_n = \frac{2\,k_n - a_n}{k_n}$: on each side of P_{n-1} we construct a broken line which is a branch of P_n, and always $l_n = \lambda^{1-2^{-n}}$.

The ordinate of a point x is cut by the lines $P_1, P_2 \ldots, P_n, \ldots$ in points with ordinates $y_1, y_2, \ldots, y_n, \ldots$; these values form a limited sequence which, if x belongs to the binary scale, consists of terms which remain equal from a definite n, but, if x is not a term of the binary scale, increases monotonely. Hence we may put

$$y = \lim y_n \equiv f(x).$$

It is clear that the curve represented by this function, is continuous and rectifiable and has the length λ.

In the points of the binary scale it is neither differentiable on the right nor on the left, for from both sides there approach points as well in a fixed non-perpendicular direction (odd points of a polygonal branch) as in a variable direction, approaching to a perpendicular (even points of a higher branch); the set of these binary points is, however, enumerable.

Any interval β contains also non-binary points which have no differential coefficient. Let us consider a part γ of this interval above which there lies exactly a complete side P_n; such a part may always be found if we choose n sufficiently large.

P_{n+1} has above γ a part consisting of legs of an infinite number of triangles of the same series for which $\varphi > \alpha$; we choose two ascending sides of them (now and in the future always counted from left to right), and call them $\delta_{0.0}$ and $\delta_{0.1}$.

Above each of these sides P_{n+1} has a part for which $\psi > \alpha$; in it we choose two descending sides above each δ: $\varepsilon_{0.00}$ and $\varepsilon_{0.01}$ (above $\delta_{0.0}$), $\varepsilon_{0.10}$ and $\varepsilon_{0.11}$ (above $\delta_{0.1}$). If we go on like this a point x of the interval γ corresponds to the binary development of any number from 0 to 1, around which we may contract an interval so that the chord of the curve above that interval successively ascends more steeply than α and descends more steeply than α; $f(x)$ is, therefore, non-differentiable in that point.

Hence it is evident that the set of these points x on γ (hence a fortiori the set of all the points on β where $f(x)$ is non-differentiable) has the same power as the continuum.